

# EQUIVARIANT RIGIDITY OF RICHARDSON VARIETIES

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ABSTRACT. We prove that Schubert and Richardson varieties in flag manifolds are uniquely determined by their equivariant cohomology classes, as well as a stronger result that replaces Schubert varieties with closures of Bialynicki-Birula cells under suitable conditions. This is used to prove a conjecture from [BCP23], stating that any two-pointed curve neighborhood representing a quantum cohomology product with a Seidel class is a Schubert variety. We pose a stronger conjecture which implies a Seidel multiplication formula in equivariant quantum  $K$ -theory, and prove this conjecture for cominuscle flag varieties.

## 1. INTRODUCTION

A Schubert variety  $\Omega$  in a flag manifold  $X = G/P$  is called *rigid* if it is uniquely determined by its class  $[\Omega]$  in the cohomology ring  $H^*(X)$ . More precisely, if  $Z \subset X$  is any irreducible closed subvariety such that  $[Z]$  is a multiple of  $[\Omega]$  in  $H^*(X)$ , then  $Z$  is a  $G$ -translate of  $\Omega$ . This problem has been studied in numerous papers, see e.g. [Hon05, Hon07, Cos11, RT12, CR13, Cos14, Cos18, HM20, LSY24] and the references therein. In this paper we show that all Schubert varieties and Richardson varieties are *equivariantly rigid*. In other words, if  $T \subset G$  is a maximal torus,  $\Omega \subset X$  is a  $T$ -stable Richardson variety, and  $Z \subset X$  is a (non-empty)  $T$ -stable closed subvariety such that the  $T$ -equivariant class  $[Z] \in H_T^*(X)$  is a multiple of  $[\Omega]$ , then  $Z = \Omega$ .

More generally, let  $T$  be an algebraic torus over an algebraically closed field, let  $X$  be a non-singular projective  $T$ -variety, and let  $\Omega \subset X$  be a  $T$ -stable closed subvariety. Let  $\Omega^T$  denote the set of  $T$ -fixed points in  $\Omega$ . We will say that  $\Omega$  is  *$T$ -convex* if, for any  $T$ -stable closed subvariety  $Z \subset X$  satisfying  $Z^T \subset \Omega$ , we have  $Z \subset \Omega$ . A fixed point  $p \in X^T$  is called *fully definite* if all  $T$ -weights of the Zariski tangent space  $T_p X$  belong to a strict half-space of the character lattice of  $T$ . We show that if all  $T$ -fixed points in  $X$  are fully definite, then any irreducible  $T$ -convex subvariety of  $X$  is also  $T$ -equivariantly rigid. Here  $\Omega$  is called  *$T$ -equivariantly rigid* if  $\Omega$  is determined by its class in the  $T$ -equivariant Chow cohomology ring of  $X$ .

Let  $\mathbb{G}_m \subset T$  be a 1-parameter subgroup such that  $X^T = X^{\mathbb{G}_m}$ , and assume that this fixed point set is finite. The associated Bialynicki-Birula decomposition of  $X$  is

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given by  $X = \bigcup_{p \in X^T} X_p^+$ , where  $X_p^+ = \{x \in X \mid \lim_{t \rightarrow 0} t.x = p\}$  is the Bialynicki-Birula cell of points attracted to  $p$  by the action of  $t \in \mathbb{G}_m$ . This decomposition is called a *stratification* if each cell closure  $\overline{X_p^+} \subset X$  is a union of smaller cells. In this case we show that all cell closures are  $T$ -convex. We arrive at the following result combining [Theorem 4.3](#) and [Proposition 5.3](#).

**Theorem.** *Let  $X$  be a non-singular projective  $T$ -variety with finitely many  $T$ -fixed points, and choose  $\mathbb{G}_m \subset T$  such that  $X^T = X^{\mathbb{G}_m}$ .*

- (a) *Assume that the Bialynicki-Birula decomposition of  $X$  is a stratification. Then each cell closure  $\overline{X_p^+}$  is  $T$ -convex.*
- (b) *Assume that all  $T$ -fixed points in  $X$  are fully definite. Then any irreducible  $T$ -convex subvariety of  $X$  is  $T$ -equivariantly rigid.*

This result applies to Schubert and Richardson varieties in flag varieties, as well as positroid varieties in Grassmannians, so these subvarieties are both  $T$ -convex and  $T$ -equivariantly rigid. However, projected Richardson varieties do not in general enjoy these properties, see [Remark 6.5](#). Our theorem also covers a class of horospherical varieties, which includes all non-singular horospherical varieties of Picard rank 1 [[Pas09](#)].

Our theorem has additional applications in quantum Schubert calculus. Let  $X = G/P$  be a complex flag manifold. A Schubert class  $[X^w]$  is called a *Seidel class* if the Weyl group element  $w$  is the minimal representative of a point in some cominuscule flag variety  $G/Q$ . Multiplication by Seidel classes in the quantum cohomology ring  $\text{QH}(X)$  is given by the identity  $[X^w] \star [X^u] = q^{d(w,u)} [X^{wu}]$ , where  $d(w,u)$  is the unique minimal degree of a rational curve connecting the opposite Schubert varieties  $X_{w_0 w}$  and  $X^u$  [[Sei97](#), [Bel04](#), [CMP09](#)]. This implies that  $[X^{wu}]$  is equal to the class of the curve neighborhood  $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ , defined as the union of all stable curves in  $X$  of degree  $d(w,u)$  connecting  $X_{w_0 w}$  to  $X^u$ . We conjectured in [[BCP23](#)] that this curve neighborhood is in fact the translated Schubert variety

$$(1) \quad \Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1}.X^{wu}.$$

This has been proved in some cases when  $X$  is cominuscule, in all cases when  $X$  is a flag variety of type A [[LLSY22](#), [Tar23](#)], and for  $X = \text{SG}(2, 2n)$  [[BPX24](#)]. Using that  $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$  and  $w^{-1}.X^{wu}$  define the same class in  $H_T^*(X)$  by an equivariant version of the Seidel multiplication formula from [[CMP09](#), [CP23](#)], the identity (1) follows from our result that Schubert varieties are equivariantly rigid.

In this paper we conjecture the more general identity

$$(2) \quad \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u) = \Gamma_e(w^{-1}(X^{wu})),$$

where the right hand side is the union of all stable curves of degree  $e$  that pass through  $w^{-1}.X^{wu}$ . This union is a Schubert variety [[BCMP13](#)] whose Weyl group element was determined in [[BM15](#)]. Let  $M_{d(w,u)+e}(X_{w_0 w}, X^u)$  denote the moduli space of 3-pointed stable maps to  $X$  of degree  $d(w,u) + e$  and genus zero, which send the first two marked points to  $X_{w_0 w}$  and  $X^u$ , respectively. We further conjecture that the evaluation map  $\text{ev}_3 : M_{d(w,u)+e}(X_{w_0 w}, X^u) \rightarrow \Gamma_e(w^{-1}(X^{wu}))$  is cohomologically trivial. This conjecture implies a Seidel multiplication formula in the equivariant quantum  $K$ -theory ring  $\text{QK}_T(X)$ . We prove this conjecture when  $X$  is a cominuscule flag variety, thereby obtaining an equivariant generalization of our Seidel multiplication formula from [[BCP23](#)]. This generalized Seidel multiplication formula has also been obtained for Grassmannians of type A in [[GKM25](#)]

using different methods. Based on suggestions from Mihail Tarigradschi, we finally apply the methods of [Tar23] to prove the identity (2) when  $X = \mathrm{GL}_n(\mathbb{C})/P$  is any flag manifold of Lie type A.

Our paper is organized as follows. In Section 2 we recall some basic facts and notation related to torus actions. In Section 3 we show that if all  $T$ -fixed points of  $X$  are fully definite, then the fixed point set  $Z^T$  of a  $T$ -stable subvariety  $Z \subset X$  is determined by its equivariant class  $[Z] \in H_T^*(X)$ . This is used in Section 4 to prove part (b) of the above theorem. Section 5 proves part (a). Section 6 interprets our theorem for flag varieties, which is used in Section 7 to prove the conjecture about curve neighborhoods from [BCP23]. Section 8 discusses the more general conjecture as well as its consequences in quantum  $K$ -theory. Finally, Section 9 interprets our theorem for certain horospherical varieties.

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## 2. TORUS ACTIONS

We work with varieties over a fixed algebraically closed field  $\mathbb{K}$ . Varieties are reduced but not necessarily irreducible. A point will always mean a closed point. The multiplicative group of  $\mathbb{K}$  is denoted  $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ . An (algebraic) torus is a group variety isomorphic to  $(\mathbb{G}_m)^r$  for some  $r \in \mathbb{N}$ .

