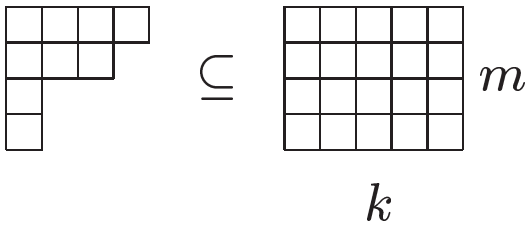
Quantum Cohomology of Grassmannians

$X = \mathrm{Gr}(m, E)$, $E = \mathbb{C}^n$, $k = n - m$

$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = E$ fixed full flag.

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m) =$ 

Schubert variety:

$\Omega_\lambda(F_\bullet) = \{V \in X \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}$

Def A *rational curve* C in X is the image of a regular map $\mathbb{P}^1 \to X$.

$\deg(C) = \#C \cap \Omega_1(F_\bullet) = \#\{V \in C \mid V \cap F_k \neq 0\}$
if F_k is in general position.

Gromov-Witten invariants

For $|\lambda| + |\mu| + |\nu| = mk + dn$ we define

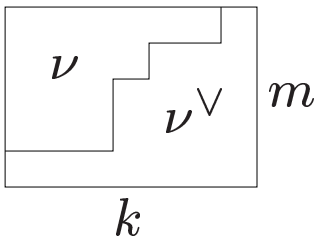
$\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = \#$ rational curves of degree d
meeting each of $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, $\Omega_\nu(H_\bullet)$
for flags $F_\bullet$, $G_\bullet$, $H_\bullet$ in general position.

(Set $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0$ if $|\lambda| + |\mu| + |\nu| \neq mk + dn$.)

Deg. zero GW invariants are computed by $H^*(X;\mathbb{Z})$:

$$H^*X = \bigoplus_{\lambda \subset \boxed{\text{m}\times\text{k}}} \mathbb{Z}\cdot\Omega_\lambda$$

$$\Omega_\lambda \cdot \Omega_\mu = \sum_\nu \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_0 \, \Omega_\nu$$



Small quantum cohomology ring

$$QH^*X = \bigoplus_{\lambda \subset \boxed{\text{m}\times\text{k}}} \mathbb{Z}[q]\cdot\sigma_\lambda$$

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu,\, d\geq 0} \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d \, q^d \, \sigma_\nu$$

AMAZING FACT (Ruan-Tian, Kontsevich-Manin): This product is associative!

Goal: Compute QH^*X with elementary methods.

Remarks:

QH^*X is graded: $\deg(q) = n$, $\deg(\sigma_\lambda) = |\lambda|$

QH^*X is a deformation of H^*X :

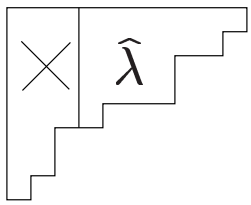
$$QH^*X/(q) \xrightarrow{\sim} H^*X \quad ; \quad \sigma_\lambda \mapsto \Omega_\lambda$$

Algebraic Geometry Sheet

$C \subset X = \mathrm{Gr}(m, E)$ rational curve of degree $d \leq k = n - m$.

Exercise $\exists$ subspace $W \subset E$ of dimension $m+d$ such that $C \subset \mathrm{Gr}(m, W) \subset \mathrm{Gr}(m, E)$.

Lemma $C \cap \Omega_\lambda(F_\bullet) \neq \emptyset$
$\Rightarrow \ W \in \Omega_{\widehat{\lambda}}(F_\bullet) \ \subset \mathrm{Gr}(m+d, E)$
where $\widehat{\lambda} = \lambda \smallsetminus$ first d columns.



Proof Take $V \in C \cap \Omega_\lambda(F_\bullet)$.
$V \subset W \ \Rightarrow \ \dim(W \cap F_{k+i-\lambda_i}) \geq i \ \forall \ i$
$\Rightarrow \ W \in \Omega_{\widehat{\lambda}}(F_\bullet)$ $\square$

Proposition If $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d \neq 0$ then
$\Omega_{\widehat{\lambda}} \cdot \Omega_{\widehat{\mu}} \cdot \Omega_{\widehat{\nu}} \neq 0$ in $H^*\,\mathrm{Gr}(m+d, E)$.

Proof Take a rational curve $C \subset X$ of degree d meeting $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, $\Omega_\nu(H_\bullet)$.
Then $W \in \Omega_{\widehat{\lambda}}(F_\bullet) \cap \Omega_{\widehat{\mu}}(G_\bullet) \cap \Omega_{\widehat{\nu}}(H_\bullet)$. $\square$

Cor If $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d \neq 0$ and $\ell(\lambda) + \ell(\mu) \leq m$ then $d = 0$.

Proof

A priori we have $|\lambda| + |\mu| + |\nu| = mk + dn$ ($\Rightarrow d \leq k$).

