Linear Algebraic Groups

Spring 2026

Class notes by Anders Buch
based on T. A. Springer's book
"Linear Algebraic Groups".

We follow parts of Springer's book with minor deviations.

See "Review of Varieties" for a brief summary of basic notions such as spaces with functions (SWF) and algebraic varieties.

Linear Algebraic Groups Jan. 20, 2026

$k = \bar{k}$ alg. closed field.

Def A LAG is a group G that is is also an affine variety, such that multiplication $\mu : G \times G \to G$ and inverse elt. fcn. $i : G \to G$ are morphisms of varieties.

Challenge: $\frac{1}{2}$ of class do research on alg. geo.

$\frac{1}{2}$ of class does not know what affine variety is!

Example: $G = GL_n$

$G = SL_n = \{ g \in GL_n \mid \det(g) = 1 \}$

$G = O(n) = \{ g \in GL_n \mid g^T g = 1 \}$

Fact: Any subgroup $G \subseteq GL_n$ defined by poly. equs. is a LAG. Every LAG is $\cong$ such a subgroup.

Students w/o alg. geo.:

Ok to ignore discussions about AG. aspects.

LAG = subgp of GL_n def. by poly equs.

Accept: AG $\Rightarrow$ "this map is surjective"

AG $\Rightarrow$ "this vector space has dim $< \infty$".

Alg. Geo. I, Fall 2026.

Example

$A \in GL_n$ any element.

$k = \bar{k} \Rightarrow A = QJQ^{-1}$, J Jordan normal form.

$J = D + N$. $D \in GL_n$ diag. $N \in \mathrm{Mat}(n \times n)$ nilpotent.

$DN = ND$.

$J = J_s J_u$:

$J_s = D$ <u>semisimple</u> part.

$J_u = D^{-1} J$ <u>unipotent</u> part.

<u>Note</u>: $J_u - I = D^{-1}J - I = D^{-1}(J - D) = D^{-1}N$

$J_u - I$ nilpotent $\Leftrightarrow$ J_u unipotent.

$A = A_s A_u$, $A_s = Q J_s Q^{-1}$ ss part.

$A_u = Q J_u Q^{-1}$ unipot. part.

<u>Fact</u>: A_s, A_u are unique.

If $G \subseteq GL_n$ LAG, then $A_s, A_u \in G$.

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Algebraic variety: separated SWF with finite open covering by affine varieties.

Irreducible variety X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed $\Rightarrow X = X_1$ or $X = X_2$.

$X = \mathrm{Spec}(R)$ affine: X irred $\Leftrightarrow$ R domain.

Connected variety X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed, $X_1 \cap X_2 = \emptyset \Rightarrow X = X_1$ or $X = X_2$.

Product of varieties $X \times Y$:

product in category of alg. varieties.

$$\begin{tikzcd}
 & & X \\
Z \arrow[urr, bend left, "\phi_X"] \arrow[r, dashed, "\exists!", "\phi"'] \arrow[drr, bend right, "\phi_Y"'] & X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
 & & Y
\end{tikzcd}$$

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ as sets!

Example: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ (not product topology!)

$X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ affine

$\Rightarrow X \times Y = \mathrm{Spec}(R \otimes_k S)$.

$f \otimes g \in R \otimes S$: $(f \otimes g)(x,y) = f(x)g(y)$.

Def SWF X is separated

$\Updownarrow$

$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$ closed.

Projective space

$$\mathbb{P}^n = \{\text{lines through } 0 \text{ in } \mathbb{A}^{n+1}\}$$
$$= \{[x_0 : x_1 : \cdots : x_n] \mid (x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}\}$$

Proj. coord. ring: $k[x_0, \ldots, x_n]$.

$f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ homogeneous polys:

$Z(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ closed.

$D_+(x_i) = \{[x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n]\} \cong \mathbb{A}^n$.

$\mathbb{P}^n = D_+(x_0) \cup \cdots \cup D_+(x_n)$ alg. var.

Dimension:

X variety.

$$\dim(X) = \max\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists\, X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d \subseteq X \\ \text{s.t. } X_i \text{ closed \& irreducible} \end{array} \right\}$$

$X = \mathrm{Spec}(R)$ irred. affine variety

$$\Rightarrow \dim(X) = \mathrm{tr.deg.}_k(K(R)).$$

Examples:
- $\dim(\mathbb{A}^n) = \mathrm{tr.deg.}_k\, k(x_1, \ldots, x_n) = n.$
- $\dim(X \times Y) = \dim(X) + \dim(Y).$

Thm $\phi : X \longrightarrow Y$ morphism of varieties.
Then $\phi(X)$ contains a dense open subset of $\overline{\phi(X)}$.

Fact: $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$.

$$\begin{array}{ccc} \{\text{morphisms } X \longrightarrow Y\} & \longleftrightarrow & \{k\text{-alg. hom. } S \longrightarrow R\} \\ \phi & \longmapsto & \phi^* \end{array}$$

Algebraic Groups

Alg. group: alg. variety G that is also a group:

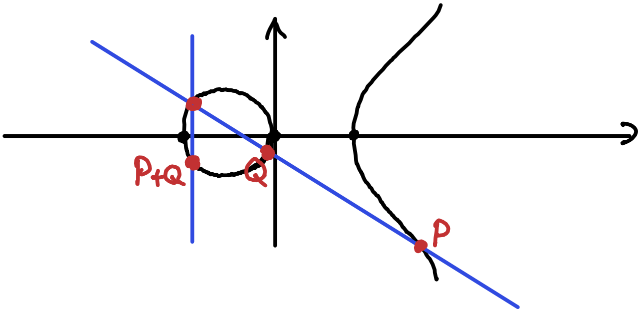
$\mu: G \times G \longrightarrow G$ and $i: G \to G$ morphisms.

Consequence: Each elt. $x \in G$ defines two automorphisms:

$G \xrightarrow{\cong} G$, $y \longmapsto x \cdot y$, $y \longmapsto y \cdot x$

Example: Elliptic curve $E = Z(zy^2 - x^3 + xz^2) \subseteq \mathbb{P}^2$

Draw in $\mathbb{A}^2 = \{z = 1\}$: $y^2 = x^3 - x$



homomorphism of alg. groups $\phi: G \longrightarrow G'$:

morphism + group hom.

products: $G \times G'$

closed subgroup: $H \subseteq G$

LAG: affine algebraic group.

Hopf algebra

G LAG. $\mu : G \times G \longrightarrow G$. $i : G \rightarrow G$.

$A = k[G] = A(G) = \mathcal{O}_G(G)$.

$k[G \times G] = A \otimes_k A$.

Comult: $\Delta = \mu^* : A \longrightarrow A \otimes A$, $f \mapsto f\mu$

$\Delta f(x,y) = f\mu(x,y) = f(x \cdot y)$

Antipode: $\iota = i^* : A \longrightarrow A$, $f \mapsto fi$

$\iota f(x) = fi(x) = f(x^{-1})$

Example

$M_n = \mathrm{Mat}(n \times n, k)$.

$k[M_n] = k[T_{ij},\ 1 \leq i,j \leq n]$.

$G = GL_n = \{ x \in M_n \mid \det(x) \neq 0 \}$

$A = k[G] = k[M_n]_{\det} = k[T_{ij},\ \det^{-1}]$.

$\mu : G \times G \longrightarrow G$ mult. $\Delta : A \longrightarrow A \otimes A$

$$(x \cdot y)_{ij} = \sum_k x_{ik} y_{kj}. \qquad \Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$$

$$\begin{gathered} (x\cdot y)_{ij} \\ \parallel \\ T_{ij}(\mu(x,y)) = \Delta T_{ij}(x,y) \end{gathered}$$

$\iota : A \longrightarrow A$, $\iota(T_{ij}) = f_{ij} : (x^{-1})_{ij} = f_{ij}(x)$.

$$\begin{gathered} (x^{-1})_{ij} \\ \parallel \\ T_{ij}(i(x)) = \iota T_{ij}(x) \end{gathered}$$

<u>Mult:</u> $m : A \otimes A \longrightarrow A, \quad f \otimes g \mapsto fg.$

$m = \delta^*, \qquad \delta : G \longrightarrow G \times G, \quad x \mapsto (x,x).$

$$\delta^*(f \otimes g)(x) = (f \otimes g)(\delta(x)) = f(x)g(x) = (fg)(x) = m(f \otimes g)(x).$$

<u>Id. elt:</u> $e \in G. \qquad e : A \longrightarrow k, \quad f \mapsto f(e).$

$\varepsilon : A \xrightarrow{e} k \xrightarrow{\subseteq} A.$

$\varepsilon f(x) = f(e).$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "id \otimes \Delta"] \\
A \otimes A \arrow[r, "\Delta \otimes id"'] & A \otimes A \otimes A
\end{tikzcd}$$

$\Updownarrow$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$$\begin{tikzcd}
A \otimes A \arrow[r, "\iota \otimes id"] & A \otimes A \arrow[d, "m"] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & A \\
A \otimes A \arrow[r, "id \otimes \iota"'] & A \otimes A \arrow[u, "m"']
\end{tikzcd}$$

$\Updownarrow$

$x^{-1} \cdot x = e = x \cdot x^{-1}.$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "id \otimes e"] \\
A \otimes A \arrow[r, "e \otimes id"'] & A
\end{tikzcd}$$

$\Updownarrow$

$e \cdot x = x = x \cdot e$

Basic results

G alg. group.

Lemma: $\exists!$ irred. comp. $G^\circ \subseteq G$ with $e \in G^\circ$.
G° is closed normal subgroup of finite index.