Let  $T = (\mathbb{G}_m)^r$  be an algebraic torus. Any rational representation  $V$  of  $T$  is a direct sum  $V = \bigoplus_{\lambda} V_{\lambda}$  of weight spaces  $V_{\lambda} = \{v \in V \mid t.v = \lambda(t)v \ \forall t \in T\}$  defined by characters  $\lambda : T \rightarrow \mathbb{G}_m$ . The *weights* of  $V$  are the characters  $\lambda$  for which  $V_{\lambda} \neq 0$ . The group of all characters of  $T$  is called the *character lattice* and is isomorphic to  $\mathbb{Z}^r$ . Given a  $T$ -variety  $X$ , we let  $X^T \subset X$  denote the closed subvariety of  $T$ -fixed points. A subvariety  $Z \subset X$  is called  $T$ -stable if  $t.z \in Z$  for all  $t \in T$  and  $z \in Z$ . In this case  $Z$  is itself a  $T$ -variety.

**Definition 2.1.** The  $T$ -fixed point  $p \in X$  is *non-degenerate* in  $X$  if  $T$  acts with non-zero weights on the Zariski tangent space  $T_p X$ . The point  $p$  is *fully definite* if all  $T$ -weights of  $T_p X$  belong to a strict half-space of the character lattice of  $T$ .

Equivalently,  $p \in X^T$  is fully definite in  $X$  if and only if there exists a cocharacter  $\rho : \mathbb{G}_m \rightarrow T$  such that  $\mathbb{G}_m$  acts with strictly positive weights on  $T_p X$  through  $\rho$ . For example, if  $X = G/P$  is a flag variety and  $T \subset G$  is a maximal torus, then all points of  $X^T$  are fully definite in  $X$  (see Section 6). Any non-degenerate  $T$ -fixed point must be isolated in  $X^T$ . Fully definite  $T$ -fixed points are called *attractive* in many sources, see e.g. [Bri97]; here we follow the terminology from [BB73].

**Remark 2.2.** If  $X$  is a normal quasi-projective  $T$ -variety, then  $X^{\mathbb{G}_m} = X^T$  holds for all general cocharacters  $\rho : \mathbb{G}_m \rightarrow T$ . Here a cocharacter is called *general* if it avoids finitely many hyperplanes in the lattice of all cocharacters. This follows because  $X$  admits an equivariant embedding  $X \subset \mathbb{P}(V)$ , where  $V$  is a rational representation of  $T$  [Kam66, Mum65, Sum74].

In the rest of this paper we let  $X$  be a non-singular  $T$ -variety. The  $T$ -equivariant Chow cohomology ring of  $X$  will be denoted  $H_T^*(X)$ , see [Ful98, AF24]. This is

an algebra over the ring  $H_T^*(\text{point})$ , which may be identified with the symmetric algebra of the character lattice of  $T$ . Given a class  $\sigma \in H_T^*(X)$  and a  $T$ -fixed point  $p \in X^T$ , we let  $\sigma_p \in H_T^*(\text{point})$  denote the pullback of  $\sigma$  along the inclusion  $\{p\} \rightarrow X$ . When  $X$  is defined over  $\mathbb{K} = \mathbb{C}$ , Chow cohomology can be replaced with singular cohomology. In fact, our arguments will only depend on equivariant classes  $[Z]_p \in H_T^*(\text{point})$  obtained by restricting the class of a  $T$ -stable closed subvariety  $Z \subset X$  to a fixed point, and these restrictions are independent of the chosen cohomology theory. Similarly, we can use cohomology with coefficients in either  $\mathbb{Z}$  or  $\mathbb{Q}$ .

### 3. EQUIVARIANT LOCAL CLASSES

Let  $Z$  be a  $T$ -variety, fix  $p \in Z^T$ , and let  $\mathfrak{m} \subset \mathcal{O}_{Z,p}$  be the maximal ideal in the local ring of  $p$ . Then the tangent cone  $C_p Z = \text{Spec}(\bigoplus \mathfrak{m}^i / \mathfrak{m}^{i+1})$  is a  $T$ -stable closed subscheme of the Zariski tangent space  $T_p Z = (\mathfrak{m} / \mathfrak{m}^2)^\vee = \text{Spec}(\text{Sym}(\mathfrak{m} / \mathfrak{m}^2))$ . The *local class* of  $Z$  at  $p$  is defined by (see [AF24, §17.4])

$$(3) \quad \eta_p Z = [C_p Z] \in H_T^*(T_p Z) = H_T^*(\text{point}).$$

When  $p$  is a non-singular point of  $Z$ , we have  $\eta_p Z = 1$ .

**Proposition 3.1.** *Let  $Z$  be a  $T$ -variety and let  $p \in Z^T$  be fully definite in  $Z$ . Then  $\eta_p Z \neq 0$  in  $H_T^*(\text{point})$ .*

*Proof.* We may assume that  $p$  is a singular point of  $Z$ , so that  $C_p Z$  has positive dimension. Choose  $\mathbb{G}_m \subset T$  such that  $\mathbb{G}_m$  acts with positive weights on  $T_p Z$ . It suffices to show that the class of  $C_p Z$  is non-zero in  $H_{\mathbb{G}_m}^*(T_p Z)$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $T_p Z$  consisting of eigenvectors of  $\mathbb{G}_m$ . Then the action of  $\mathbb{G}_m$  is given by  $t.v_i = t^{a_i} v_i$  for positive integers  $a_1, \dots, a_n > 0$ . Set  $A = \prod_{i=1}^n a_i$ , and let  $\mathbb{G}_m$  act on  $U = \mathbb{K}^n$  by  $t.u = t^A u$ . Then the map  $\phi : T_p Z \rightarrow U$  defined by

$$\phi(c_1 v_1 + \dots + c_n v_n) = (c_1^{A/a_1}, \dots, c_n^{A/a_n})$$

is a finite  $\mathbb{G}_m$ -equivariant morphism. By [EG98, Thm. 4] we obtain

$$H_{\mathbb{G}_m}^*(U \setminus \{0\}) \otimes \mathbb{Q} = H^*(\mathbb{P}U) \otimes \mathbb{Q},$$

where  $\mathbb{P}U = (U \setminus \{0\}) / \mathbb{G}_m \cong \mathbb{P}^{n-1}$  is the projective space of lines in  $U$ , and

$$\phi_* [C_p Z] |_{U \setminus \{0\}} = \deg(\phi) [\phi(C_p Z \setminus \{0\}) / \mathbb{G}_m] \in H^*(\mathbb{P}U) \otimes \mathbb{Q}.$$

The result now follows from the fact that every non-empty closed subvariety of projective space defines a non-zero Chow class.  $\square$

**Corollary 3.2.** *Let  $X$  be a non-singular  $T$ -variety,  $Z \subset X$  a  $T$ -stable closed subvariety, and  $p \in Z^T$  a  $T$ -fixed point of  $Z$ . If  $p$  is non-degenerate in  $X$  and fully definite in  $Z$ , then  $[Z]_p \neq 0 \in H_T^*(\text{point})$ .*

*Proof.* By [AF24, Prop. 17.4.1] we have  $[Z]_p = c_m(T_p X / T_p Z) \cdot \eta_p Z$ , where  $m = \dim T_p X - \dim T_p Z$ . The result therefore follows from Proposition 3.1, noting that  $T$  acts with non-zero weights on  $T_p X / T_p Z$ .  $\square$

The following example rules out some potential generalizations of Corollary 3.2.

**Example 3.3.** Let  $\mathbb{G}_m$  act on  $\mathbb{A}^4$  by

$$t.(a, b, c, d) = (ta, tb, t^{-1}c, t^{-1}d).$$

Set  $Z = V(ad - bc) \subset \mathbb{A}^4$ , and let  $p = (0, 0, 0, 0)$  be the origin in  $\mathbb{A}^4$ . Then  $T_p Z = T_p \mathbb{A}^4 = \mathbb{A}^4$  and  $C_p Z = Z$ . Since  $\mathbb{G}_m$  acts trivially on the equation  $ad - bc$ , we have  $\eta_p Z = [Z] = 0$  in  $H_{\mathbb{G}_m}^*(\mathbb{A}^4)$  (see [AF24, §2.3]).

#### 4. RIGIDITY OF CONVEX SUBVARIETIES

Let  $T$  be an algebraic torus and let  $X$  be a non-singular  $T$ -variety. We will show in Section 6 that Schubert varieties and Richardson varieties in a flag variety  $X$  satisfy the following two definitions.

