Gromov-Witten invariants on Grassmannians

Joint work with
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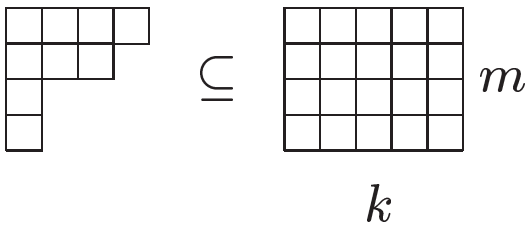
$X = \mathrm{Gr}(m, \mathbb{C}^n)$, $k = n - m$

$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$ fixed full flag.

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m) =$  $\subseteq$ m

k

Schubert variety:

$\Omega_\lambda(F_\bullet) = \{V \in X \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}$

Definition A *rational curve* C in X is the image of a regular map $\mathbb{P}^1 \to X$.

$\deg(C) = \#C \cap \Omega_1(F_\bullet) = \#\{V \in C \mid V \cap F_k \neq 0\}$
if F_k is in general position.

Gromov-Witten invariant

For $|\lambda| + |\mu| + |\nu| = mk + dn$ we define

$\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = \#$ rational curves of degree d
meeting each of $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, $\Omega_\nu(H_\bullet)$
for flags $F_\bullet$, $G_\bullet$, $H_\bullet$ in general position.

Computing Gromov-Witten invariants

Fix $d \geq 0$.

Set $Y_d = \mathsf{F}\ell(m-d, m+d; \mathbb{C}^n)$
$= \{(A, B) \mid A^{m-d} \subset B^{m+d} \subset \mathbb{C}^n\}$.

Given $\Omega_\lambda(F_\bullet) \subset X$, define

$\widetilde{\Omega}_\lambda(F_\bullet) = \{(A, B) \in Y_d \mid \exists V \in \Omega_\lambda(F_\bullet) : A \subset V \subset B\}$

$$\begin{array}{ccccc} & & \mathsf{F}\ell(m-d, m, m+d; \mathbb{C}^n) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ Y_d & & & & X \end{array}$$

$\widetilde{\Omega}_\lambda(F_\bullet) = \pi_1(\pi_2^{-1}(\Omega_\lambda(F_\bullet)))$.

$\Longrightarrow$ • $\widetilde{\Omega}_\lambda(F_\bullet) \subset Y_d$ is a Schubert variety.

• $\operatorname{codim} \widetilde{\Omega}_\lambda(F_\bullet) \geq |\lambda| - d^2$.

Theorem (B-K-T)

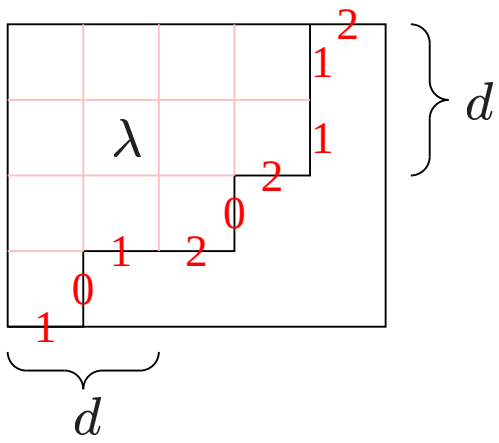
$$\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = \int_{Y_d} \widetilde{\Omega}_\lambda \cdot \widetilde{\Omega}_\mu \cdot \widetilde{\Omega}_\nu$$

Application

Schubert varieties in $Y_d = \mathrm{F}\ell(m-d, m+d; \mathbb{C}^n)$ are indexed by *012-strings* with $m-d$ "0"s, $2d$ "1"s, and $k-d$ "2"s.

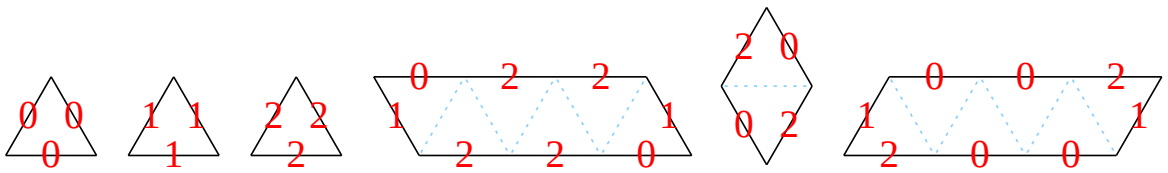
Example

$X = \mathrm{Gr}(4, \mathbb{C}^9)$, $d = 2$, and $\lambda = (4,4,3,1)$. We have $\widetilde{\Omega}_\lambda(F_\bullet) = \Omega_{101202112}(F_\bullet)$ in $Y_d = \mathrm{F}\ell(2,6;\mathbb{C}^9)$.