Pf $X, Y \subseteq G$ irred. comps, $e \in X$, $e \in Y$.

$X \times Y$ irred. $\Rightarrow$ $XY = \mu(X \times Y)$ irred.

$\Rightarrow$ $\overline{XY}$ irred.

$X, Y \subseteq \overline{XY} \Rightarrow X = Y = \overline{XY}$.

X closed under mult.

$i^{-1}(X)$ irred comp., $e \in i^{-1}(X) \Rightarrow i^{-1}(X) = X$.

$\therefore$ $G^\circ = X$ closed subgroup.

$[G : G^\circ] = \#$ irred. comps $< \infty$.

□

Cor G irred $\Leftrightarrow$ G connected.

Pf G connected but not irred.

$\Rightarrow \exists x \in G$, x in two irred. comps

$\Rightarrow e$ in two comps.

$$\begin{array}{rccc} x : & G & \xrightarrow{\cong} & G \\ & y & \longmapsto & xy \end{array}$$

□

Cor $H \subseteq G$ closed subgp. with $[G:H] < \infty$
$\Rightarrow G^\circ \subseteq H$.

Pf $H^\circ \subseteq G^\circ$ closed with $[G^\circ : H^\circ] < \infty$.

□ $G^\circ = x_1 H^\circ \cup x_2 H^\circ \cup \cdots \cup x_\ell H^\circ$. G° irred $\Rightarrow G^\circ = H^\circ$.

Lemma $U, V \subseteq G$ dense open subsets $\Rightarrow UV = G$

Proof Let $x \in G$.

$U,\ xV^{-1} \subseteq G$ dense open.

$\Rightarrow U \cap xV^{-1} \neq \emptyset$. $\quad xv^{-1} \in U$ for some $v \in V$.

□

Lemma $H \subseteq G$ any subgroup.

(1) $\overline{H} \subseteq G$ is a closed subgroup. (EXER)

(2) If H contains nonempty open subset of $\overline{H}$, then $H = \overline{H}$. $\quad (H \cdot H = \overline{H}.)$

Prop $\phi : G \longrightarrow G'$ hom. of alg. groups.

(1) $\operatorname{Ker}(\phi) \subseteq G$ closed normal subgp.

(2) $\phi(G) \subseteq G'$ closed subgp

(3) $\phi(G^\circ) = \phi(G)^\circ$

Pf: (2): $\phi(G)$ contains dense open subset of $\overline{\phi(G)}$.

(3): $[\phi(G) : \phi(G^\circ)] < \infty \Rightarrow \phi(G)^\circ \subseteq \phi(G^\circ)$.

□

Prop $\{\phi_i : X_i \longrightarrow G\}_{i\in I}$ family of morphisms.

Assume X_i irred. and $e \in Y_i = \phi_i(X_i)$ $\forall i \in I$.

$H \leq G$ smallest closed subgroup with $Y_i \subseteq H$ $\forall i$

(1) H is connected.

(2) $H = Y_{a(1)}^{\varepsilon(1)} Y_{a(2)}^{\varepsilon(2)} \cdots Y_{a(n)}^{\varepsilon(n)}$ for some $a(1), \dots, a(n) \in I$, $\varepsilon(1), \dots, \varepsilon(n) \in \{\pm 1\}$.

Eg. $H = Y_1 Y_2^{-1} Y_3 Y_1 Y_2$

Proof WLOG: $\forall i \in I$ $\exists j \in I$: $Y_i^{-1} = Y_j$.

Given $a = (a(1), \dots, a(n)) \in I^n$,

set $Y_a = Y_{a(1)} Y_{a(2)} \cdots Y_{a(n)}$.

Then $\overline{Y_a} \subseteq G$ irred. closed subset.

$Y_b \cdot Y_c = Y_{(b,c)}$.

EXER: $\overline{Y_b} \cdot \overline{Y_c} \subseteq \overline{Y_{(b,c)}}$.

Choose a such that $\dim \overline{Y_a}$ is maximal.

$\forall b$: $\overline{Y_a} \subseteq \overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$.

$\dim \overline{Y_a} = \dim \overline{Y_{(a,b)}}$, both closed irred

$\Rightarrow \overline{Y_a} = \overline{Y_{(a,b)}}$.

$\therefore$ $H = \overline{Y_a} \leq G$ connected closed subgroup.

$\square$

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G alg. group.

G-variety: Variety X with action (morphism) $a: G \times X \to X$.

$a(g,x) = g.x$.

Orbit of x: $G.x$

X homogeneous $\Leftrightarrow$ $X = G.x_0$.

G-equivariant morphism: $\phi: X \longrightarrow Y$, $\phi(g.x) = g.\phi(x)$.

Rational representation:

Alg. group hom. $r: G \longrightarrow GL(V)$, $\dim(V) < \infty$.

$G \circlearrowright V$: $g.v = r(g)(v)$.

Lemma X G-variety.

(1) $G.x \subseteq \overline{G.x}$ is open $\forall x \in X$.

(2) $\exists$ closed orbits in X.

Pf (1) $G \longrightarrow X$, $g \mapsto g.x$.

Image $G.x \supseteq$ dense open $U \subseteq \overline{G.x}$.

$G.x = \bigcup_{g \in G} g.U \subseteq \overline{G.x}$ open.

(2) $\overline{G.x} - G.x$ = union of orbits.

Choose $x \in X$ with $\dim G.x$ minimal.

$\square$

<u>Now</u>: G LAG, X affine G-variety.

$a : G \times X \longrightarrow X$

$a^* : k[X] \longrightarrow k[G] \otimes k[X]$. $\quad a^*(f)(g,x) = f(g.x)$.

<u>Def</u>: $s : G \longrightarrow GL(k[X])$ $\quad$ <u>rep. of abstract gps.</u>

$$(s(g)f)(x) = f(g^{-1}.x)$$

<u>Relation</u>: let $f \in k[X]$.

$$a^*(f) = \sum_{i=1}^{n} u_i \otimes f_i \in k[G] \otimes k[X].$$

$$(s(g)f)x = f(g^{-1}.x) = a^*(f)(g^{-1},x) = \sum u_i(g^{-1}) f_i(x)$$

$$\Rightarrow s(g)f = \sum u_i(g^{-1}) f_i$$

Assume $V \subseteq k[X]$, $\dim(V) < \infty$.

<u>Lemma</u> $\exists V \subseteq W \subseteq k[X]$: $\dim(W) < \infty$, $s(g)(W) \subseteq W$ $\forall g \in G$.

<u>Pf</u> WLOG $V = \mathrm{Span}\{f\}$.

Relation $\Rightarrow$ $s(g)f \in \mathrm{Span}\{f_1, \dots, f_n\}$ $\forall g \in G$.

$\Rightarrow \dim\left(W = \mathrm{Span}\{s(g)f \mid g \in G\}\right) < \infty$.

□

<u>Lemma</u> $V \subseteq k[X]$ $s(G)$-stable $\Leftrightarrow a^*(V) \subseteq k[G] \otimes V$.

$\Rightarrow$ V rational rep. of G, $s : G \times V \longrightarrow V$.

- similar.

Action by translation

$G \circlearrowright G$, $\quad g.x = gx$ (left), $\quad g.x = xg^{-1}$ (right)

$\lambda : G \longrightarrow GL(k[G])$, $\quad (\lambda(g)f)(x) = f(g^{-1}x)$

$\rho : G \longrightarrow GL(k[G])$, $\quad (\rho(g)f)(x) = f(xg)$.

Note: λ and ρ are faithful (= injective):

$\lambda(g)$ determines $G \longrightarrow G, x \longmapsto g^{-1}x$

determines $g \in G$.

Thm $G \cong$ closed subgroup $\subseteq GL_n$.

Pf Choose $f_1, \dots, f_n \in k[G]$:

- $k[G] = k[f_1, \dots, f_n]$
- $V = \mathrm{Span}\{f_1, \dots, f_n\}$ is $\rho(G)$-stable
- $\{f_1, \dots, f_n\}$ lin. indep.

$\exists\ m_{ij} \in k[G]$, $\quad 1 \le i,j \le n$ such that

$$\rho(g) f_j = \sum_{i=1}^{n} m_{ij}(g) f_i$$

Check: $a : G \times G \longrightarrow G$, $\quad a(g,x) = xg$.

$a^*(f_j) = \sum_i m_{ij} \otimes f_i$ for some $m_{ij} \in k[G]$.

$(\rho(g) f_j)(x) = f_j(xg) = a^*(f_j)(g,x) = \sum_i m_{ij}(g) f_i(x)$.

$\phi : G \longrightarrow GL_n$, $\phi(g) = (m_{ij}(g))_{i,j}$ alg. gp. hom.

$k[GL_n] = k[T_{ij}, \det^{-1}]$.

$$\begin{aligned} \phi^* : k[T_{ij}, \det^{-1}] &\longrightarrow k[G] \\ T_{ij} &\longmapsto m_{ij} \\ \det^{-1} &\longmapsto \det(m_{ij})^{-1}. \end{aligned}$$

Surjective:

$$f_j(g) = f_j(eg) = (\rho(g) f_j)(e) = \sum_i m_{ij}(g) f_i(e)$$

$$f_j = \sum_i f_i(e) m_{ij} \in \operatorname{Im}(\phi^*).$$

$\therefore$ $k[G] = k[GL_n]/I$, $I = \ker(\phi^*)$

$\Updownarrow$ $G \cong Z(I) \subseteq GL_n$.

□

Jordan decomposition

V vector space, $\dim(V)<\infty$.