**Definition 4.1.** A  $T$ -stable closed subvariety  $\Omega \subset X$  is  *$T$ -equivariantly rigid* if it is uniquely determined by its  $T$ -equivariant cohomology class up to a constant. More precisely, if  $Z \subset X$  is any  $T$ -stable closed subvariety such that  $[Z] = c[\Omega]$  holds in  $H_T^*(X)$  for some  $0 \neq c \in \mathbb{Q}$ , then  $Z = \Omega$ .

**Definition 4.2.** A  $T$ -stable closed subvariety  $\Omega \subset X$  is  *$T$ -convex* if, for any  $T$ -stable closed subvariety  $Z \subset X$  satisfying  $Z^T \subset \Omega$ , we have  $Z \subset \Omega$ .

When the action of  $T$  is clear from the context, we frequently drop  $T$  from the notation and write simply *equivariantly rigid* and *convex*. Both notions are properties of the  $T$ -equivariant embedding  $\Omega \subset X$ ; for example, any  $T$ -variety is convex as a subvariety of itself. Intersections of  $T$ -convex subvarieties are again  $T$ -convex (with the reduced scheme structure). Most of this paper concerns applications of the following observation.

**Theorem 4.3.** *Let  $X$  be a non-singular projective  $T$ -variety such that all fixed points  $p \in X^T$  are fully definite in  $X$ . Then any irreducible  $T$ -convex subvariety of  $X$  is  $T$ -equivariantly rigid.*

*Proof.* Let  $\Omega \subset X$  be irreducible and convex, and let  $Z \subset X$  be any  $T$ -stable closed subvariety such that  $[Z] = c[\Omega]$  holds in  $H_T^*(X)$ , with  $0 \neq c \in \mathbb{Q}$ . Then Corollary 3.2 shows that  $Z^T = \Omega^T = \{p \in X^T : [Z]_p \neq 0\}$ . Since  $\Omega$  is convex, we obtain  $Z \subset \Omega$ . Finally, the assumption  $[Z] = c[\Omega]$  implies that  $Z$  and  $\Omega$  have the same dimension, so we must have  $Z = \Omega$ .  $\square$

**Example 4.4.** Let  $X$  be a non-singular projective  $T$ -variety, let  $H_T^*(X)$  be the  $T$ -equivariant Chow cohomology ring, and let  $\mathcal{L}$  be a  $T$ -equivariant line bundle. Given a section  $f \in \Gamma(X, \mathcal{L})$ , the associated divisor  $D = Z(f)$  is  $T$ -stable if and only if  $f$  is semi-invariant, that is,  $f \in \Gamma(X, \mathcal{L})_\lambda$  for some character  $\lambda$ . In this case  $f$  is an equivariant section of  $\mathcal{L} \otimes \mathbb{K}_{-\lambda}$ , hence  $[D] = c_1(\mathcal{L}) - c_1(\mathbb{K}_\lambda) \in H_T^*(X)$ . Moreover, the  $T$ -stable effective Cartier divisors  $D'$  satisfying  $[D'] = [D]$  are in bijective correspondence with  $\mathbb{P}(\Gamma(X, \mathcal{L})_\lambda)$ . It follows that if  $D$  is reduced and  $\dim \Gamma(X, \mathcal{L}^{\otimes m})_{m\lambda} = 1$  for all  $m \in \mathbb{N}$ , then  $D$  is  $T$ -equivariantly rigid. This observation can be used to produce examples of equivariantly rigid subvarieties that are not convex. For example, if  $T = (\mathbb{G}_m)^{n+1}$  acts on  $\mathbb{P}^n$  through the standard action on  $\mathbb{K}^{n+1}$ , then any reduced  $T$ -stable divisor  $D \subset \mathbb{P}^n$  is equivariantly rigid, but  $D$  is convex only if it is irreducible, see Theorem 6.3. We have not found an example of an irreducible  $T$ -stable subvariety that is equivariantly rigid but not convex.

## 5. RIGIDITY OF BIALYNICKI-BIRULA CELLS

The multiplicative group  $\mathbb{G}_m$  is identified with the complement of the origin in  $\mathbb{A}^1$ . Given a morphism of varieties  $f : \mathbb{G}_m \rightarrow X$ , we write  $\lim_{t \rightarrow 0} f(t) = p$  if  $f$  can be extended to a morphism  $\bar{f} : \mathbb{A}^1 \rightarrow X$  such that  $\bar{f}(0) = p$ . This limit is unique when it exists, and it always exists when  $X$  is complete.

Let  $X$  be a non-singular projective  $\mathbb{G}_m$ -variety such that  $X^{\mathbb{G}_m}$  is finite. Then each fixed point  $p \in X^{\mathbb{G}_m}$  defines the (positive) Bialynicki-Birula cell

$$X_p^+ = \{x \in X \mid \lim_{t \rightarrow 0} t.x = p\}.$$

A negative cell is similarly defined by  $X_p^- = \{x \in X \mid \lim_{t \rightarrow 0} t^{-1}.x = p\}$ . By [BB73, Thm. 4.4], these cells form a locally closed decomposition of  $X$ ,

$$(4) \quad X = \bigcup_{p \in X^{\mathbb{G}_m}} X_p^+,$$

that is, a disjoint union of locally closed subsets. In addition, each cell  $X_p^+$  is isomorphic to an affine space.

**Lemma 5.1.** *For any  $\mathbb{G}_m$ -stable closed subset  $Z \subset X$ , we have  $Z \subset \bigcup_{p \in Z^{\mathbb{G}_m}} X_p^+$ .*

*Proof.* For any point  $x \in Z$ , we have  $x \in X_p^+$ , where  $p = \lim_{t \rightarrow 0} t.x \in Z^{\mathbb{G}_m}$ .  $\square$

**Definition 5.2.** A locally closed decomposition  $X = \bigcup X_i$  is called a *stratification* if each subset  $X_i$  is non-singular and its closure  $\overline{X_i}$  is a union of subsets  $X_j$  of the decomposition.

The Bialynicki-Birula decomposition (4) typically fails to be a stratification, for example when  $X$  is the blow-up of  $\mathbb{P}^2$  at the point  $[0, 1, 0]$ , where  $\mathbb{G}_m$  acts on  $\mathbb{P}^2$  by  $t.[x, y, z] = [x, ty, t^2z]$ , see [BB76, Ex. 1]. Lemma 5.1 shows that the Bialynicki-Birula decomposition is a stratification if and only if  $X_q^+ \subset \overline{X_p^+}$  holds for each fixed point  $q \in (\overline{X_p^+})^{\mathbb{G}_m}$ . It was proved in [BB76, Thm. 5] that the decomposition is a stratification when each positive cell  $X_p^+$  meets each negative cell  $X_q^-$  transversally. In particular, this holds when  $X = G/P$  is a flag variety and  $\mathbb{G}_m \subset G$  is a general 1-parameter subgroup, see [McG02, Ex. 4.2] or Lemma 6.1. When both the positive and negative Bialynicki-Birula decompositions are stratifications, all cells  $X_p^+$  and  $X_q^-$  of complementary dimensions meet transversally, hence the positive and negative cell closures form a pair of Poincare dual bases of the cohomology ring  $H^*(X)$ , see [BP22, Lemma 3.11]. In this paper we utilize the following application, which is a consequence of Lemma 5.1.

**Proposition 5.3.** *Assume that the Bialynicki-Birula decomposition of  $X$  is a stratification. Then each cell closure  $\overline{X_p^+} \subset X$  is  $\mathbb{G}_m$ -convex.*

**Corollary 5.4.** *Let  $T$  be an algebraic torus and  $X$  a non-singular projective  $T$ -variety such that all fixed points  $p \in X^T$  are fully definite in  $X$ . Assume that  $X^T = X^{\mathbb{G}_m}$  for some 1-parameter subgroup  $\mathbb{G}_m \subset T$ , such that the associated Bialynicki-Birula decomposition of  $X$  is a stratification. Then each cell closure  $\overline{X_p^+}$  is  $T$ -convex and  $T$ -equivariantly rigid.*

*Proof.* The cell  $X_p^+$  is  $T$ -stable because  $T$  is commutative and  $p \in X^T$ . The result now follows from Theorem 4.3 and Proposition 5.3.  $\square$

**Question 5.5.** We do not know whether [Proposition 5.3](#) and [Corollary 5.4](#) are true without the assumption that the Bialynicki-Birula decomposition of  $X$  is a stratification. It would be very interesting to settle this question.