Conjecture (Knutson) If u, v, w are 012-strings then

$$\int_{Y_d} \Omega_u \cdot \Omega_v \cdot \Omega_w \;=\; \#\ \textit{012-puzzles} \text{ with } u,\ v,\ w \text{ as border labels.}$$

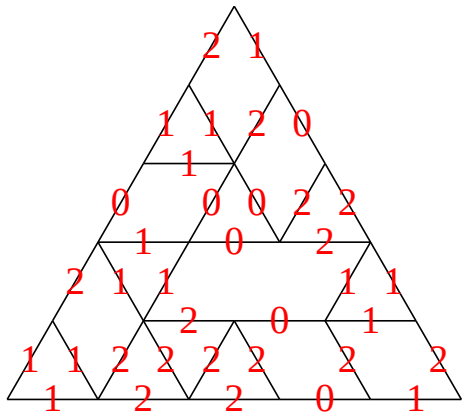


Example $X = \mathrm{Gr}(2, \mathbb{C}^5)$, $d = 1$, $Y_d = \mathrm{F}\ell(1,3;\mathbb{C}^5)$.

$$\langle \Omega_{(2,2)}, \Omega_{(2,1)}, \Omega_{(3,1)} \rangle_1 =$$

$$\int_{Y_d} \widetilde{\Omega}_{(2,2)} \cdot \widetilde{\Omega}_{(2,1)} \cdot \widetilde{\Omega}_{(3,1)} =$$

$$\int_{Y_d} \Omega_{12012} \cdot \Omega_{10212} \cdot \Omega_{10221} = 1$$



Definition (B) Given $C \subset X$ we set

$$\ker(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n \quad \text{and} \quad \operatorname{span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$$

Lemma (B)

$$\dim(\ker(C)) \geq m - d \quad \text{and} \quad \dim(\operatorname{span}(C)) \leq m + d \ .$$

Theorem (B-K-T)

Suppose $|\lambda| + |\mu| + |\nu| = mk + nd$.
For general flags $F_\bullet$, $G_\bullet$, $H_\bullet$ there is a bijection

$$\left\{ \begin{array}{c} \text{rat. curves in } X \\ \text{of degree } d \\ \text{meeting } \Omega_\lambda(F_\bullet), \\ \Omega_\mu(G_\bullet),\ \Omega_\nu(H_\bullet) \end{array} \right\} \longleftrightarrow \widetilde{\Omega}_\lambda(F_\bullet) \cap \widetilde{\Omega}_\mu(G_\bullet) \cap \widetilde{\Omega}_\nu(H_\bullet)$$

$$C \quad \longmapsto \quad (\ker(V), \operatorname{span}(V))$$

Proof

Set $\mathfrak{C} = \{$ curves in X of degree d meeting $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, and $\Omega_\nu(H_\bullet)$ $\}$

$$\mathfrak{I} = \widetilde{\Omega}_\lambda(F_\bullet) \cap \widetilde{\Omega}_\mu(G_\bullet) \cap \widetilde{\Omega}_\nu(H_\bullet) \subset Y_d$$

Step 1: The map $\mathfrak{C} \to \mathfrak{I}$ given by $C \mapsto (\ker(C), \operatorname{span}(C))$ is well defined.

$$\operatorname{codim} \widetilde{\Omega}_\lambda + \operatorname{codim} \widetilde{\Omega}_\mu + \operatorname{codim} \widetilde{\Omega}_\nu$$
$$\geq |\lambda|+|\mu|+|\nu|-3d^2 = mk+nd-3d^2 = \dim Y_d$$

$\Longrightarrow$ $\mathfrak{I}$ is a finite set.

Let $C \in \mathfrak{C}$.

Suppose $(A, B) \in Y_d$. Then
$A \subset \ker(C)$ and $\operatorname{span}(C) \subset B \Longrightarrow (A, B) \in \mathfrak{I}$.

If $m - d < \dim(\ker(C))$ or $\dim(\operatorname{span}(C)) < m + d$ then there are infinitely many points $(A, B) \in \mathfrak{I}$, a contradiction.

Conclude: $(\ker(C), \operatorname{span}(C)) \in \mathfrak{I}$

Lemma Let $P, Q, R \in Z = \mathrm{Gr}(d, \mathbb{C}^{2d})$.
Suppose $P \cap Q = Q \cap R = R \cap P = 0$.
Then there exists exactly one rational curve $C \subset Z$ of degree d such that $P, Q, R \in C$.