$a \in \mathrm{End}_k(V)$.

a **semi-simple**: $\exists$ basis of eigenvectors.

a **nilpotent**: $a^n = 0$ for some $n \geq 0$.

a **unipotent**: $a-1$ nilpotent.

Note: $\mathrm{char}(k) = p > 0$: a unipotent $\Leftrightarrow a^{p^s} = 1$, $s \in \mathbb{N}$.

$M_n = \mathrm{Mat}(n \times n) = \mathrm{End}(k^n)$.

Lemma $S \subseteq M_n$ set of pairwise commuting matrices.

(1) $\exists\, x \in GL_n$: xSx^{-1} upper Δ.

(2) All elts. of S semi-simple $\Rightarrow xSx^{-1}$ diagonal.

Proof Simultaneous Jordan decomp (almost)! $\square$

Lemma

(1) $a, b \in \mathrm{End}(V)$, $ab = ba$.
a, b both ss/nilpot/unipot $\Rightarrow$ so is ab.

(2) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot/unipot
$\Rightarrow$ so is $a \oplus b \in \mathrm{End}(V \oplus W)$ and $a \otimes b \in \mathrm{End}(V \otimes W)$.

(3) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot
$\Rightarrow$ so is $a \otimes 1 + 1 \otimes b \in \mathrm{End}(V \otimes W)$.

Note: $a \otimes 1 + 1 \otimes b - 2(1 \otimes 1)$ is nilpotent!

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Prop $a \in \mathrm{End}(V)$, $\dim(V) < \infty$.

(1) $\exists!\, a_s, a_n \in \mathrm{End}(V)$: a_s ss, a_n nilpot, $a_s a_n = a_n a_s$, $a = a_s + a_n$.

(2) $a_s, a_n \in k[a]$ (polys. in a).

(3) $W \subseteq V$, $a(W) \subseteq W \Rightarrow a_s(W) \subseteq W$ and $a_n(W) \subseteq W$.

$(a|_W)_s = a_s|_W$ and $(a|_W)_n = a_n|_W$.

$\bar{a} : V/W \longrightarrow V/W$. $(\bar{a})_s = \overline{a_s}$ and $(\bar{a})_n = \overline{a_n}$.

(4) $\phi : V \longrightarrow W$ linear, $b \in \mathrm{End}(W)$.

$\phi a = b \phi \Rightarrow \phi a_s = b_s \phi$ and $\phi a_n = b_n \phi$.

Proof of (4):

$$\begin{tikzcd}
a & a \oplus b & b \\
V \arrow[r, "v \mapsto (v, \phi(v))"'] & V \oplus W \arrow[r] & W
\end{tikzcd}$$

Cor $a \in GL(V)$, $\dim(V) < \infty$.

$\exists!\, a_s, a_u \in GL(V)$: a_s ss, a_u unipotent.

$a = a_s a_u = a_u a_s$

Proof $a = a_s + a_n = a_s(1 + a_s^{-1} a_n)$.

Locally finite

V any vector space, $a \in \mathrm{End}(V)$.

a is locally finite: $\forall v \in V \ \exists v \in W \subseteq V : a(W) \subseteq W, \dim(W) < \infty$.

Assume a locally finite.

a semi-simple: $a|_W$ simisimple $\forall W \subseteq V$, $a(W) \subseteq W$, $\dim < \infty$

a locally nilpotent: $a|_W$ nilpot. —"—

a locally unipotent: $a|_W$ unipot. —"—

Example $a = J_1(0) \oplus J_2(0) \oplus J_3(0) \oplus \cdots$ locally nilpot., not nilpot.

Cor $a \in \mathrm{End}(V)$ loc. finite $\Rightarrow$

$\exists!\ a_s, a_n \in \mathrm{End}(V)$ loc. finite: a_s SS, a_n loc. nilpot.

$$a = a_s + a_n, \quad a_s a_n = a_n a_s.$$

$$a_s|_W = (a|_W)_s, \quad a_n|_W = (a|_W)_n.$$

Cor $a \in GL(V)$ loc. finite

$\exists!\ a_s, a_u \in GL(V)$ loc. finite: a_s SS, a_u loc. unipot.

$$a = a_s a_u = a_u a_s.$$

G LAG.

Recall: $\rho(g) : k[G] \longrightarrow k[G]$ locally finite $\forall g \in G$.

Lemma $G = GL(V)$, $\dim(V) < \infty$.
$g \in G$ is ss/unipot $\Leftrightarrow$ $\rho(g)$ ss / loc. unipot.

Proof

For $f \in V^*$, **def.** $\tilde{f} : V \longrightarrow k[G]$, $\tilde{f}(v)(g) = f(gv)$

$\tilde{f}(gv) = \rho(g)\, \tilde{f}(v)$:

$$\tilde{f}(gv)(h) = f(hgv) = \tilde{f}(v)(hg) = \big(\rho(g)\, \tilde{f}(v)\big)(h).$$

$$\rho(g)_s\, \tilde{f}(v) \underset{\substack{\uparrow \\ \text{Prop (4)}}}{=} \tilde{f}(g_s v) = \rho(g_s)\, \tilde{f}(v).$$

$k[G]$ gen. by $\{ \tilde{f}(v) \mid f \in V^*,\ v \in V \}$

□ $\Rightarrow$ $\rho(g)_s = \rho(g_s)$.

Def $g \in G$ is semi-simple if $\rho(g)$ semi-simple.
$g \in G$ is unipotent if $\rho(g)$ loc. unipot.

Thm G LAG, $g \in G$.

(1) $\exists !\, g_s, g_u \in G$: g_s ss, g_u unipot., $g = g_s g_u = g_u g_s$.

(2) $\phi : G \longrightarrow G'$ alg. group hom.
$\Rightarrow$ $\phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$.

(3) $G = GL_n$ $\Rightarrow$ g_s, g_u are as above.

Recall: G simple: G solvable $\Leftrightarrow$ nilpotent $\Leftrightarrow$ abelian.

G nilpotent $\Leftrightarrow$ $G/Z(G)$ nilpotent.

$H \lhd G$ normal: G solvable $\Leftrightarrow$ G/H, H solvable.

G abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable.

Exer $U_n = \boxed{\begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix}} \subseteq B_n = \boxed{\begin{matrix} * & & * \\ & \ddots & \\ 0 & & * \end{matrix}} \subsetneq GL_n$

nilpotent

solvable, not nilpotent.

Burnside's Thm

V vector space over $k = \bar{k}$, $\dim(V) < \infty$.

$R \subseteq \operatorname{End}_k(V)$ subalgebra.

V simple R-module $\Rightarrow R = \operatorname{End}_k(V)$.

Cor $G \subseteq GL(V)$ closed subgroup, $\dim(V) < \infty$.

If V irred. G-rep. then $\operatorname{End}_k(V) = \operatorname{span}(G)$.

Cor $G \subseteq GL_n$ consists of unipotent elts.

$\Rightarrow xGx^{-1} \subseteq U_n$ for some $x \in GL_n$.

Proof Enough: $xGx^{-1} \subseteq B_n$.

$V = k^n$ not irred: $\exists\, 0 \neq W \subsetneq V$, W G-stable.

Images of G in $GL(W)$, $GL(V/W)$ consist of unipot. elts.

Use induction on n.

WLOG V irreducible.

For $g, h \in G$: $\operatorname{Tr}((1-g)h) = \operatorname{Tr}(h) - \operatorname{Tr}(gh) = n - n = 0$.

Burnside $\Rightarrow \operatorname{Tr}((1-g)h) = 0 \;\; \forall\, h \in \operatorname{End}(V) \Rightarrow g = e$.

$\square$

Def G is **unipotent** if all elts of G are unipot.

Cor unipotent $\Rightarrow$ nilpotent.

Prop G unipotent, X affine G-variety.

$\Rightarrow$ $G.x \subseteq X$ is closed $\forall x \in X$.

Proof

$O \subseteq X$ orbit.

WLOG: $X = \overline{O}$. $\Rightarrow$ $O \subseteq X$ is open.

$Y = X \setminus O \subseteq X$ closed, G-stable.

$s : G \longrightarrow GL(k[X])$, $(s(g)f)(x) = f(g^{-1}.x)$

G acts locally finitely on $k[X]$.

$G.I(Y) \subseteq I(Y)$.

$\exists\, 0 \neq f \in I(Y) : s(g)f = f \;\forall g \in G$:

$\exists\, 0 \neq W \subseteq I(Y)$, $\dim(W) < \infty$, $G.W \subseteq W$.

G unipotent $\Rightarrow$ $s(G) \subseteq GL(W)$ unipotent

WLOG: $s(G) \subseteq U_m \subseteq GL_m = GL(W)$.

$\Rightarrow$ f constant on O

$\Rightarrow$ f constant on X

$I(Y) \supseteq \langle f \rangle = k[X] \Rightarrow Y = \emptyset$.

□

Def G LAG.

$$\left.\begin{aligned} G_s &= \{g \in G \mid g \text{ is semi-simple}\} \\ G_u &= \{g \in G \mid g \text{ is unipotent}\} \end{aligned}\right\} \text{ subsets, usually not subgroups.}$$

Note: $G_u \subseteq G$ is closed.

$$(GL_n)_u = \{g \in GL_n \mid \chi_g(t) = (t-1)^n\}$$

Thm G commutative LAG.

(1) G_s and G_u are closed subgroups.

(2) $\mu : G_s \times G_u \xrightarrow{\cong} G$ (product map).

Proof

$(gh)_s = g_s h_s = gh \Rightarrow G_s$ subgroup.

WLOG: $G \subseteq B_n \subseteq GL_n$.