**Example 5.6.** Let  $X$  be a non-singular projective toric variety, with torus  $T \subset X$ , and choose  $\mathbb{G}_m \subset T$  such that  $X^T = X^{\mathbb{G}_m}$ . We show that the conclusion of [Corollary 5.4](#) holds, even though the Bialynicki-Birula decomposition is rarely a stratification. All fixed points  $p \in X^T$  are fully definite in  $X$ , as the weights of  $T_p X$  form a basis of the character lattice of  $T$ . The  $T$ -orbits  $O_\tau \subset X$  correspond to the cones  $\tau$  of the fan defining  $X$ , and we have  $O_\sigma \subset \overline{O_\tau}$  if and only if  $\tau$  is a face of  $\sigma$ , see [[Ful93](#), §3.1]. In particular, the  $T$ -fixed points in  $X$  correspond to the maximal cones  $\sigma$ . Since  $X$  is complete, each cone  $\tau$  is the intersection of the maximal cones  $\sigma$  corresponding to the  $T$ -fixed points in  $\overline{O_\tau}$ . Since all cell closures  $\overline{X_p^+}$  are  $T$ -orbit closures, it suffices to show that each orbit closure  $\overline{O_\tau}$  is  $T$ -convex. Let  $Z \subset X$  be a  $T$ -stable closed subvariety such that  $Z^T \subset \overline{O_\tau}$ . We may assume that  $Z$  is irreducible, in which case  $Z = \overline{O_\kappa}$  is also a  $T$ -orbit closure. Since  $\kappa$  is the intersection of the maximal cones given by the fixed points in  $Z^T$ , we obtain  $\tau \subset \kappa$  and  $\overline{O_\tau} \subset \overline{O_\kappa}$ , as required. Now assume that  $X$  has dimension two. By [[BB73](#), Cor. 1 of Thm. 4.5], there is a unique repulsive fixed point  $b \in X^{\mathbb{G}_m}$  with  $X_b^+ = \{b\}$ , and a unique attractive fixed point  $a \in X^{\mathbb{G}_m}$  such that  $X_a^+$  is a dense open subset of  $X$ . For all other fixed points  $p \in X^{\mathbb{G}_m} \setminus \{a, b\}$ , the cell  $X_p^+ \cong \mathbb{A}^1$  is a line. If the Bialynicki-Birula decomposition of  $X$  is a stratification, then  $b \in \overline{X_p^+}$  for all  $p \in X^{\mathbb{G}_m}$ . The  $T$ -fixed point  $b$  corresponds to a maximal cone  $\sigma$ , and  $b$  is connected to exactly two  $T$ -stable lines corresponding to the rays forming the boundary of this cone. We deduce that  $X$  contains at most four  $T$ -fixed points. Higher dimensional toric varieties for which the Bialynicki-Birula decomposition is not a stratification can be constructed by taking products. We do not know if the cell closures  $\overline{X_p^+}$  are  $\mathbb{G}_m$ -convex when  $X$  is a toric variety.<sup>1</sup>

## 6. RIGIDITY OF RICHARDSON VARIETIES

Let  $X = G/P = \{g.P \mid g \in G\}$  be a flag variety defined by a connected reductive linear algebraic group  $G$  and a parabolic subgroup  $P$ . Fix a maximal torus  $T$  and a Borel subgroup  $B$  such that  $T \subset B \subset P \subset G$ . The opposite Borel subgroup  $B^- \subset G$  is defined by  $B^- \cap B = T$ . Let  $\Phi$  be the root system of non-zero weights of  $T_1 G$ , the tangent space of  $G$  at the identity element. The positive roots  $\Phi^+$  are the non-zero weights of  $T_1 B$ . Let  $W = N_G(T)/T$  be the Weyl group of  $G$ ,  $W_P = N_P(T)/T$  the Weyl group of  $P$ , and let  $W^P \subset W$  be the subset of minimal representatives of the cosets in  $W/W_P$ . The set of  $T$ -fixed points in  $X$  is given by  $X^T = \{w.P \mid w \in W\}$ , where each point  $w.P$  depends only on the coset  $wW_P$  in  $W/W_P$ . Each fixed point  $w.P$  defines the *Schubert varieties*  $X_w = \overline{Bw.P}$  and  $X^w = \overline{B^-w.P}$ . For  $w \in W^P$  we have  $\dim(X_w) = \text{codim}(X^w, X) = \ell(w)$ . The Bruhat order  $\leq$  on  $W^P$  is defined by

$$u \leq w \Leftrightarrow X_u \subset X_w \Leftrightarrow X^u \supset X^w \Leftrightarrow X^u \cap X_w \neq \emptyset.$$

A *Richardson variety* is any non-empty intersection  $X_w^u = X_w \cap X^u$  of opposite Schubert varieties in  $X$ . More generally, any  $G$ -translate of  $X_w^u$  will be called a

<sup>1</sup>Teddy Gonzales and Chayim Lowen have recently produced several examples showing that  $\overline{X_p^+}$  may not be  $\mathbb{G}_m$ -convex when  $X$  is a non-singular projective toric variety [[GL25](#)].

Richardson variety. Any Richardson variety is reduced, irreducible, and rational, see [Deo77] and [BK05, §2].

Recall that a cocharacter  $\rho : \mathbb{G}_m \rightarrow T$  is *strongly dominant* if  $\langle \alpha, \rho \rangle > 0$  for all positive roots  $\alpha \in \Phi^+$ , where  $\langle \alpha, \rho \rangle \in \mathbb{Z}$  is defined by  $\alpha(\rho(t)) = t^{\langle \alpha, \rho \rangle}$  for  $t \in \mathbb{G}_m$ . The following lemma is well known, see e.g. [McG02, Ex. 4.2] or [BP22, Cor. 3.14].

**Lemma 6.1.** *Let  $\rho : \mathbb{G}_m \rightarrow T$  be a strongly dominant cocharacter. Then the associated Bialynicki-Birula cells of  $X$  are given by  $X_p^+ = B.p$ , for  $p \in X^T$ .*

*Proof.* Let  $\mathbb{G}_m$  act on  $G$  by conjugation through  $\rho$ . The fixed point set for this action is [Spr98, (7.1.2), (7.6.4)]

$$T = \{g \in G \mid tgt^{-1} = g \ \forall t \in \mathbb{G}_m\},$$

and the corresponding Bialynicki-Birula cell is [Spr98, (8.2.1)]

$$B = \{g \in G \mid \lim_{t \rightarrow 0} tgt^{-1} \in T\}.$$

This implies  $B.p \subset X_p^+$  for any fixed point  $p \in X^{\mathbb{G}_m}$ . We deduce from (4) that the positive Bialynicki-Birula cells in  $X$  are the  $B$ -orbits.  $\square$

**Lemma 6.2.** *Let  $Y$  be any  $G$ -variety, and  $\Omega \subset Y$  a  $T$ -stable closed subvariety. Any  $T$ -stable  $G$ -translate of  $\Omega$  has the form  $w.\Omega$ , with  $w \in N_G(T)$ .*

*Proof.* Let  $\Omega' = g.\Omega$  be a  $T$ -stable translate, and let  $H \subset G$  be the stabilizer of  $\Omega'$ . Since  $T$  and  $gTg^{-1}$  are maximal tori in  $H$ , we can choose  $h \in H$  such that  $T = hgTg^{-1}h^{-1}$ . We obtain  $hg \in N_G(T)$  and  $\Omega' = h.\Omega' = hg.\Omega$ , as required.  $\square$

**Theorem 6.3.** *Any  $T$ -stable Richardson variety in the flag variety  $X = G/P$  is  $T$ -convex and  $T$ -equivariantly rigid.*

*Proof.* It follows from Proposition 5.3 and Lemma 6.1 that all Schubert varieties  $X_w$  and  $X^u$  are convex. This implies that every Richardson variety  $X_w^u = X_w \cap X^u$  is convex, hence all  $T$ -stable Richardson varieties in  $X$  are convex by Lemma 6.2. The  $B$ -fixed point  $p = 1.P$  is fully definite in  $X$  because the weights of  $T_p X$  are a subset of the negative roots of  $G$ . Since  $W$  acts transitively on  $X^T$ , this implies that all  $T$ -fixed points in  $X$  are fully definite. The result therefore follows from Theorem 4.3.  $\square$

Let  $E = G/B$  denote the variety of complete flags, and let  $\pi : E \rightarrow X$  be the natural projection. A *projected Richardson variety* in  $X$  is the image  $\Pi_w^u(X) = \pi(E_w^u)$  of a Richardson variety in  $E$ . Projected Richardson varieties in the Grassmannian  $X = \text{Gr}(m, n)$  of type A, obtained as images of Richardson varieties in  $\text{Fl}(n)$ , are also called *positroid varieties*.

