Quantum cohomology of partial flag manifolds

Anders Skovsted Buch

$a = (0 < a_1 < a_2 < \cdots < a_k < n)$

$X = \mathsf{F}\ell(a; \mathbb{C}^n)$
$= \{V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n \mid \dim V_j = a_j\}$

$S_n(a) \subset S_n$ permutations w with descent positions among $\{a_1, \ldots, a_k\}$.
i.e. $i \notin \{a_j\} \Rightarrow w(i) < w(i+1)$

Schubert varieties: $w \in S_n(a)$

$\Omega_w \subset X$ subset of $V_\bullet \in X$ such that
$\dim(V_j \cap \mathbb{C}^p) \geq \#\{t \leq a_j : w(t) > n - p\}$

$$H^*(X) = \bigoplus_{w \in S_n(a)} \mathbb{Z} \cdot [\Omega_w]$$

$[\Omega_w]$ is Poincare dual to $[\Omega_{w^\vee}]$,
$w^\vee = w_0\, w\, w_a \in S_n(a)$
$w_a \in S_n$ longest perm. generated by s_i , $i \notin \{a_j\}$.

Gromov-Witten invariants

A (rational) *curve* $C \subset X$ is the image of a regular map $\mathbb{P}^1 \to X$.

C has *multidegree* $d = (d_1, \ldots, d_k)$ if

$$[C] \;=\; \sum_{i=1}^{k} d_i \, [\Omega_{s_{a_i}^\vee}] \;\in\; H_2(X)$$

Let $u, v, w \in S_n(a)$ be such that

$\ell(u) + \ell(v) + \ell(w) = \dim X + \sum (a_{i+1} - a_{i-1}) d_i$

$\langle \Omega_u, \Omega_v, \Omega_w \rangle_d$:= # rational curves $C \subset X$ of multidegree d meeting fixed general translates of Ω_u , Ω_v , Ω_w .

(Small) Quantum cohomology

Define $\mathbb{Z}[q] \;=\; \mathbb{Z}[q_1, \ldots, q_k]$

$$QH^*(X) \;=\; \bigoplus_{w \in S_n(a)} \mathbb{Z}[q] \cdot \sigma_w$$

$$\sigma_u \cdot \sigma_v \;=\; \sum_{w,d} \langle \Omega_u, \Omega_v, \Omega_{w^\vee} \rangle_d \; q^d \, \sigma_w$$

Pieri formula

u , w , $\alpha = s_r s_{r+1} \cdots s_m \in S_n$

Write $u \xrightarrow{\alpha} w$ if and only if

$\exists\ b_1, \ldots, b_\ell, c_1, \ldots, c_\ell$, $\ell = \ell(\alpha)$, such that

- $b_i \leq m < c_i$ for all i
- $w = u\, t_{b_1 c_1} \cdots t_{b_\ell c_\ell}$
- $\ell(u\, t_{b_1 c_1} \cdots t_{b_i c_i}) = \ell(u) + i$ for all i
- $b_1, \ldots, b_\ell$ are distinct.

If $m = a_j$ then $\alpha \in S_n(a)$

$[\Omega_\alpha] = c_\ell(\mathcal{V}_j^\vee) \in H^*(X)$,

$\mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k \subset \mathbb{C}^n \times X$ tautological flag.

Thm (Lascoux, Schützenberger , Sottile)

$$[\Omega_\alpha] \cup [\Omega_u] = \sum_{u \xrightarrow{\alpha} w} [\Omega_w]$$

Quantum Pieri formula

$d = (d_1, \ldots, d_k)$, $0 \leq a_1 - d_1 \leq \cdots \leq a_k - d_k$.

Define $\gamma_d := \tau_1 \tau_2 \cdots \tau_k \in S_n$, where

$$\tau_i(p) = \begin{cases} p + a_{i+1} - a_i & \text{if } a_i - d_i < p \leq a_i \\ p - d_i & \text{if } a_i < p \leq a_{i+1} \\ p & \text{otherwise} \end{cases}$$

d is a *Pieri sequence with maximum at position* j if $d_1 \leq \cdots \leq d_{j-1} \leq d_j \geq d_{j+1} \geq \cdots \geq d_k$ and $|d_i - d_{i+1}| \leq 1 \quad \forall\ 0 \leq i \leq k$. $(d_0 = d_{k+1} = 0)$

Theorem (Ciocan-Fontanine)
Let $\alpha = s_r s_{r+1} \cdots s_{a_j}$ and $u \in S_n(a)$. Then

$$\sigma_\alpha \cdot \sigma_u = \sum_{w,d} q^d \sigma_w$$

the sum is over all Pieri sequences d with max. at position j and $w \in S_n(a)$ such that $\ell(u\gamma_d) = \ell(u) - \ell(\gamma_d)$, $\ell(w^\vee \gamma_d) = \ell(w^\vee) - \ell(\gamma_d)$, and $u\,\gamma_d \xrightarrow{\overline{\alpha}} w\, w_a\, \gamma_d\, w_b$;
$\overline{\alpha} = s_r s_{r+1} \cdots s_{b_j}$, $b = (a_1 - d_1, \ldots, a_k - d_k)$.

Quantum Giambelli formula

Let $w \in S_n(a)$.

Choose j maximal s.t. $w(a_j+1) \neq a_j+1$.