$G_s = G \cap D_n$, $G_u = G \cap U_n$ closed. $D_n = \boxed{\begin{matrix} * & & 0 \\ & * & \\ 0 & & * \end{matrix}}$

$\mu : G_s \times G_u \longrightarrow G$, $(g,h) \mapsto gh$ is bijective
(unique Jordan decomp.)

$\mu^{-1}(g) = (g_s, g_s^{-1} g)$ is morphism of varieties.

□

Cor G commutative & connected $\Rightarrow$ so are G_s, G_u.

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Prop G connected LAG, $\dim(G)=1$.

(1) G is commutative.

(2) $G=G_s$ or $G=G_u$.

(3) $G=G_u$ and $\mathrm{char}(k)=p>0 \Rightarrow g^p=e \quad \forall g\in G$.

Pf

(1) Assume G not commutative.

$(G,G)\neq e$ connected closed subgroup $\Rightarrow (G,G)=G$.

Let $g\in G\setminus Z(G)$.

$G=\overline{\{xgx^{-1} \mid x\in G\}}$

$G\subseteq GL_n$: $\chi_g(t)$ constant for $g\in G$.

$\Rightarrow \chi_g(t)=(t-1)^n \quad \forall g\in G$

$\Rightarrow G$ is unipotent $\Rightarrow$ solvable $\Rightarrow (G,G)\neq G$ ↯

(2) clear.

(3) $G^{(p^h)}=\langle g^{p^h} \mid g\in G\rangle \subseteq G$ connected closed subgp.

$G\subseteq U_n$: $g^{p^h}=e$ for $h\geq n$.

Must have $G^{(p)}\neq G \Rightarrow G^{(p)}=e$.

□

Characters & cocharacters

$\mathbb{G}_m = k^\times$ mult. group.

Character: $\chi : G \longrightarrow \mathbb{G}_m$ alg. gp. hom.

$X^*(G) = \{\chi : G \longrightarrow \mathbb{G}_m\} \subseteq k[G]^\times$ (abelian) subgroup.

Dedekind: $X^*(G) \subseteq k[G]$ lin. independent.

Pf

Equation with n minimal: $\sum_{i=1}^{n} \lambda_i \chi_i(g) = 0$.

$$\Rightarrow \sum_{i=1}^{n} \lambda_i \chi_n(h) \chi_i(g) = 0$$

$$\sum_{i=1}^{n} \lambda_i \chi_i(h) \chi_i(g) = 0$$

$$\Rightarrow \sum_{i=1}^{n-1} \lambda_i (\chi_n(h) - \chi_i(h)) \chi_i(g) = 0$$

Choose $h \in G$ with $\chi_n(h) \neq \chi_1(h)$. ↯

□

Cocharacter: $\lambda : \mathbb{G}_m \longrightarrow G$ alg. group hom.

$X_*(G) = \{\lambda : \mathbb{G}_m \longrightarrow G\}$ set of cocharacters.

G commutative $\Rightarrow X_*(G)$ (abelian) group.

For $n \in \mathbb{Z}$: $(n.\lambda)(a) = \lambda(a)^n$. Prop. (1) $\Rightarrow n.\lambda \in X_*(G)$

$-\lambda = (-1).\lambda$.

Example: $X^*(\mathbb{G}_m) = X_*(\mathbb{G}_m) = \mathbb{Z}$.

$\chi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ alg. group hom.

$\chi(a) = a^n$ for some $n \in \mathbb{Z}$.

Note: $k[\mathbb{G}_m] = k[t, t^{-1}]$ has basis $\{t^n : n \in \mathbb{Z}\}$.

$D_n = (\mathbb{G}_m)^n$. $\quad X^*(D_n) \cong \mathbb{Z}^n \cong X_*(D_n)$.

$k[D_n] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has basis $X^*(D_n)$.

Def G is diagonalizable $\Leftrightarrow$ $G \subseteq D_n$ closed.

G is a torus $\Leftrightarrow$ $G \cong D_n$

Thm G LAG. TFAE:

(1) G is diagonalizable.

(2) $X^*(G)$ is a basis of $k[G]$.

(3) $G \circlearrowright V$ rational rep. $\Rightarrow$ V direct sum of 1-dim. reps.

Proof

(1) $\Rightarrow$ (2): $G \subseteq D_n$. $k[D_n] \twoheadrightarrow k[G]$.

$k[G]$ spanned by image of $X^*(D_n)$.

$\therefore$ $k[G] = \operatorname{Span}(X^*(G))$.

(2) $\Rightarrow$ (3): $\phi : G \longrightarrow GL(V)$ rat. rep.

$\exists!\, A_\chi \in \operatorname{End}(V)$ for $\chi \in X^*(G)$:

$$\phi(g) = \sum_\chi \chi(g) A_\chi$$

($\phi : G \longrightarrow GL(V) \subseteq \operatorname{End}(V) = M_n$

$\phi(g) = (\phi_{ij}(g)) \in M_n$, $\phi_{ij} \in k[G] = \operatorname{Span} X^*(G)$.)

Note: $A_\chi \neq 0$ for finitely many χ.

$1_V = \phi(e) = \sum_{\chi} A_\chi$

$g, h \in G$: $\sum_{\chi} \chi(g)\chi(h) A_\chi = \phi(gh) = \phi(g)\phi(h)$

$= \sum_{\chi,\psi} \chi(g)\psi(h) A_\chi A_\psi$

$X^*(G \times G)$ linearly indep. $\Rightarrow A_\chi A_\psi = \delta_{\chi,\psi} A_\chi$

$\therefore\ V = \bigoplus_{\chi} A_\chi(V)$.

<u>Note</u>: $\phi(g).v = \chi(g)v$ for $v \in A_\chi(V)$.

(3) $\Rightarrow$ (1): $G \subseteq GL(V) = GL_n$ closed. Clear.

$\square$

<u>Cor</u> Assume G is diagonalizable.

(1) $X^*(G)$ f.g. abelian group.

(2) $k[G] = k[X^*(G)]$ group algebra.

(3) $\mathrm{char}(k) = p > 0 \Rightarrow X^*(G)$ has no p-torsion.

<u>Pf</u>

(1) $G \subseteq D_n$ closed $\Rightarrow \mathbb{Z}^n = X^*(D_n) \twoheadrightarrow X^*(G)$.

(3) $\chi^p = 1 \Rightarrow \chi(g)^p = 1 \in k\ \forall g \Rightarrow \chi = 1$.

$\square$

Diagonalizable LAG $\longleftrightarrow$ f.g. abelian group

M f.g. abelian group.

$k[M]$ = k-vector space with basis $\{e(m) : m \in M\}$.

$$e(m)\,e(n) = e(m+n).$$

Assume M has no p-torsion.

$\Leftrightarrow$ $k[M]$ reduced f.g. k-alg.

$\mathcal{G}(M) = \mathrm{Spec}(k[M])$ affine variety.

$$\Delta : k[M] \longrightarrow k[M] \otimes k[M], \quad \Delta(e(m)) = e(m) \otimes e(m).$$

$$\iota : k[M] \longrightarrow k[M] \qquad \iota(e(m)) = e(-m).$$

$$\varepsilon : k[M] \longrightarrow k \qquad \varepsilon(e(m)) = 1.$$

Prop

(1) $\mathcal{G}(M)$ is diagonalizable LAG.

(2) $X^*(\mathcal{G}(M)) = M$

(3) G diagonalizable LAG $\Rightarrow$ $\mathcal{G}(X^*(G)) = G$.

Note: M_1, M_2 f.g. abelian groups.

$$k[M_1 \oplus M_2] = k[M_1] \otimes_k k[M_2]$$

$$\mathcal{G}(M_1 \oplus M_2) = \mathcal{G}(M_1) \times \mathcal{G}(M_2).$$

Exer: M finite $\Rightarrow$ $\mathcal{G}(M) \cong M$.

Cor G diagonalizable LAG.

(1) $G \cong D_n \times F$, F finite abelian w/o p-torsion.

(2) G torus $\Leftrightarrow$ G connected $\Leftrightarrow$ $X^*(G)$ free abelian.

Prop (Rigidity)

G, H diagonalizable LAGs. V connected affine var.

$\phi : V \times G \longrightarrow H$ morphism.

Assume $g \longmapsto \phi(v,g)$ is alg. gp. hom. $\forall v \in V$.

Then $\phi(v,g)$ is independent of v.

Proof

Let $\psi \in X^*(H) \subseteq k[H]$.

$$\phi^*(\psi) = \sum_{\chi \in X^*(G)} f_{\chi,\psi} \otimes \chi \in k[V] \otimes k[G].$$

$$\psi(\phi(v,g)) = \sum_{\chi} f_{\chi,\psi}(v)\, \chi(g)$$

$v \in V$ fixed: LHS $\in X^*(G)$.

$$\Rightarrow \quad f_{\chi,\psi}(v) = \begin{cases} 1 & \text{if } \chi = \text{LHS} \\ 0 & \text{else.} \end{cases}$$

V connected $\Rightarrow$ $f_{\chi,\psi}$ constant.

□

G alg. group, $H \subseteq G$ closed subgroup.

$Z_G(H) = \{ g \in G \mid gh = hg \ \forall h \in H \}$

$N_G(H) = \{ g \in G \mid gHg^{-1} = H \}$

<u>Exer</u> $G = GL_n$

$Z_G(D_n) = D_n$

$N_G(D_n) = S_n D_n$, $S_n \subseteq G$ perm. matrices.

$N_G(D_n) / Z_G(D_n) = S_n$ Weyl group of GL_n.

<u>Cor</u> G LAG, $H \subseteq G$ diagonalizable closed subgroup. Then $N_G(H)^\circ = Z_G(H)^\circ$ and $N_G(H)/Z_G(H)$ is finite.