**Corollary 6.4.** *Let  $X = \text{Gr}(m, n)$  be a Grassmannian of type A, and let  $T = (\mathbb{G}_m)^n$  act on  $X$  through the diagonal action on  $\mathbb{K}^n$ . Then all positroid varieties in  $X$  are  $T$ -convex and  $T$ -equivariantly rigid.*

*Proof.* It was proved in [KLS13] that any positroid variety  $\Omega$  is defined by Plucker equations. Equivalently,  $\Omega$  is an intersection of  $T$ -stable Schubert divisors, so  $\Omega$  is convex by Theorem 6.3 and equivariantly rigid by Theorem 4.3.  $\square$

**Remark 6.5.** Corollary 6.4 does not hold for projected Richardson varieties in arbitrary flag varieties  $X = G/P$ . Each simple root  $\beta$  defines a projected Richardson divisor  $D_\beta = \Pi_{w_0^P}^{s_\beta}(X)$ , where  $w_0^P$  denotes the longest element in  $W^P$ . It frequently

happens that two distinct divisors  $D_{\beta'}$  and  $D_{\beta''}$  have the same  $T$ -equivariant cohomology and  $K$ -theory classes, which implies that these divisors are not equivariantly rigid. For example, this is the case for the quadric hypersurfaces of dimensions 7 and 8, of Lie types  $B_4$  and  $D_5$ , and the two-step flag variety  $\text{Fl}(1, 4; 5)$  of type  $A_4$ . For other flag varieties  $X$ , all projected Richardson varieties have distinct equivariant classes, but some projected Richardson divisor  $D_\beta$  contains all  $T$ -fixed points in  $X$ , which rules out that  $D_\beta$  is convex. For example, this is the case for the Lagrangian Grassmannian  $\text{LG}(2, 4)$  of type  $C_2$  and the maximal orthogonal Grassmannian  $\text{OG}(4, 8)$  of type  $D_4$ . This is a special case of [BP22, Lemma 3.1], which can be used to produce many more examples.

Any element  $u \in W$  has a unique factorization  $u = u^P u_P$  for which  $u^P \in W^P$  and  $u_P \in W_P$ , called the *parabolic factorization* with respect to  $P$ . This factorization is *reduced* in the sense that  $\ell(u) = \ell(u^P) + \ell(u_P)$ . The parabolic factorization of the longest element  $w_0 \in W$  is  $w_0 = w_0^P w_{0,P}$ , where  $w_0^P$  and  $w_{0,P}$  are the longest elements in  $W^P$  and  $W_P$ , respectively. Since  $w_0$  and  $w_{0,P}$  are self-inverse, we have  $w_{0,P} = w_0 w_0^P$ . As preparation for the next section, we prove the following identity of Schubert varieties.

**Lemma 6.6.** *Let  $Q \subset G$  be a parabolic subgroup containing  $B$  and set  $w = w_0^Q$ . Then  $w^{-1} \cdot X^w = X_{w_0 w}$ .*

*Proof.* Since  $X_{w_0, Q}$  is a  $Q$ -stable Schubert variety, we have  $X_{w_0, Q} = w_{0, Q} \cdot X_{w_0, Q}$ . By translating both sides by  $w = w_0^Q$ , we obtain  $w \cdot X_{w_0 w} = w_0 \cdot X_{w_0 w} = X^w$ .  $\square$

## 7. SEIDEL NEIGHBORHOODS

In this section we prove a conjecture about curve neighborhoods from [BCP23]. Since this conjecture and its proof relies on the moduli space of stable maps, we will restrict our attention to varieties defined over the field  $\mathbb{K} = \mathbb{C}$  of complex numbers. As in Section 6, we let  $X = G/P$  denote a flag variety.

For any effective degree  $d \in H_2(X, \mathbb{Z})$ , we let  $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$  denote the Kontsevich moduli space of 3-pointed stable maps to  $X$  of degree  $d$  and genus zero, see [FP97]. The evaluation map  $\text{ev}_i : M_d \rightarrow X$ , defined for  $1 \leq i \leq 3$ , sends a stable map to the image of the  $i$ -th marked point in its domain. Given two opposite Schubert varieties  $X_v$  and  $X^u$ , the *Gromov-Witten variety*  $M_d(X_v, X^u)$  is the variety of stable maps that send the first two marked points to  $X_v$  and  $X^u$ :

$$M_d(X_v, X^u) = \text{ev}_1^{-1}(X_v) \cap X_2^{-1}(X^u) \subset M_d.$$

The *curve neighborhood*  $\Gamma_d(X_v, X^u)$  is the union of all stable curves of degree  $d$  in  $X$  connecting  $X_v$  and  $X^u$ :

$$\Gamma_d(X_v, X^u) = \text{ev}_3(M_d(X_v, X^u)) \subset X.$$

Let  $\mathbb{Z}[q] = \text{Span}_{\mathbb{Z}}\{q^d : d \in H_2(X, \mathbb{Z}) \text{ effective}\}$  be the semigroup ring defined by the effective curve classes on  $X$ . The equivariant quantum cohomology ring of  $X$  is an algebra over  $H_T^*(\text{point}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ , which is defined by  $\text{QH}_T(X) = H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$  as a module. The *quantum product* of two opposite Schubert classes is given by

$$[X_v] \star [X^u] = \sum_{d \geq 0} q^d \text{ev}_{3,*}[M_d(X_v, X^u)],$$

where the sum is over all effective degrees  $d \in H_2(X; \mathbb{Z})$ .

A simple root  $\gamma \in \Phi^+$  is called *cominuscule* if, when the highest root is written in the basis of simple roots, the coefficient of  $\gamma$  is one. The flag variety  $G/Q$  is cominuscule if  $Q$  is a maximal parabolic subgroup corresponding to a cominuscule simple root  $\gamma$ , that is,  $s_\gamma$  is the unique simple reflection in  $W^Q$ . Let  $W^{\text{comin}} \subset W$  be the subset of point representatives of cominuscule flag varieties of  $G$ , together with the identity element:

$$W^{\text{comin}} = \{w_0^Q \mid G/Q \text{ is cominuscule}\} \cup \{1\}.$$

This is a subgroup of  $W$ , which is isomorphic to the quotient of the coweight lattice of  $\Phi$  modulo the coroot lattice [Bou81, Prop. VI.2.6]. The isomorphism sends  $w_0^Q$  to the class of the fundamental coweight  $\omega_\gamma^\vee$  corresponding to  $Q$ . Notice that  $\gamma$  is the unique simple root for which  $w_0^Q \cdot \gamma < 0$ . In the following we set  $d(w_0^Q, u) = \omega_\gamma^\vee - u^{-1} \cdot \omega_\gamma^\vee \in H_2(X; \mathbb{Z})$  for any  $u \in W$ . Here we identify the group  $H_2(X, \mathbb{Z})$  with a quotient of the coroot lattice, by mapping each simple coroot  $\beta^\vee$  to the curve class  $[X_{s_\beta}]$  if  $s_\beta \in W^P$ , and to zero otherwise.

The *Seidel representation* of  $W^{\text{comin}}$  on  $\text{QH}(X)/\langle q-1 \rangle$  is defined by  $w \cdot [X^u] = [X^w] \star [X^u]$  for  $w \in W^{\text{comin}}$  and  $u \in W$ . In fact, we have [Sei97, Bel04, CMP09]

$$(5) \quad [X^w] \star [X^u] = q^{d(w,u)} [X^{wu}]$$

in the (non-equivariant) quantum ring  $\text{QH}(X)$ . This implies that  $d(w, u)$  is the unique minimal degree  $d$  for which  $\Gamma_d(X_{w_0w}, X^u)$  is not empty [FW04, BCLM20]. More generally, it was proved in [CMP09, CP23] that the identity

$$(6) \quad [X^w] \star [w \cdot X^u] = q^{d(w,u)} [X^{wu}]$$

holds in the equivariant quantum cohomology ring  $\text{QH}_T(X)$ . We will discuss generalizations to quantum  $K$ -theory in Section 8.

It follows from (5) and the definition of the quantum product in  $\text{QH}(X)$  that  $[\Gamma_{d(w,u)}(X_{w_0w}, X^u)] = [X^{wu}]$  holds in  $H^*(X)$ . Conjecture 3.11 from [BCP23] asserts that  $\Gamma_{d(w,u)}(X_{w_0w}, X^u)$  is in fact equal to the translated Schubert variety  $w^{-1} \cdot X^{wu}$ . This is proved below as a consequence of Theorem 6.3 and (6). This result was known when  $X = G/P$  is cominuscule and  $w = w_0^P$  [BCP23], when  $X$  is a Grassmannian of type A and  $[X^w]$  is a special Seidel class [LLSY22, Cor. 4.6], when  $X$  is any flag variety of type A [Tar23], and when  $X$  is the symplectic Grassmannian  $\text{SG}(2, 2n)$  [BPX24, Thm. 8.1].