Define $u \in S_n(a)$ by

$$u(p) = \begin{cases} w(p) & \text{if } i \leq a_j \text{ and } w(p) < w(a_j+1) \\ w(p)-1 & \text{if } i \leq a_j \text{ and } w(p) > w(a_j+1) \\ w(p+1)-1 & \text{if } i > a_j \end{cases}$$

Set $\alpha = s_{w(a_j+1)} \cdots s_{a_j-1} s_{a_j} \in S_n(a)$. Then

$$(*) \qquad \sigma_w = \sigma_u \cdot \sigma_\alpha - \sum_{u \xrightarrow{\alpha} v \neq w} \sigma_v$$

can be used recursively to write $\sigma_w \in QH^*(X)$ as a polynomial in the Chern classes σ_α .

Computing Gromov-Witten invariants

$\langle \Omega_u, \Omega_v, \Omega_w \rangle_d$ = coeff. of $q^d \sigma_{w^\vee}$ in $\sigma_u \cdot \sigma_v$

Use (*) to write $\sigma_u = P(\{\sigma_\alpha\})$

Compute $P \cdot \sigma_v$ using quantum Pieri formula.

Geometry

$C \subset X = \mathsf{F}\ell(a;\mathbb{C}^n)$ curve of multidegree d

Define $$K_i = \bigcap_{V_\bullet \in C} V_i \quad \subset \mathbb{C}^n$$

Lemma $\dim(K_i) \geq a_i - d_i$

Kernel of C :

$$K_\bullet = \{K_1 \subset K_2 \subset \cdots \subset K_k\} \in \mathsf{F}\ell(b;\mathbb{C}^n)\ ,$$

where $b_i = \dim K_i$

Given subvariety $Z \subset X$, define

$$\widehat{Z} = \{W_\bullet \in \mathsf{F}\ell(b;\mathbb{C}^n) \mid \exists V_\bullet \in Z : W_i \subset V_i\ \forall i\}$$

Notice: $C \cap Z \neq \emptyset \ \Rightarrow\ K_\bullet \in \widehat{Z}$

Set $e_i = a_i - b_i$. Then $e_i \leq d_i$

Lemma

- $\operatorname{codim}(\widehat{Z}) \geq \operatorname{codim}(Z) - \ell(\gamma_e)$
- $\operatorname{codim}(\widehat{\Omega}_u) = \operatorname{codim}\Omega_u - \ell(\gamma_e)$

$$\Rightarrow \quad \widehat{\Omega}_u = \Omega_{u\gamma_e} \subset \mathsf{F}\ell(b;\mathbb{C}^n)$$

Special case of quantum Pieri formula

(enough to prove quantum Giambelli.)

Let u , w , $\alpha = s_r s_{r+1} \cdots s_{a_j} \in S_n(a)$,

$\ell(u) + \ell(w) + \ell(\alpha) = \dim X + \sum(a_{i+1} - a_{i-1})d_i$

Suppose $\langle \Omega_u, \Omega_w, \Omega_\alpha \rangle_d \neq 0$

I will show that d is a Pieri sequence with maximum at position j , and

$$\int_{\mathrm{F}\ell(a-d;\mathbb{C}^n)} [\Omega_{u\gamma_d}] \cup [\Omega_{w\gamma_d}] \cup [\Omega_{\bar{\alpha}}] \neq 0$$

where $\bar{\alpha} = s_r s_{r+1} \cdots s_{a_j - d_j}$

(this says that $u\,\gamma_d \xrightarrow{\bar{\alpha}} w^\vee\, w_a\, \gamma_d\, w_{a-d}$.)

Let $C \subset X$ be a curve of multidegree d meeting Ω_u , Ω_v , and Ω_α .

$K_\bullet \in \widehat{\Omega}_u \cap \widehat{\Omega}_w \cap \widehat{\Omega}_\alpha \subset \mathsf{F}\ell(b;\mathbb{C}^n)$. Set $e = a - b$.

$\widehat{\Omega}_\alpha = \Omega_{\overline{\alpha}}$, $\overline{\alpha} = s_r s_{r+1} \cdots s_{b_j}$

$$\ell(u) + \ell(w) + \ell(\alpha) - 2\ell(\gamma_e) - e_j \leq \dim \mathsf{F}\ell(b;\mathbb{C}^n)$$

$$\iff e_j + \sum e_i e_{i+1} - \sum e_i^2 \geq \sum (a_{i+1} - a_{i-1})(d_i - e_i)$$

Lemma $e = (e_1, \ldots, e_k)$ non-negative integers.

$e_j + \sum e_i e_{i+1} - \sum e_i^2 \geq 0 \iff$

$e_j + \sum e_i e_{i+1} - \sum e_i^2 = 0 \iff$

e is a Pieri sequence with max. at position j.

Conclude:

- $d = e$ is a Pieri sequence.
- $\operatorname{codim} \widehat{\Omega}_u = \operatorname{codim} \Omega_u - \ell(\gamma_d)$
  $\Rightarrow \widehat{\Omega}_u = \Omega_{u\gamma_d} \subset \mathsf{F}\ell(b;\mathbb{C}^n)$
- $\displaystyle\int_{\mathsf{F}\ell(b;\mathbb{C}^n)} [\Omega_{u\gamma_d}] \cup [\Omega_{w\gamma_d}] \cup [\Omega_{\bar{\alpha}}] \neq 0$