<u>Proof</u>

The morphism

$$N_G(H)^\circ \times H \longrightarrow H, \quad (g, h) \longmapsto ghg^{-1}$$

is independent of g.

$\Rightarrow \quad ghg^{-1} = h \quad \forall \ g \in N_G(H), \ h \in H$

$\Rightarrow \quad N_G(H)^\circ \subseteq Z_G(H).$

□

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T torus.

$X^*(T) = \{\chi : T \to \mathbb{G}_m\}$ group of characters.

$X_*(T) = \{\lambda : \mathbb{G}_m \to T\}$ group of cocharacters.

<u>Pairing</u>: $X^*(T) \times X_*(T) \longrightarrow X^*(\mathbb{G}_m) = \mathbb{Z}$

$$(\chi, \lambda) \longmapsto \chi\lambda \longleftrightarrow \langle \chi, \lambda \rangle$$

$\chi\lambda(a) = a^{\langle \chi, \lambda\rangle}$ for $a \in \mathbb{G}_m$.

<u>Exer</u>: Perfect pairing.

$\mathbb{G}_m = k^\times = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$.

<u>Def</u> $\phi : \mathbb{G}_m \to Z$ morphism, $z \in Z$ point.

$\lim\limits_{a \to 0} \phi(a) = z \iff \exists$ morphism $\tilde{\phi} : \mathbb{A}^1 \to Z$:

$\tilde{\phi}(a) = \phi(a)$ for $a \in \mathbb{G}_m$, $\tilde{\phi}(0) = z$.

$\lim\limits_{a \to \infty} \phi(a) = z \iff \lim\limits_{a \to 0} \phi(a^{-1}) = z$.

Always exist if Z is projective (or complete).

Assume Z affine.

$\phi^* : k[Z] \longrightarrow k[t, t^{-1}]$.

$\lim\limits_{a \to 0} \phi(a)$ exists $\iff \phi^*(k[Z]) \subseteq k[t]$

$\iff \forall f \in k[Z] : f\phi \in k(\mathbb{A}^1)$ is defined at 0.

Def T torus, V T-variety, $\lambda \in X_*(T)$.

$$V(\lambda) = \{ v \in V \mid \lim_{a \to 0} \lambda(a).v \text{ exists} \}$$

Note: $V(-\lambda) = \{ v \in V \mid \lim_{a \to \infty} \lambda(a).v \text{ exists} \}$

Lemma T torus, V affine T-variety, $\lambda \in X_*(T)$.

(1) $V(\lambda) \subseteq V$ is closed.

(2) $V(\lambda) \cap V(-\lambda) = V^{\lambda(\mathbb{G}_m)} = \{ v \in V \mid \lambda(a).v = v \ \forall a \in \mathbb{G}_m \}$.

Proof

$T \circlearrowright k[V]$ locally finite.

$(s(t).f)(v) = f(t^{-1}.v)$.

$$k[V] = \bigoplus_{\chi} k[V]_{\chi}$$

$$f = \sum_{\chi} f_{\chi} \quad \Rightarrow \quad s(t).f = \sum_{\chi} \chi(t) f_{\chi}.$$

Let $v \in V$.

$\phi : \mathbb{G}_m \longrightarrow V, \quad \phi(a) = \lambda(a).v$

$$\phi^*(f)(a) = f(\lambda(a).v) = (s(\lambda(a)^{-1}).f)(v)$$
$$= \sum_{\chi} a^{-\langle \chi, \lambda \rangle} f_{\chi}(v).$$

Defined at $a = 0$ $\Leftrightarrow$ $f_{\chi}(v) = 0$ when $\langle \chi, \lambda \rangle > 0$.

$$V(\lambda) = Z\Big(\bigoplus_{\langle \chi, \lambda \rangle > 0} k[V]_{\chi} \Big) \qquad \text{(not ideal!)}$$

$$f(\lambda(a).v) = \sum_{\chi} a^{-\langle \chi, \lambda \rangle} f_\chi(v)$$

$$\begin{aligned} v \in V(\lambda) \cap V(-\lambda) &\Leftrightarrow \forall f \in k[V] : f_\chi(v) = 0 \text{ for } \langle \chi, \lambda \rangle \neq 0 \\ &\Leftrightarrow \forall f \in k[V] : f(\lambda(a).v) = f(v) \\ &\Leftrightarrow v \in V^{\lambda(\mathbb{G}_m)} \end{aligned}$$

$\square$

Example

$\mathbb{G}_m \circlearrowright \mathbb{A}^2, \quad a.(x,y) = (ax, a^{-1}y).$

$\mathbb{G}_m \circlearrowright k[\mathbb{A}^2] = k[X,Y].$

$(a.f)(x,y) = f(a^{-1}.(x,y)) = f(a^{-1}x, ay).$

$a.X = a^{-1}X, \quad a.Y = aY$

$\lambda = id : \mathbb{G}_m \longrightarrow \mathbb{G}_m.$

$(x,y) \in \mathbb{A}^2(\lambda) \Leftrightarrow \lim\limits_{a \to 0} a.(x,y)$ exists $\Leftrightarrow y = 0.$

$\mathbb{A}^2(\lambda) = Z(Y), \quad \mathbb{A}^2(-\lambda) = Z(X).$

$\mathbb{A}^2(\lambda) \cap \mathbb{A}^2(-\lambda) = \{(0,0)\} = (\mathbb{A}^2)^{\mathbb{G}_m}$

$k[\mathbb{A}^2]_\chi = \operatorname{Span}\{X^i Y^j \mid j - i = \chi\}$

$\bigoplus_{\langle \chi, \lambda \rangle > 0} k[\mathbb{A}^2]_\chi = \operatorname{Span}\{X^i Y^j \mid j - i > 0\}$ (not ideal!)

Generates $I(\mathbb{A}^2(\lambda)) = \langle Y \rangle \subseteq k[\mathbb{A}^2].$

Quiz

G LAG, $g \in G$ torsion elt. $g^m = e$.

$g = g_s g_u = ?$

$p = \mathrm{char}(k) = 0$: $g = g_s$.

Assume $p > 0$:

$p \nmid m \Rightarrow g = g_s$.

$m = p^j \Rightarrow g = g_u$.

$m = n p^j$, $p \nmid n$.

g^n is unipotent.

g^{p^j} is semi-simple.

$an + bp^j = 1$, $a, b \in \mathbb{Z}$.

$g = (g^{bp^j})(g^{an}) = g_s g_u$.

Additive functions

G LAG. $p = \mathrm{char}(k)$.

$\mathbb{G}_a = k$ (additive group)

Def Additive functions on G:

$$\mathcal{A}(G) = \{ f \in k[G] \mid f : G \longrightarrow \mathbb{G}_a \text{ group hom.} \}$$

Example $G = \mathbb{G}_a^n$ vector group.

$k[G] = k[T_1, \dots, T_n]$

$f \in k[G]$ additive $\iff f(xy) = f(x) + f(y) \quad \forall\, x, y \in G$

$\iff f(T_1 + U_1, \dots, T_n + U_n) = f(T_1, \dots, T_n) + f(U_1, \dots, U_n)$.

Claim:

$$\mathcal{A}(G) = \begin{cases} \mathrm{Span}_k\{T_1, \dots, T_n\} & \text{if } p = 0 \\ \mathrm{Span}_k\{ T_i^{p^j} \mid 1 \le i \le n,\ j \ge 0 \} & \text{if } p > 0 \end{cases}$$

Proof

$$\frac{\partial f}{\partial T_i}(T_1 + U_1, \dots, T_n + U_n) = \frac{\partial f}{\partial T_i}(T_1, \dots, T_n)$$

$$\Rightarrow \frac{\partial f}{\partial T_i}(U_1, \dots, U_n) = c_i \in k \text{ constant.}$$

$$g = f - \sum_{i=1}^{n} c_i T_i . \qquad \frac{\partial g}{\partial T_i} = 0 \quad \forall\, i$$

$p = 0$: $g = 0$

$p > 0$: $g = h(T_1^p, \dots, T_n^p)$, $\quad h \in k[G]$

$g \in \mathcal{A}(G) \Rightarrow h \in \mathcal{A}(G)$.

Induction on $\deg(f)$.

□

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Def G LAG.

$G \cong \mathbb{G}_a^n$: G is a *vector group*

$G \subseteq \mathbb{G}_a^n$ closed: G is an *elementary unipotent group.*

Theorem G LAG. TFAE:

(1) G is elementary unipotent.

(2) G is unipotent, abelian, and $pG = 0$.

(3) $G = \mathbb{G}_a^n \times F$, F finite elementary unipotent.

(4) $k[G]$ is generated by $\mathcal{A}(G)$ as k-algebra.

Note $p = 0 \Rightarrow F = 0$

$p > 0 \Rightarrow F = (\mathbb{Z}/p\mathbb{Z})^m$

Cor G connected LAG, $\dim(G) = 1$

$\Rightarrow G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.

Module structure on $\mathcal{A}(G)$

$\mathcal{A}(G) \subseteq k[G]$ vector subspace.

$p = 0$: $R = k$, $\mathcal{A}(G)$ is an R-module.

Assume $p>0$:

$\mathcal{A}(\mathbb{G}_a) = \mathrm{Span}_k\{T^{p^j} \mid j \geq 0\}$, $\dim \mathcal{A}(\mathbb{G}_a) = \infty$.

$f \in \mathcal{A}(G) \Rightarrow f^p \in \mathcal{A}(G)$.