**Theorem 7.1.** *Let  $X = G/P$  be a complex flag variety. For  $w \in W^{\text{comin}}$  and  $u \in W$  we have  $\Gamma_{d(w,u)}(X_{w_0w}, X^u) = w^{-1} \cdot X^{wu}$ .*

*Proof.* By applying  $w^{-1}$  to both sides of (6) and using Lemma 6.6, we obtain

$$[X_{w_0w}] \star [X^u] = q^{d(w,u)} [w^{-1} \cdot X^{wu}]$$

in  $\text{QH}_T(X)$ . By definition of the quantum product, this implies that

$$[w^{-1} \cdot X^{wu}] = \text{ev}_{3,*} [M_{d(w,u)}(X_{w_0w}, X^u)] = c [\Gamma_{d(w,u)}(X_{w_0w}, X^u)]$$

holds in  $H_T^*(X)$ , where  $c$  is the degree of the map  $\text{ev}_3 : M_{d(w,u)}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)}(X_{w_0w}, X^u)$ . The result therefore follows from Theorem 6.3.  $\square$

8. SEIDEL PRODUCTS IN QUANTUM  $K$ -THEORY

In this section we discuss a generalization of the Seidel multiplication formula to quantum  $K$ -theory. We start by briefly recalling the definition of quantum  $K$ -theory. A more detailed discussion can be found in [BCMP18a, §2].

Let  $X = G/P$  be a flag variety defined over  $\mathbb{K} = \mathbb{C}$ . The equivariant  $K$ -theory ring  $K^T(X)$  is an algebra over the representation ring  $\Gamma = K^T(\text{point})$ . The equivariant quantum  $K$ -theory ring  $\text{QK}_T(X)$  was originally constructed by Givental and Lee [Giv00, Lee04]. This ring is an algebra over the formal power series ring  $\Gamma[[q]] = \Gamma[[q_\beta : s_\beta \in W^P]]$ , which has one variable  $q_\beta$  for each simple reflection  $s_\beta$  in  $W^P$ . As a module over  $\Gamma[[q]]$  we have  $\text{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[[q]]$ . The *undeformed product* of two opposite Schubert classes in  $\text{QK}_T(X)$  is defined by

$$[\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}] = \sum_{d \geq 0} q^d \text{ev}_{3,*} [\mathcal{O}_{M_d(X_v, X^u)}].$$

Let  $\Psi : \text{QK}_T(X) \rightarrow \text{QK}_T(X)$  be the  $\Gamma[[q]]$ -linear map defined by

$$\Psi([\mathcal{O}_{X^w}]) = \sum_{d \geq 0} q^d [\mathcal{O}_{\Gamma_d(X^w)}],$$

where the curve neighborhood  $\Gamma_d(X^w) = \text{ev}_2(\text{ev}_1^{-1}(X^w))$  is defined using the evaluation maps from  $M_d$ . This curve neighborhood is a Schubert variety in  $X$  by [BCMP13, Prop. 3.2(b)], whose Weyl group element was determined in [BM15]. By [BCMP18a, Prop. 2.3], Givental's *quantum  $K$ -theory product*  $\star$  is given by

$$(7) \quad [\mathcal{O}_{X_v}] \star [\mathcal{O}_{X^u}] = \Psi^{-1}([\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}]).$$

The following conjecture is the  $K$ -theoretic analogue of the Seidel multiplication formula (6) in  $\text{QH}_T(X)$  proved in [CMP09, CP23].

**Conjecture 8.1.** *For  $w \in W^{\text{comin}}$  and  $u \in W$  we have*

$$[\mathcal{O}_{X_{w_0w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1}.X^{wu}}] \quad \text{and} \quad [\mathcal{O}_{X^w}] \star [\mathcal{O}_{w.X^u}] = q^{d(w,u)} [\mathcal{O}_{X^{wu}}]$$

in  $\text{QK}_T(X)$ .

The two identities in Conjecture 8.1 are equivalent by Lemma 6.6. The non-equivariant case of this conjecture was proved in [BCP23, Cor. 3.7] when  $X$  is a cominuscule flag variety. When  $X$  is a Grassmannian of type A, the conjecture is equivalent to [GKM25, Cor. 10.4]. We will prove Conjecture 8.1 for cominuscule flag varieties below, based on the following conjectural generalization of Theorem 7.1. Recall that a morphism  $\pi : Z \rightarrow Y$  is called *cohomologically trivial* if  $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$  and  $R^j \pi_* \mathcal{O}_Z = 0$  for  $j \geq 1$ .

**Conjecture 8.2.** *Let  $w \in W^{\text{comin}}$ ,  $u \in W$ , and let  $e \in H_2(X, \mathbb{Z})$  be effective.*

(a) *We have  $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1}.X^{wu})$ .*

(b) *The evaluation map  $\text{ev}_3 : M_{d(w,u)+e}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$  is cohomologically trivial.*

Conjecture 8.2 is a variant of the quantum-equals-classical theorem for Gromov-Witten invariants as stated in [BCMP18b, Thm. 4.1], see also [Xu24, Thm. 1.2]. The conjecture is true for  $e = 0$ ; part (a) is equivalent to Theorem 7.1, and part (b) holds because the map  $\text{ev}_3 : M_{d(w,u)}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)}(X_{w_0w}, X^u)$  is birational

by [Bel04, CMP09], and  $M_{d(w,u)}(X_{w_0w}, X^u)$  has rational singularities by [BCMP13, Cor. 3.1]. For  $e \geq 0$ , Theorem 7.1 implies that

$$(8) \quad \Gamma_e(w^{-1}.X^{wu}) = \Gamma_e(\Gamma_{d(w,u)}(X_{w_0w}, X^u)) \subset \Gamma_{d(w,u)+e}(X_{w_0w}, X^u),$$

and  $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$  is irreducible by [BCMP13, Cor. 3.8]. Conjecture 8.2(a) is therefore true if and only if  $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$  and  $\Gamma_e(X^{wu})$  have the same dimension. We prove below that Conjecture 8.2(a) is true when  $X = \mathrm{GL}(n)/P$  is any flag variety of Lie type A. Conjecture 8.1 follows from Conjecture 8.2 by the following observation.

**Lemma 8.3.** *Given  $w \in W^{\mathrm{comin}}$  and  $u \in W$ , the identity  $[\mathcal{O}_{X_{w_0w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1}.X^{wu}}]$  holds in  $\mathrm{QK}_T(X)$  if and only if*

$$(9) \quad \mathrm{ev}_{3,*}[\mathcal{O}_{M_{d(w,u)+e}(X_{w_0w}, X^u)}] = [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}]$$

*holds in  $K_T(X)$  for all effective degrees  $e \in H_2(X, \mathbb{Z})$ .*

*Proof.* Both assertions are equivalent to the identity

$$[\mathcal{O}_{X_{w_0w}}] \odot [\mathcal{O}_{X^u}] = \sum_{e \geq 0} q^{d(w,u)+e} [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}]$$

by the definition (7) of the quantum product in  $\mathrm{QK}_T(X)$ .  $\square$

**Theorem 8.4.** *Conjecture 8.1 and Conjecture 8.2 are true when  $X$  is a cominuscule flag variety.*

*Proof.* Assume that  $X$  is cominuscule. Then Conjecture 8.2(b) is a special case of [BCMP18b, Thm. 4.1], and Conjecture 8.2(a) follows from Theorem 7.1 and [BCMP22, Cor. 8.24], noting that  $q^{d(w,u)}$  is the maximal power of  $q$  occurring in the quantum cohomology product  $[X_{w_0w}] \star [X^u]$  by [Bel04, CMP09]. This proves Conjecture 8.2, which implies Conjecture 8.1 by Lemma 8.3.  $\square$

We finish this section by proving that Conjecture 8.2(a) can be reduced to the case where  $X$  is a flag variety of Picard rank 1. In particular, Conjecture 8.2(a) follows from Theorem 8.4 in type A. These results were proved for  $e = 0$  in [Tar23]. We thank Mihail Tarigradschi for suggesting that his methods might apply to the general case of our conjecture.