Proof Let $\{r_1, \ldots, r_d\}$ be a basis for $R \subset \mathbb{C}^{2d}$.
Let $p_i \in P$ and $q_i \in Q$ be the projections of $r_i \in \mathbb{C}^{2d} = P \oplus Q$. Then C is the image of $\mathbb{P}^1 \to Z$; $(s:t) \mapsto \mathrm{span}\{s\,p_1 + t\,q_1, \ldots, s\,p_d + t\,q_d\}$.
□

Step 2: Let $(A_0, B_0) \in \mathfrak{I} \subset Y_d$. There is exactly one curve $C \in \mathfrak{C}$ with kernel A_0 and span B_0.

Set $Z = Gr(d, B_0/A_0) = \{V \in X \mid A_0 \subset V \subset B_0\}$.

Claim: Each of $\Omega_\lambda(F_\bullet) \cap Z$, $\Omega_\mu(G_\bullet) \cap Z$, and $\Omega_\nu(H_\bullet) \cap Z$ are single points in Z which satisfy the Lemma.

Enough to prove:

If $P \in \Omega_\lambda(F_\bullet) \cap Z$ and $Q \in \Omega_\mu(G_\bullet) \cap Z$ then $P \cap Q = A_0$.

Set:

$W = \mathsf{F}\ell(m-d, m-d+1, m+d; \mathbb{C}^n)$

$\widehat{\Omega}_\lambda = \{(A, A', B) \in W \mid \exists V \in \Omega_\lambda(F_\bullet) : A' \subset V \subset B\}$

$\widehat{\Omega}_\mu = \{(A, A', B) \in W \mid \exists V \in \Omega_\mu(G_\bullet) : A' \subset V \subset B\}$

$\widetilde{\Omega}_\nu = \{(A, A', B) \in W \mid \exists V \in \Omega_\nu(H_\bullet) : A \subset V \subset B\}$

$\operatorname{dim} W = \operatorname{dim} Y_d + 2d - 1$

$\operatorname{codim} \widehat{\Omega}_\lambda \geq |\lambda| - d^2 + d$

$\operatorname{codim} \widehat{\Omega}_\mu \geq |\mu| - d^2 + d$

$\operatorname{codim} \widetilde{\Omega}_\nu \geq |\nu| - d^2$

$\operatorname{codim} \widehat{\Omega}_\lambda + \operatorname{codim} \widehat{\Omega}_\mu + \operatorname{codim} \widetilde{\Omega}_\nu > \operatorname{dim} W$

$\Longrightarrow \widehat{\Omega}_\lambda \cap \widehat{\Omega}_\mu \cap \widetilde{\Omega}_\nu = \emptyset$

Recall $P \in \Omega_\lambda(F_\bullet) \cap Z$ and $Q \in \Omega_\mu(G_\bullet) \cap Z$;
$Z = \mathsf{Gr}(d, B_0/A_0) \subset X$.

If $P \cap Q \neq A_0$ then take $A' \subset \mathbb{C}^n$ of dimension $m - d + 1$ such that $A_0 \subset A' \subset P \cap Q$.

Then $(A_0, A', B_0) \in \widehat{\Omega}_\lambda \cap \widehat{\Omega}_\mu \cap \widetilde{\Omega}_\nu$, contradiction.

□

Example

Let $X = \mathrm{F}\ell(\mathbb{C}^5)$ (full flag variety).

Let $\pi_i : X \to \mathrm{Gr}(i, \mathbb{C}^5)$ be the projection.

For $C \subset X$ set

$\ker(C) = (A_1 \subset A_2 \subset A_3 \subset A_4)$, $A_i = \ker(\pi_i(C))$

$\mathrm{span}(C) = (B_1 \subset B_2 \subset B_3 \subset B_4)$, $B_i = \mathrm{span}(\pi_i(C))$

Let $F_\bullet$, $G_\bullet$, $H_\bullet$ be three general points in X.

Then there are exactly two rational curves $C \subset X$ of multidegree $\underline{d} = (2, 3, 3, 2)$ containing these points. (I.e. $\langle \mathrm{point}, \mathrm{point}, \mathrm{point} \rangle_{\underline{d}} = 2$.)

Both of these curves have

$\ker(C) = (0\,,\, 0\,,\, 0\,,\, F_4 \cap G_4 \cap H_4)$ and

$\mathrm{span}(C) = (F_1 + G_1 + H_1\,,\, \mathbb{C}^5\,,\, \mathbb{C}^5\,,\, \mathbb{C}^5)$.

Conclude:

For flag varieties it is not in general true that the curves counted by a Gromov-Witten invariant are distinguished by their kernels and spans.