$R = k[T]$ as *additive group*

$$(aT^i)\cdot(bT^j) := a\, b^{p^j}\, T^{i+j}.$$

$$Tb = b^p T.$$

Properties:

(1) R associative, non-commutative.

(2) R is "Euclidean": division algorithm works.

(3) All left/right ideals are principal.

(4) Any f.g. left R-module is direct sum of cyclic modules.

R-module structure on $\mathcal{A}(G)$:

$a.f = af$ for $a \in k$, $f \in \mathcal{A}(G)$.

$T.f = f^p$

$(aT^i) = a\, f^{p^i}$

Exer: $\mathcal{A}(\mathbb{G}_a^n) =$ free left R-module, basis $\{T_1, \dots, T_n\}$.

Thm G elementary unipotent.

(1) $\mathcal{A}(G)$ f.g. left R-module.

(2) G connected $\Leftrightarrow$ $\mathcal{A}(G)$ free left R-module.

Derivations

R com. ring. A com. R-algebra. M A-module.

R-derivation $D : A \longrightarrow M$:

(1) R-linear.

(2) $D(ab) = a.D(b) + b.D(a)$, $a, b \in A$.

Note: If D satisfies (2), then (1) $\Leftrightarrow$ $D(R) = 0$.

$\mathrm{Der}_R(A, M) = \{ D : A \to M \text{ } R\text{-derivation} \}$

A-module: $(b.D + D')(a) = b.D(a) + D'(a)$.

Example $A = k[x_1, \dots, x_n]$, $D : A \longrightarrow M$ any k-derivation.

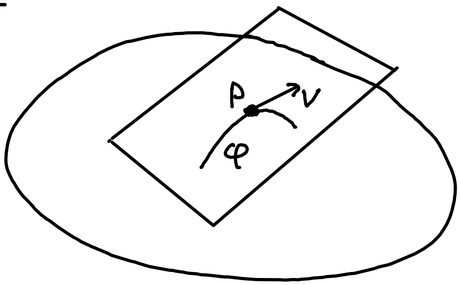
$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i).$$

$\mathrm{Der}_k(A, M) \cong M^{\oplus n}$ as A-module.

$\phi : A \longrightarrow B$ R-algebra hom, N B-module.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_A(B, N) \arrow[r] & \mathrm{Der}_R(B, N) \arrow[r, "\phi_0"] & \mathrm{Der}_R(A, N). \\ & & D \arrow[r, mapsto] & D \circ \phi \end{tikzcd}$$

Tangent and cotangent vectors



X manifold, $p \in X$.

Tangent vector $v \in T_p X$:

Equiv. class of param. curves
$\varphi : \mathbb{R} \longrightarrow X$ with $\varphi(0) = p$.

Given C^∞ $f : X \longrightarrow \mathbb{R}$:

$$D_v(f) = \frac{d}{dt} f(\varphi(t))\Big|_{t=0}.$$

$C^\infty(X)$-module: $\mathbb{R}(p) = \mathbb{R}$, $f \cdot a = f(p)a$.

$D_v \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), \mathbb{R}(p)) =: T_p X$.

$df_p \in (T_p X)^*$ cotangent vector: $df_p(v) = D_v(f)$.

Local ring of variety

X irred. variety, $p \in X$.

$\mathcal{F} = \{ (U, f) \mid p \in U \subseteq X \text{ open}, f : U \to k \text{ regular} \}$

Equiv. rel: $(U, f) \sim (U', f') \Leftrightarrow f|_{U \cap U'} = f'|_{U \cap U'}$

Local ring at p: $\mathcal{O}_{X,p} = \mathcal{F}/\sim \; = \{ f \in k(X) \mid f \text{ def. at } p \}$

$\mathfrak{m}_x = \{ f \in \mathcal{O}_{X,p} \mid f(p) = 0 \} \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \cong k$ is $\mathcal{O}_{X,p}$-module: $f \cdot a = f(p)a$

Example: X affine, $p \in X$.

$I(p) \subseteq k[X]$ max. ideal.

$$\mathcal{O}_{X,p} = k[X]_{I(p)} = \left(k[X] - I(p)\right)^{-1} k[X].$$

this def. is valid when X not irred.

Zariski tangent space

$T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$ tangent space.

$T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$. cotangent space.

Note: $D \in T_pX$, $f \in \mathfrak{m}_p^2 \Rightarrow D(f) = 0$.

$g, h \in \mathfrak{m}_p \Rightarrow D(gh) = g(p)D(h) + h(p)D(g) = 0$.

Note:

$$\begin{array}{ccc} \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) & \xrightarrow{\cong} & \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* \\ D & \longmapsto & \left[f + \mathfrak{m}_p^2 \mapsto D(f)\right] \\ \left[f \mapsto \bar{D}(f - f(p) + \mathfrak{m}_p^2)\right] & \longleftarrow\!\shortmid & \bar{D} \end{array}$$

$\therefore$ **Perfect pairing**

$$\begin{aligned} T_p^*X \times T_pX &\longrightarrow k \\ (f + \mathfrak{m}_p^2, D) &\longmapsto D(f) \end{aligned}$$

Def $p \in X$ is a non-sing. point if $\dim_k(T_pX) = \dim(X)$.

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X irred. variety, $p \in X$.

$\mathcal{O}_{X,p} = \{f \in k(X) \mid f \text{ def. at } p\}$

$\mathfrak{m}_p \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$. $\quad T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$

Exer: $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) =$ min. # generators of ideal $\mathfrak{m}_p$.

Principal Ideal Theorem:

min. # gens of $\mathfrak{m}_p \geq \dim \mathcal{O}_{X,p}$ (Krull dim.)

Def $\mathcal{O}_{X,p}$ is a regular local ring if $\mathfrak{m}_p$ is gen. by $\dim(\mathcal{O}_{X,p})$ elts.

X irred. $\Rightarrow$ $\dim(X) = \dim(\mathcal{O}_{X,p})$.

$\therefore \dim_k(T_p^*X) \geq \dim(X)$.

Def $p \in X$ non-sing point $\Leftrightarrow$ $\mathcal{O}_{X,p}$ regular local

$\Leftrightarrow \dim_k(T_p^*X) = \dim(X)$.

Theorem $X_{sing} \subsetneq X$ proper closed subset.

Exer X affine, $p \in X$. $k[X] \longrightarrow \mathcal{O}_{X,p}$ k-alg. hom.

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) \xrightarrow{\cong} \mathrm{Der}_k(k[X], k(p))$$

$$I(p)/I(p)^2 \xrightarrow{\cong} \mathfrak{m}_p/\mathfrak{m}_p^2$$

$$f/g(p) + I(p)^2 \longleftarrow\!\shortmid \; f/g + \mathfrak{m}_p^2$$

$f, g \in k[X]$, $f(p) = 0$, $g(p) \neq 0$.

Differentiation:

$\phi : X \longrightarrow Y$ morphism. $\phi^* : \mathcal{O}_{Y,\phi(p)} \longrightarrow \mathcal{O}_{X,p}$

$$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y.$$
$$D \longmapsto D\phi^*$$

$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$: $d(\psi\phi)_p = d\psi_{\phi(p)} \circ d\phi_p$

Differentials

R com. ring. A com. R-algebra.

$\exists$ *universal R-derivation* $d_A : A \longrightarrow \Omega_{A/R}$:

For any R-derivation $D : A \longrightarrow M$
$\exists!$ A-linear map $\tilde{D} : \Omega_{A/R} \longrightarrow M$ s.t. $D = \tilde{D} \circ d_A$.

$$\begin{tikzcd}
A \arrow[rr, "D"] \arrow[dr, "d_A"'] & & M \\
& \Omega_{A/R} \arrow[ur, dashed, "\exists!\ \tilde{D}"] &
\end{tikzcd}$$

Construction:

$$\Omega_{A/R} = \left(\text{free } A\text{-module gen. by } \{d_A(b) : b \in A\}\right)$$
$$\Big/ \left\langle \begin{array}{l|l} d_A(a+b) = d_A(a) + d_A(b) & a, b \in A \\ d_A(ab) = a\, d_A(b) + b\, d_A(a) & r \in R \\ d_A(r) = 0 & \end{array} \right\rangle$$

X affine variety, $p \in X$.

<u>Notation</u>: M $k[X]$-module.

$$M(p) = M/I(p)M = M \otimes_{k[X]} k(p).$$

$$M \longrightarrow M(p), \quad m \mapsto m(p) = m + I(p)M.$$

<u>Exer</u> $\mathrm{Hom}_{k[X]}(M, k(p)) = \mathrm{Hom}_k(M(p), k)$

$$M \longrightarrow M(p) \longrightarrow k(p).$$

<u>Def</u>: $\Omega_X = \Omega_{k[X]/k} = \{\text{covector fields on } X\}$

$d = d_X : k[X] \longrightarrow \Omega_X$ universal k-derivation.

$$T_pX = \mathrm{Der}_k(k[X], k(p)) = \mathrm{Hom}_{k[X]}(\Omega_X, k(p))$$
$$= \mathrm{Hom}_k(\Omega_X(p), k) = \Omega_X(p)^*$$

$$\therefore \ \Omega_X(p) = T_p^*X = m_p/m_p^2$$
$$df(p) \longleftrightarrow f - f(p) + m_p^2.$$

<u>Exer</u> $k[\mathbb{A}^n] = k[T_1, \dots, T_n]$.

$\Omega_{\mathbb{A}^n} = \mathrm{Span}_{k[\mathbb{A}^n]}\{dT_1, \dots, dT_n\}$ (free!)