Now $$\begin{aligned} |\widehat{\lambda}| + |\widehat{\mu}| + |\widehat{\nu}| &\geq |\lambda| + |\mu| + |\nu| - 2md \\ &= mk + dk - dm \\ &= \dim \mathrm{Gr}(m+d, E) + d^2 \end{aligned}$$

□

Set $\sigma_i = 0$ for $i < 0$ and $k < i < n$.

Quantum Giambelli formula: (Bertram)

$$\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{m \times m} \in QH^*X$$

Proof

Define a *linear* map

$\phi : H^*X \to QH^*X$ by $\phi(\Omega_\lambda) = \sigma_\lambda$.

Cor $\Rightarrow \sigma_p \cdot \sigma_\mu = \phi(\Omega_p \cdot \Omega_\mu)$ for $\ell(\mu) \leq m-1$.

$\Rightarrow \sigma_{p_1} \cdot \sigma_{p_2} \cdots \sigma_{p_m} = \phi(\Omega_{p_1} \cdot \Omega_{p_2} \cdots \Omega_{p_m})$

$\Rightarrow \det(\sigma_{\lambda_i + j - i}) = \phi(\det(\Omega_{\lambda_i + j - i})) = \phi(\Omega_\lambda) = \sigma_\lambda$

□

Def $\quad c_i = \sigma_{(1^i)} \in QH^*X,\ 1 \le i \le m.$

(i.e. $c_i = c_i(S^\vee),\ S \subset E \otimes \mathcal{O}_X$ tautological subbundle.)

For $p \ge n$, set $\quad \sigma_p = \sum_{i=1}^{m} (-1)^{i-1} c_i \cdot \sigma_{p-i}$.

(same equation holds for all $p \ge 1$.)

Lemma $\quad$ If $\quad p \ge k+1 \quad$ then $\quad \sigma_p = (-1)^{m-1}\, q\, \sigma_{p-n}$

Proof

$k+1 \le p < n$: $0 = 0$.

$p = n$: $\sigma_n = (-1)^{m-1}\, c_m \cdot \sigma_k = (-1)^{m-1}\, q$

(because the only line meeting

$\Omega_{(1^m)}(F_\bullet),\ \Omega_{(k)}(G_\bullet),\ \Omega_{(k^m)}(H_\bullet)$ is

$C = \{V \in X \mid H_m \cap F_{n-1} \subset V \subset H_m + G_1\}$)

$p > n$: Induction on p.

□

Corollary (Siebert-Tian, Witten)

$QH^*X = \mathbb{Z}[c_1, \ldots, c_m, q]/(\sigma_{k+1}, \ldots, \sigma_{n-1}, \sigma_n + (-1)^m q)$

Def If $I = (I_1, I_2, \ldots, I_m)$ is a sequence of arbitrary integers, set

$$\sigma_I = \det(\sigma_{I_i+j-i})_{m\times m} \in QH^*X$$

Note: Using moves $\sigma_{I,a,b,J} = \sigma_{I,b-1,a+1,J}$ and $\sigma_{I,a,a+1,J} = 0$ we always have:

$$\sigma_I = 0 \quad \text{or} \quad \sigma_I = \pm\sigma_\lambda$$

Corollary $\rho = (\rho_1 \geq \cdots \geq \rho_m)$ any partition.

For $1 \leq j \leq m$ choose $I_j \in \mathbb{Z}$ such that

$I_j \equiv \rho_j \pmod{n}$ and $j - m \leq I_j < j - m + n$.

Then $\sigma_\rho = (-1)^{d(m-1)}\, q^d\, \sigma_I$, $nd = |\rho| - \sum I_j$.

Note: This $\sigma_I =$ zero or $\pm\sigma_\nu$, $\nu \subset \boxed{\text{m}\times\text{k}}$.

The Corollary is equivalent to the Rim-hook algorithm of Bertram, Ciocan-Fontanine, Fulton (cf. also Goodman-Wenzl, Walton, Kac.)

Application

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\ell(\rho)\leq m} c^\rho_{\lambda\mu}\, \sigma_\rho = \sum_{\nu\subset\boxed{\text{m}\times\text{k}}} \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee}\rangle_d\, q^d\, \sigma_\nu$$

With **Andrew Kresch** and **Harry Tamvakis** :

Fix $d \geq 0$ and set $Y = \mathsf{F}\ell(m-d, m+d; \mathbb{C}^n)$.

Given a Schubert variety $\Omega_\lambda(F_\bullet) \subset \mathsf{Gr}(m, \mathbb{C}^n)$, set

$$\tilde{\Omega}_\lambda(F_\bullet) = \{(K, W) \in Y \mid \exists V \in \Omega_\lambda(F_\bullet) : K \subset V \subset W\}$$

This is a Schubert variety in Y.