Recall that  $X = G/P$ . Let  $Q_1, Q_2 \subset G$  be parabolic subgroups such that  $P = Q_1 \cap Q_2$ . Set  $Y_i = G/Q_i$  and let  $\pi_i : X \rightarrow Y_i$  be the projection, for  $i \in \{1, 2\}$ . Given a degree  $d \in H_2(X, \mathbb{Z})$ , we also let  $d$  denote the image  $\pi_{i,*}(d)$  of this degree in  $H_2(Y_i, \mathbb{Z})$ . Let  $\Gamma_d(Y_{i,v}, Y_i^u) \subset Y_i$  be the union of all stable curves of degree  $d$  in  $Y_i$  that connect the Schubert varieties  $Y_{i,v} = \pi_i(X_v)$  and  $Y_i^u = \pi_i(X^u)$ , for  $u, v \in W$ . The following result generalizes [BB05, Thm. 2.6.1] and [Tar23, Lemma 4].

**Lemma 8.5.** *We have  $\Gamma_d(X^u) = \pi_1^{-1}(\Gamma_d(Y_1^u)) \cap \pi_2^{-1}(\Gamma_d(Y_2^u))$ .*

*Proof.* Let  $\mathrm{dist}_X(X_v, X^u)$  denote the unique minimal degree of a rational curve in  $X$  connecting  $X_v$  and  $X^u$ . It follows from [BCLM20, Thm. 5] that this degree is uniquely determined by  $\pi_{i,*}(\mathrm{dist}_X(X_v, X^u)) = \mathrm{dist}_{Y_i}(Y_{i,v}, Y_i^u)$  for  $i \in \{1, 2\}$ . Using that  $v.P \in \Gamma_d(X^u)$  holds if and only if  $d \geq \mathrm{dist}_X(X_v, X^u)$ , we deduce that  $\Gamma_d(X^u)$  and  $\pi_1^{-1}(\Gamma_d(Y_1^u)) \cap \pi_2^{-1}(\Gamma_d(Y_2^u))$  contain the same  $T$ -fixed points. The lemma follows from this, as both sets are  $B^-$ -stable subvarieties of  $X$ .  $\square$

The following result implies that [Conjecture 8.2\(a\)](#) follows from the case where  $X$  has Picard rank 1. It was proved for  $e = 0$  in [[Tar23](#), Thm. 3].

**Theorem 8.6.** *Let  $X = G/P$ ,  $Y_1 = G/Q_1$ , and  $Y_2 = G/Q_2$  be flag varieties such that  $P = Q_1 \cap Q_2$ . Let  $w \in W^{\text{comin}}$ ,  $u \in W$ , and let  $e \in H_2(X, \mathbb{Z})$  be any effective degree. If  $\Gamma_{d(w,u)+e}(Y_{i,w_0w}, Y_i^u) = \Gamma_e(w^{-1}.Y_i^{wu})$  holds for  $i \in \{1, 2\}$ , then  $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1}.X^{wu})$ .*

*Proof.* The assumptions and [Lemma 8.5](#) imply that

$$\begin{aligned} \Gamma_{d(w,u)+e}(X_{w_0w}, X^u) &\subset \pi_1^{-1}(\Gamma_{d(w,u)+e}(Y_{1,w_0w}, Y_1^u)) \cap \pi_2^{-1}(\Gamma_{d(w,u)+e}(Y_{2,w_0w}, Y_2^u)) \\ &= \pi_1^{-1}(\Gamma_e(w^{-1}.Y_1^{wu})) \cap \pi_2^{-1}(\Gamma_e(w^{-1}.Y_2^{wu})) = \Gamma_e(w^{-1}.X^{wu}), \end{aligned}$$

and the opposite inclusion holds by [\(8\)](#).  $\square$

**Corollary 8.7.** *[Conjecture 8.2\(a\)](#) is true when  $X = \text{GL}(n)/P$  has Lie type A.*

*Proof.* This follows from [Theorem 8.4](#) and [Theorem 8.6](#), noting that all flag varieties of type A with Picard rank 1 are Grassmannians, and therefore cominuscule.  $\square$

## 9. HOROSPHERICAL VARIETIES OF PICARD RANK 1

In this section we interpret [Theorem 4.3](#) and [Proposition 5.3](#) for a class of horospherical varieties that includes all non-singular projective horospherical varieties of Picard rank 1 (except flag varieties) by Pasquier's classification [[Pas09](#)]. Let  $G$  be a connected reductive linear algebraic group,  $B \subset G$  a Borel subgroup, and  $T \subset B$  a maximal torus. Let  $V_1$  and  $V_2$  be irreducible rational representations of  $G$ , and let  $v_i \in V_i$  be a highest weight vector of weight  $\lambda_i$ , for  $i \in \{1, 2\}$ . We assume that  $\lambda_1 \neq \lambda_2$ . Define

$$X = \overline{G.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2).$$

If  $X$  is normal, then  $X$  is a horospherical variety of rank 1, see [[Tim11](#), Ch. 7]. We will assume that  $X$  is non-singular and  $\mathbb{K} = \mathbb{C}$ , even though many claims hold more generally; this implies that  $X$  is fibered over a flag variety  $G/P_{12}$  with non-singular horospherical fibers of Picard rank 1, see [Remark 9.5](#). Any  $G$ -translate of a  $B$ -orbit closure in  $X$  will be called a *Schubert variety*. Our next result uses the action of  $T \times \mathbb{G}_m$  on  $X$  defined by  $(t, z).[u_1 + u_2] = t.[u_1 + zu_2]$ , for  $u_i \in V_i$ . We have  $X^{T \times \mathbb{G}_m} = X^T$ , and a Schubert variety is  $T$ -stable if and only if it is  $T \times \mathbb{G}_m$ -stable.

**Theorem 9.1.** *Any  $T$ -stable Schubert variety in  $X$  is  $T \times \mathbb{G}_m$ -convex and  $T \times \mathbb{G}_m$ -equivariantly rigid.*

Before proving [Theorem 9.1](#), we sketch elementary proofs of some basic facts about  $X$ , which are also consequences of general results about spherical varieties, see [[Tim11](#), [Per14](#), [Pas09](#)] and the references therein.

Given an element  $[u_1 + u_2] \in \mathbb{P}(V_1 \oplus V_2)$ , we will always assume  $u_i \in V_i$ , and  $i$  will always mean an element from  $\{1, 2\}$ . We consider  $\mathbb{P}(V_i)$  as a subvariety of  $\mathbb{P}(V_1 \oplus V_2)$ . Let  $\pi_i : \mathbb{P}(V_1 \oplus V_2) \setminus \mathbb{P}(V_{3-i}) \rightarrow \mathbb{P}(V_i)$  denote the projection from  $V_{3-i}$ , defined by  $\pi_i([u_1 + u_2]) = [u_i]$ . Set  $X_0 = G.[v_1 + v_2] \subset \mathbb{P}(V_1 \oplus V_2)$ ,  $X_i = G.[v_i] \subset \mathbb{P}(V_i)$ , and  $X_{12} = G.([v_1], [v_2]) \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ . Since  $v_i$  is a highest weight vector, the stabilizer  $P_i = G_{[v_i]}$  is a parabolic subgroup containing  $B$ . It follows that  $X_i \cong G/P_i$  and  $X_{12} \cong G/(P_1 \cap P_2)$  are flag varieties. In particular,  $X_i$  is closed in  $\mathbb{P}(V_i)$ , and  $X_{12}$  is closed in  $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ . Notice also that  $X_0 \cong G/H$ , where  $H \subset P_1 \cap P_2$  is the kernel of the character  $\lambda_1 - \lambda_2 : P_1 \cap P_2 \rightarrow \mathbb{G}_m$ . This shows

that  $X_0$  is a  $\mathbb{G}_m$ -bundle over  $X_{12}$ , so  $X$  is a non-singular projective horospherical variety of rank 1 (but not necessarily of Picard rank 1, see [Remark 9.5](#)).

Let  $W$  be the Weyl group of  $G$ , and recall the notation from [Section 6](#).