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i$$

$$T_p^*\mathbb{A}^n = \mathrm{Span}_k\{dT_1, \dots, dT_n\}$$

$$df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(p)\, dT_i$$

Exer $X \subseteq \mathbb{A}^n$ closed. $I(X) = \langle f_1, .., f_m \rangle \subseteq k[\mathbb{A}^n]$.

$t_i = \overline{T_i} \in k[X] = k[\mathbb{A}^n]/I(X)$.

$$\Omega_X = \mathrm{Span}_{k[X]}\{dt_1, .., dt_n\} \Big/ \langle \overline{df_1}, .., \overline{df_m} \rangle$$

$$= \left(\Omega_{\mathbb{A}^n} / \langle df_1, .., df_m \rangle \right) \otimes_{k[\mathbb{A}^n]} k[X].$$

Notation: $\overline{df} = \sum_{i=1}^{n} \overline{\frac{\partial f}{\partial T_i}}\, dt_i \in \mathrm{Span}_{k[X]}\{dt_1, .., dt_n\}$

$T_p^* X = T_p^* \mathbb{A}^n \Big/ \langle df_1(p), .., df_m(p) \rangle$

$T_p X = \langle df_1(p), .., df_m(p) \rangle^{\perp} \subseteq T_p \mathbb{A}^n$

Jacobi matrix: $J = \left(\frac{\partial f_i}{\partial T_j} \right) \in \mathrm{Mat}(n \times m, k[X])$

$$\begin{tikzcd} k[X]^{\oplus m} \arrow[r, "J"] & k[X]^{\oplus n} \arrow[r] & \Omega_X \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} k^{\oplus m} \arrow[r, "J(p)"] & k^{\oplus n} \arrow[r] & T_p^* X \arrow[r] & 0 \end{tikzcd}$$

$\therefore$ rank $J(p) \leq n - \dim(X)$

Equality $\Leftrightarrow$ $p \in X$ nonsing. point.

Cor $X_{sing} = \{ \mathrm{rank}(J) < \mathrm{codim}(X, \mathbb{A}^n) \} \subseteq X$ closed.

Vector fields: $\mathrm{Der}_k(k[X], k[X]) = \mathrm{Hom}_{k[X]}(\Omega_X, k[X])$

X non-singular $\Rightarrow$ Ω_X locally free $k[X]$-module

$\Rightarrow \mathrm{Hom}_{k[X]}(\Omega_X, k[X])(p) = \mathrm{Hom}_k(\Omega_X(p), k) = T_p X.$

Separable field extensions

E/F field extension. $(F \subseteq E)$ $p = \mathrm{char}(F)$.

Def: E/F is separably algebraic if

$\forall a \in E\ \exists f \in F[T] : f(a) = 0$ and f has no multible roots.

Note: $b \in E$ is a multiple root $\Leftrightarrow$ $f(b) = f'(b) = 0$.

$p = 0 \Rightarrow E/F$ always separable.

WLOG: $f \in F[T]$ irred.

$$f(b) = f'(b) = 0 \Rightarrow f'(T) = 0 \in F[T].$$

$$p = 0 \Rightarrow f'(T) \neq 0.$$

Def: Transcendence basis of E/F:

$B \subseteq E$ such that B is alg. indep. $/F$

and $E/F(B)$ is algebraic.

$\mathrm{tr.deg}_F(E) = \# B$

Def E/F is separably generated if $\exists$ tr. basis B

such that $E/F(B)$ is separably algebraic.

Def F is perfect if $p = 0$ or $\forall r \in F\ \exists s \in F : s^p = r$.

alg. closed $\Rightarrow$ perfect.

Theorem F perfect $\Rightarrow$ E/F separably generated.

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Rational functions

X irred. variety.

$k(X) = \{(U,f) \mid \emptyset \neq U \subseteq X,\ f: U \to k \text{ regular}\}/\sim$

$\quad = \{f : X \dashrightarrow k\}$ field of rat. fcns. on X.

X affine $\Rightarrow$ $k(X) = K(k[X])$ field of fractions.

$\phi : X \longrightarrow Y$ morphism of irred. varieties.

Def ϕ is dominant if $\overline{\phi(X)} = Y$.

Assume $\phi : X \longrightarrow Y$ dominant.

$\phi^* : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ is injective.

$\phi^* : k(Y) \longrightarrow k(X)$, $\phi^* f = f\phi : X \to Y \dashrightarrow k$

Def: ϕ is separable if $k(X)/k(Y)$ is separably generated.

Thm $\phi : X \longrightarrow Y$ morphism of irred. varieties.

(1) Assume $p \in X$ is non-sing, $\phi(p) \in Y$ is non-sing., and $d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y$ is surjective.
Then ϕ is dominant and separable.

(2) Assume ϕ is dominant and separable.
Then assumption of (1) holds for all points p in dense open $\subseteq X$.

Let G be a connected alg. group.

<u>Cor</u> Any homogeneous G-variety X is irred. and non-singular.

<u>Cor</u> $\phi: X \longrightarrow Y$ equivariant morphism of homogeneous G-varieties. TFAE:

(1) ϕ is separable.

(2) $d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y$ is surjective for some $p \in X$.

(3) $d\phi_p$ is surjective for all $p \in X$.

<u>Cor</u> $\phi: G \longrightarrow G'$ surjective homomorphism of alg. groups.
ϕ separable $\Leftrightarrow$ $d\phi_e$ surjective.

<u>Tangent spaces</u>

X affine variety, $p \in X$.

$k(p) = k[X]/I(p)$.

$d_X : k[X] \longrightarrow \Omega_X = \Omega_{k[X]/k}$.

$$\Omega_X \longrightarrow \Omega_X(p) = T_p^*X = I(p)/I(p)^2$$

$$d_Xf \longmapsto d_Xf(p) \longleftrightarrow f - f(p) + I(p)^2$$

$T_pX = \operatorname{Der}_k(k[X], k(p))$

Perfect pairing: $T_p^*X \times T_pX \longrightarrow k$

$$(f + I(p)^2, D) \longmapsto D(f)$$

Differentiation

$\phi : X \longrightarrow Y$ morphism of affine varieties, $p \in X$.

$$\begin{tikzcd}
k[Y] \arrow[r, "\phi^*"] \arrow[d, "d_Y"'] & k[X] \arrow[d, "d_X"] \\
\Omega_Y \arrow[r, "\phi^*"] \arrow[d] & \Omega_X \arrow[d] \\
T^*_{\phi(p)}Y \arrow[r, "\phi^*"] & T^*_p X
\end{tikzcd}$$

$$\phi^*(d_Y(f)) = d_X(\phi^* f)$$

$$\phi^*(f + I(\phi(p))^2) = \phi^*(f) + I(p)^2.$$

$$\phi^*(d_Y f(\phi(p))) = d_X(\phi^* f)(p)$$

$$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$$
$$D \longmapsto D\phi^*$$

$D \in T_p X$, $u \in T^*_{\phi(p)} Y \Rightarrow (u, d\phi_p D) = (\phi^* u, D)$

Products

Let $(p,q) \in X \times Y$

$D \in T_{(p,q)}(X \times Y) = \mathrm{Der}_k(k[X] \otimes k[Y], k(p,q))$

$D(f \otimes g) = g(q) D(f \otimes 1) + f(p) D(1 \otimes g)$.

$j_q : X \longrightarrow X \times Y$, $j_p : Y \longrightarrow X \times Y$

$T_{(p,q)}(X \times Y) = T_p X \oplus T_q Y = dj_q(T_p X) \oplus dj_p(T_q Y)$

Lemma G alg. group. $X, Y \in T_eG$.

$\mu : G \times G \longrightarrow G$ mult. $\quad i : G \longrightarrow G$ inverse.

$d\mu_{(e,e)} : T_eG \oplus T_eG \longrightarrow T_eG, \quad (X,Y) \longmapsto X+Y$

$di_e : T_eG \longrightarrow T_eG, \quad X \longmapsto -X$.

Proof

$$\begin{tikzcd} G \arrow[r, "j_1"] & G \times G \arrow[r, "\mu"] & G \\ x \arrow[r, mapsto] & (x,e) \arrow[r, mapsto] & x \end{tikzcd}$$

$$d\mu(X,0) = d\mu(dj_1(X)) = d(\mu j_1)(X) = X.$$

$$\begin{tikzcd} G \arrow[r, "\phi"] & G \times G \arrow[r, "\mu"] & G \\ x \arrow[r, mapsto] & (x, x^{-1}) \arrow[r, mapsto] & e \end{tikzcd}$$

$$\begin{tikzcd} T_eG \arrow[r, "d\phi"] & T_eG \oplus T_eG \arrow[r, "d\mu"] & T_eG \\ X \arrow[r, mapsto] & (X, di_e(X)) \arrow[r, mapsto] & X + di_e(X) = 0. \end{tikzcd}$$

□

Adjoint Representation

G LAG.