Thm For general flags $F_\bullet$, $G_\bullet$, $H_\bullet$ there is a bijection

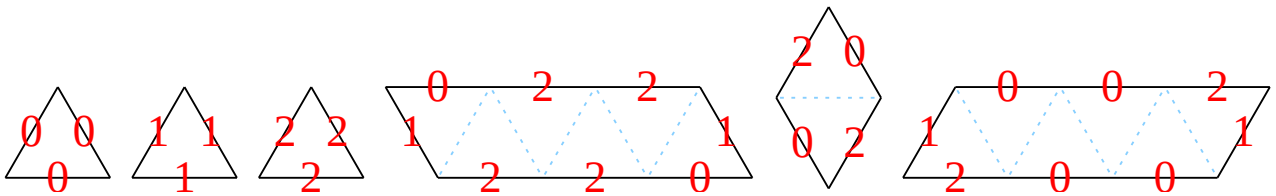
$$\left\{\begin{array}{c}\text{rat. curves in } X \\ \text{of degree } d \\ \text{meeting } \Omega_\lambda(F_\bullet), \\ \Omega_\mu(G_\bullet),\ \Omega_\nu(H_\bullet)\end{array}\right\} \longleftrightarrow \tilde{\Omega}_\lambda(F_\bullet) \cap \tilde{\Omega}_\mu(G_\bullet) \cap \tilde{\Omega}_\nu(H_\bullet)$$

$$C \quad \longmapsto \quad \left(\bigcap_{V \in C} V \quad , \quad \sum_{V \in C} V \right)$$

Consequence: $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = \int_Y \tilde{\Omega}_\lambda \cdot \tilde{\Omega}_\mu \cdot \tilde{\Omega}_\nu$

Reviving Knutson's conjecture

A *012-puzzle* is a decomposition of an equilateral triangle into puzzle pieces of the following types (up to rotation), with matching labels:



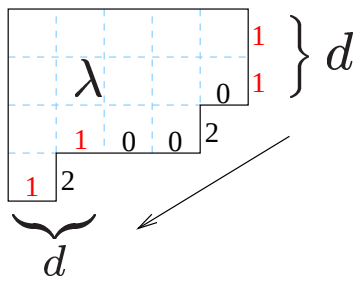
Recall: Schubert varieties in $Y = \mathrm{F}\ell(a, b; \mathbb{C}^n)$ are indexed by *012-strings* with
a "0"s, $b - a$ "1"s, and $n - b$ "2"s.

Conjecture (Knutson) If u, v, w are 012-strings then

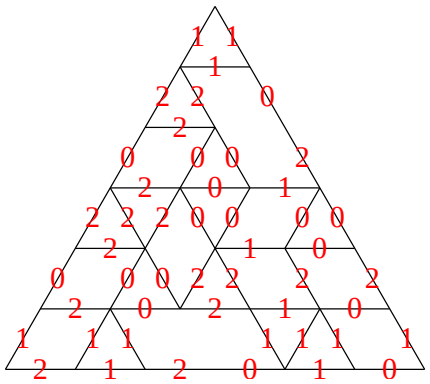
$$\int_Y \Omega_u \cdot \Omega_v \cdot \Omega_w = \#\ \text{012-puzzles with } u,\ v,\ w \text{ as border labels.}$$

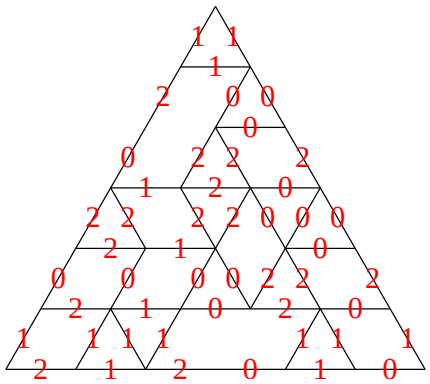
Example $X = \mathrm{Gr}(3, \mathbb{C}^6)$, $d = 1$, $Y = \mathrm{F}\ell(2, 4; \mathbb{C}^6)$

$$\langle \Omega_{(3,2,1)}, \Omega_{(3,2,1)}, \Omega_{(2,1)} \rangle_1$$
$$= \int_Y \tilde{\Omega}_{(3,2,1)} \cdot \tilde{\Omega}_{(3,2,1)} \cdot \tilde{\Omega}_{(2,1)}$$
$$= \int_Y \Omega_{102021} \cdot \Omega_{102021} \cdot \Omega_{010212}$$
$$= 2$$



$$\tilde{\Omega}_\lambda = \Omega_{110200121}$$





Knutson's conjecture has been verified for all $\mathrm{F}\ell(a, b; \mathbb{C}^n)$ with $n \leq 12$.

The resulting quantum Littlewood-Richardson rule has been proved in all cases where a quantum LR rule had previously been obtained by Bégin, Kirillov, Mathieu, Tudose, Walton.