**Lemma 9.2.** *We have  $X = X_0 \cup X_1 \cup X_2$ . The  $B$ -orbit closures in  $X$  are*

$$\begin{aligned} \overline{Bw.[v_i]} &= \bigcup_{w' \leq w} Bw'.[v_i] \quad \text{for } w \in W^{P_i} \text{ and } i \in \{1, 2\}, \text{ and} \\ \overline{Bw.[v_1 + v_2]} &= \bigcup_{w' \leq w} (Bw'.[v_1 + v_2] \cup Bw'.[v_1] \cup Bw'.[v_2]) \quad \text{for } w \in W^{P_1 \cap P_2}. \end{aligned}$$

*Proof.* Set  $\mathbb{P}_0 = \mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$ . Since  $\lambda_1 \neq \lambda_2$ , it follows that  $\overline{T.[v_1 + v_2]}$  is the line through  $[v_1]$  and  $[v_2]$  in  $\mathbb{P}(V_1 \oplus V_2)$ . This implies  $X_0 = (\pi_1 \times \pi_2)^{-1}(X_{12})$ , hence  $X_0$  is closed in  $\mathbb{P}_0$ , and  $X_0 = X \cap \mathbb{P}_0$ . We also have  $X_i \subset X \cap \mathbb{P}(V_i) \subset \pi_i^{-1}(X_i) \cap \mathbb{P}(V_i) = X_i$ , which proves the first claim. To finish the proof, it suffices to show  $w'.[v_i] \in \overline{Bw.[v_1 + v_2]}$  if and only if  $w' \leq w$  (when  $w' \in W^{P_i}$ ). The implication ‘if’ holds because  $w'.[v_i] \in \overline{Tw'.[v_1 + v_2]}$ , and ‘only if’ holds because  $\pi_i(\overline{Bw.[v_1 + v_2]} \setminus X_{3-i}) \subset \overline{Bw.[v_i]}$ .  $\square$

Define an alternative action of  $P_i$  on  $V_{3-i}$  by  $p \bullet u = \lambda_i(p)^{-1}p.u$ , and use this action to form the space

$$G \times^{P_i} V_{3-i} = \{[g, u] : g \in G, u \in V_{3-i}\} / \{[gp, u] = [g, p \bullet u] : p \in P_i\}.$$

Define a morphism of varieties  $\phi_i : G \times^{P_i} V_{3-i} \rightarrow \mathbb{P}(V_1 \oplus V_2)$  by  $\phi_i([g, u]) = g.[v_i + u]$ . This is well defined since  $p.(v_i + u) = \lambda_i(p)(v_i + p \bullet u)$  holds for  $p \in P_i$  and  $u \in V_{3-i}$ . Set  $E_i = (P_i \bullet v_{3-i}) \cup \{0\} \subset V_{3-i}$ . Noting that  $E_i$  is the cone over  $P_i.[v_{3-i}] \cong P_i/(P_1 \cap P_2)$ , it follows that  $E_i$  is closed in  $V_{3-i}$ .

**Lemma 9.3.** *The restricted map  $\phi_i : G \times^{P_i} E_i \rightarrow X_0 \cup X_i$  is an isomorphism of varieties. In particular,  $E_i \subset V_{3-i}$  is a linear subspace.*

*Proof.* Assume  $\phi_i([g, u]) = \phi_i([g', u'])$ , and set  $p = g^{-1}g'$ . We obtain  $p \in P_i$  and  $[v_i + u] = p.[v_i + u'] = [v_i + p \bullet u']$  in  $\mathbb{P}(V_1 \oplus V_2)$ , hence  $[g, u] = [g, p \bullet u'] = [gp, u'] = [g', u']$  in  $G \times^{P_i} V_{3-i}$ . We deduce that  $\phi_i : G \times^{P_i} E_i \rightarrow X_0 \cup X_i$  is bijective, so the lemma follows from Zariski’s main theorem, using that  $X_0 \cup X_i$  is non-singular.  $\square$

Fix a strongly dominant cocharacter  $\rho : \mathbb{G}_m \rightarrow T$ . For  $a \in \mathbb{Z}$ , define  $\rho_a : \mathbb{G}_m \rightarrow T \times \mathbb{G}_m$  by  $\rho_a(z) = (\rho(z), z^a)$ . The resulting action of  $\mathbb{G}_m$  on  $X$  is given by  $\rho_a(z).[u_1 + u_2] = \rho(z).[u_1 + z^a u_2]$ .

**Lemma 9.4.** *All  $T$ -fixed points in  $X$  are fully definite for the action of  $T \times \mathbb{G}_m$ .*

*Proof.* [Lemma 9.3](#) shows that  $[v_1]$  has a  $T \times \mathbb{G}_m$ -stable open neighborhood in  $X$  isomorphic to  $B^-[v_1] \times E_1$ , where the action is given by  $(t, z).(x, u) = (t.x, t \bullet zu)$ . If  $a$  is sufficiently negative, then  $\mathbb{G}_m$  acts through  $\rho_a$  on  $T_{[v_1]}X = T_{[v_1]}X_1 \oplus E_1$  with strictly negative weights, hence  $[v_1]$  is fully definite in  $X$  for the action of  $T \times \mathbb{G}_m$ . A symmetric argument shows that  $[v_2]$  is fully definite. The result follows from this, since all  $T$ -fixed points in  $X$  are obtained from  $[v_1]$  or  $[v_2]$  by the action of the Weyl group  $W$ .  $\square$

*Proof of [Theorem 9.1](#).* For  $a$  sufficiently negative, it follows from [Lemma 6.1](#) that the Bialynicki-Birula cells of  $X$  defined by  $\rho_a$  are

$$X_{w.[v_1]}^+ = Bw.[v_1] \quad \text{and} \quad X_{w.[v_2]}^+ = Bw.[v_1 + v_2] \cup Bw.[v_2].$$

These cells form a stratification of  $X$  by Lemma 9.2, so Proposition 5.3 implies that  $\overline{Bw.[v_1]}$  and  $\overline{Bw.[v_1 + v_2]}$  are  $T \times \mathbb{G}_m$ -convex for  $w \in W$ . A symmetric argument applies to  $\overline{Bw.[v_2]}$ , hence all  $T$ -stable Schubert varieties in  $X$  are  $T \times \mathbb{G}_m$ -convex by Lemma 6.2. The result now follows from Theorem 4.3 and Lemma 9.4.  $\square$

**Remark 9.5.** The exact sequence of [Per14, Thm. 3.2.4] implies that  $\text{Pic}(X)$  is a free abelian group of rank equal to the rank of  $X$  (which is one) plus the number of  $B$ -stable prime divisors in  $X$  that do not contain a  $G$ -orbit. Any  $B$ -stable prime divisor meeting  $X_0$  has the form  $D = \overline{Bw_0 s_\beta.[v_1 + v_2]}$ , where  $\beta$  is a simple root, and Lemma 9.2 shows that  $D$  contains  $X_i$  if and only if  $\beta$  is a root of  $P_i$ . Let  $P_{12} \subset G$  be the parabolic subgroup generated by  $P_1$  and  $P_2$ . We obtain  $\text{Pic}(X) \cong \mathbb{Z} \oplus \text{Pic}(G/P_{12})$ . Let  $\pi : X \rightarrow G/P_{12}$  be the map defined by  $\pi(g.[v_1 + v_2]) = \pi(g.[v_i]) = g.P_{12}$ . This is a  $G$ -equivariant morphism of varieties, as its restriction to  $X_0 \cup X_i$  is the composition of  $\pi_i : X_0 \cup X_i \rightarrow G/P_i$  with the projection  $G/P_i \rightarrow G/P_{12}$ . The fibers of  $\pi$  are translates of  $\pi^{-1}(1.P_{12}) = \overline{L.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$ , where  $L$  is the Levi subgroup of  $P_{12}$  containing  $T$ . Moreover,  $\pi^{-1}(1.P_{12})$  is a non-singular projective horospherical variety of Picard rank 1, so it is either a flag variety or one of the non-homogeneous spaces from Pasquier's classification [Pas09].

**Question 9.6.** Let  $X$  be any projective  $G$ -horospherical variety fibered over a flag variety  $G/P$  with non-singular horospherical fibers of Picard rank 1. Is it true that  $X$  is isomorphic to an orbit closure  $\overline{G.[v_1 + v_2]} \subset \mathbb{P}(V)$ , where  $V$  is a rational representation of  $G$ , and  $v_1, v_2 \in V$  are highest weight vectors?

**Example 9.7.** Let  $X$  be the blow-up of  $\mathbb{P}^2$  at a point  $p$ , let  $\pi : X \rightarrow \mathbb{P}^1$  be the morphism defined by projection from  $p$ , and set  $G = \text{SL}(2, \mathbb{C})$ . Then  $X$  is  $G$ -horospherical and fibered over  $\mathbb{P}^1$  with fiber  $\mathbb{P}^1$ . This variety  $X$  is isomorphic to  $\overline{G.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$ , where  $v_1$  is a highest weight vector in  $V_1 = \mathbb{C}^2$ , and  $v_2$  is a highest weight vector in  $V_2 = \text{Sym}^2(\mathbb{C}^2)$ .

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