$$\hat{\lambda}, \hat{\rho} : G \longrightarrow \mathrm{Aut}_{var}(G)$$

$$\hat{\lambda}(x)(y) = xy, \quad \hat{\rho}(x)(y) = yx^{-1}.$$

$$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[G])$$

$$\lambda(x) = \hat{\lambda}(x^{-1})^*, \quad \rho(x) = \hat{\rho}(x^{-1})^*$$

$$(\lambda(x)f)(y) = f(x^{-1}y), \quad (\rho(x)f)(y) = f(yx).$$

<u>Note</u>: $\lambda(x)\rho(y) = \rho(y)\lambda(x) \quad \forall\, x, y \in G.$

$$\lambda(x) = \hat{\lambda}(x^{-1})^* : T^*_{x^{-1}y}G \longrightarrow T^*_yG$$

$$\rho(x) = \hat{\rho}(x^{-1})^* : T^*_{yx}G \longrightarrow T^*_yG$$

$$\lambda(x).df(x^{-1}y) = d(\lambda(x).f)(y).$$

$$\rho(x).df(yx) = d(\rho(x).f)(y).$$

$$\mathrm{Int} : G \longrightarrow \mathrm{Aut}(G)$$

$$\mathrm{Int}(x) = \hat{\lambda}(x)\hat{\rho}(x). \quad \mathrm{Int}(x)(y) = xyx^{-1}.$$

$$\mathrm{Int}(x)^* = \lambda(x^{-1})\rho(x^{-1}) : k[G] \longrightarrow k[G]$$

$$(\mathrm{Int}(x)^*.f)(y) = f(xyx^{-1}).$$

$$\mathrm{Ad} : G \longrightarrow GL(T_eG)$$

$$\mathrm{Ad}(x) = d\,\mathrm{Int}(x)_e$$

$$\mathrm{Ad}(x).X = X\,\mathrm{Int}(x)^* = X\,\lambda(x^{-1})\rho(x^{-1}).$$

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Dual adjoint representation

$$\mathrm{Ad}^*: G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[G])$$

$$\mathrm{Ad}^*(x) = \mathrm{Int}(x^{-1})^* = \lambda(x)\rho(x) : k[G] \longrightarrow k[G].$$

$$\mathrm{Ad}^*: G \longrightarrow GL(T_e^*G)$$

$$\mathrm{Ad}^*(x) = \mathrm{Int}(x^{-1})^* = \lambda(x)\rho(x) : T_e^*G \longrightarrow T_e^*G$$

$$\mathrm{Ad}^*(x).df(e) = \lambda(x).d(\rho(x).f)(x^{-1})$$

$$= d(\mathrm{Ad}^*(x).f)(e).$$

For $u \in T_e^*G$, $X \in T_eG$: $(\mathrm{Ad}^*(x).u, X) = (u, \mathrm{Ad}(x^{-1}).X)$

because $(\mathrm{Int}(x^{-1})^*u, X) = (u, d\mathrm{Int}(x^{-1})_e . X)$

Rationality

$\mu^2 : G\times G\times G \longrightarrow G$ mult.

$$(\mu^2)^*f = \sum_i f_i \otimes g_i \otimes h_i : \quad f(xyz) = \sum_i f_i(x)g_i(y)h_i(z)$$

$$(\mathrm{Ad}^*(x).f)(y) = f(\mathrm{Int}(x^{-1})(y)) = f(x^{-1}yx)$$

$$\mathrm{Ad}^*(x).f = \sum_i f_i(x^{-1})h_i(x)g_i$$

$$\mathrm{Ad}^*(x).df(e) = \sum_i f_i(x^{-1})h_i(x)\,dg_i(e).$$

$\therefore$ $\mathrm{Ad}^*: G \longrightarrow GL(T_e^*)$, $\mathrm{Ad}: G \longrightarrow GL(T_eG)$

are rational representations of G.

Lie algebra

G LAG.

$\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G]) = \{$ tangent vector fields on $G \}$

$D, D' \in \mathcal{D}_G \Rightarrow [D, D'] = DD' - D'D \in \mathcal{D}_G$

$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k[G]}(\mathcal{D}_G)$

$$\left.\begin{aligned} \lambda(x).D &= \lambda(x)\, D\, \lambda(x^{-1}) \\ \rho(x).D &= \rho(x)\, D\, \rho(x^{-1}) \end{aligned}\right\} \text{ translation of vector fields.}$$

$L(G) = \{ D \in \mathcal{D}_G \mid \lambda(x).D = D \ \forall x \in G \} \subseteq \mathcal{D}_G$ Lie subalg.

Note: $\rho(x).L(G) = L(G)$.

Def $X \in T_eG$, $f \in k[G]$, $y \in G$: $(\overline{X}f)(y) = X(\lambda(y^{-1}).f)$

Lemma $\overline{X} \in L(G)$

Proof

$\overline{X}f \in k[X]$: $\mu^*(f) = \sum g_i \otimes h_i$: $f(xy) = \sum g_i(x) h_i(y)$.

$\lambda(x^{-1}).f = \sum g_i(x) h_i \in k[G]$

$X(\lambda(x^{-1}).f) = \sum g_i(x) X(h_i)$ reg. fcn. of $x \in G$.

$\overline{X} \in \mathcal{D}_G$: $\overline{X}(fg) = f \cdot (\overline{X}g) + g \cdot (\overline{X}f)$

$\overline{X} \in L(G)$: $(\lambda(x)\, \overline{X}\, \lambda(x^{-1}).f)(y) = (\overline{X}\, \lambda(x^{-1}).f)(x^{-1}y)$

$= X(\lambda(y^{-1}x)\, \lambda(x^{-1}).f) = X(\lambda(y^{-1}).f) = (\overline{X}f)(y)$

□

Def $\alpha : D_G \longrightarrow T_eG$, $(\alpha D).f = (Df)(e)$

Prop $\alpha : L(G) \xrightarrow{\cong} T_eG$ iso. of vector spaces with inverse $X \mapsto \overline{X}$.

Proof

$\alpha(\overline{X}) = X$: $\alpha(\overline{X}).f = (\overline{X}f)(e) = X(\lambda(e^{-1}).f) = X(f)$.

$D \in L(G) \Rightarrow \overline{\alpha D} = D$:

$$(\overline{\alpha D}.f)(x) = (\alpha D)(\lambda(x^{-1}).f) = D(\lambda(x^{-1}).f)(e)$$

$$= (\lambda(x^{-1})D.f)(e) = Df(x).$$

□

Lemma $\alpha \circ \rho(y) \circ \alpha^{-1} = Ad(y) : T_eG \longrightarrow T_eG$

$$\begin{tikzcd} L(G) \arrow[r, "\alpha"] \arrow[loop below, "\rho(y)"] & T_eG \arrow[loop below, "Ad(y)"] \end{tikzcd}$$

Proof

$$(\alpha \circ \rho(y) \circ \alpha^{-1})(X)(f) = ((\rho(y).\overline{X}).f)(e)$$

$$= (\rho(y)\overline{X}\rho(y^{-1}).f)(e) = (\overline{X}\rho(y^{-1}).f)(y)$$

$$= X(\lambda(y^{-1})\rho(y^{-1}).f) = (Ad(y).X)(f).$$

□

Lie algebra of subgroup

G LAG, $H \subseteq G$ closed subgroup.

$k[H] = k[G]/I(H)$

$T_eH = \{X \in T_eG \mid X(I(H)) = 0\} \subseteq T_eG$

Def: $\mathcal{D}_{G,H} = \{D \in \mathcal{D}_G \mid D(I(H)) \subseteq I(H)\} \subseteq \mathcal{D}_G$ Lie subalg.

Lie algebra hom: $\phi : \mathcal{D}_{G,H} \longrightarrow \mathcal{D}_H$:

$D \in \mathcal{D}_{G,H}$: $D : k[G] \longrightarrow k[G]$ k-derivation,

$\phi D : k[H] \longrightarrow k[H]$, $(\phi D)(\bar{f}) = \overline{Df}$.

Lemma $\phi : \mathcal{D}_{G,H} \cap L(G) \xrightarrow{\cong} L(H)$ iso. of Lie algebras.

Proof

$$\begin{tikzcd}
\mathcal{D}_{G,H} \arrow[r, "\alpha_G"] \arrow[d, "\phi"'] & T_eG \\
\mathcal{D}_H \arrow[r, "\alpha_H"'] & T_eH \arrow[u, "\cup\!\text{l}"']
\end{tikzcd}$$

Note: $x \in H \Rightarrow$
$\lambda(x)(I(H)) \subseteq I(H)$.

$\phi(\mathcal{D}_{G,H} \cap L(G)) \subseteq L(H)$:

$\lambda(x)\,D = D\,\lambda(x)$: $k[G] \longrightarrow k[G]$ $\forall x \in G$

$\Rightarrow \lambda(x)\,\phi(D) = \phi(D)\,\lambda(x)$: $k[G]/I(H) \longrightarrow k[G]/I(H)$ $\forall x \in H$

$$\begin{tikzcd}
\mathcal{D}_{G,H} \cap L(G) \arrow[r, "\alpha_G", "\subseteq"'] \arrow[d, "\phi"', "\cap\!\text{l}"] & T_eG \\
L(H) \arrow[r, "\cong", "\alpha_H"'] & T_eH \arrow[u, "\cup\!\text{l}"']
\end{tikzcd}$$

Show: $X \in T_eH \Rightarrow \bar{X} \in \mathcal{D}_{G,H}$

$X \in T_eH$, $f \in I(H)$, $y \in H$:

$(\bar{X}f)(y) = X(\lambda(y^{-1}).f) = 0$ since $\lambda(y^{-1}).f \in I(H)$.

$\therefore$ $\bar{X}(I(H)) \subseteq I(H)$

□

Lie algebra homomorphism

$\phi : G \longrightarrow H$ homomorphism of LAGs.

$d\phi = d\phi_e : L(G) \longrightarrow L(H)$, $d\phi(\bar{X}) = \overline{d\phi_e(X)}$

Lemma

$D \in L(G) \Rightarrow D \circ \phi^* = \phi^* \circ d\phi(D) : k[H] \longrightarrow k[G]$

$$\begin{tikzcd}
k[H] \arrow[r, "\phi^*"] \arrow[d, "d\phi(D)"'] & k[G] \arrow[d, "D"] \\
k[H] \arrow[r, "\phi^*"] & k[G]
\end{tikzcd}$$

Proof

$X \in T_eG$, $f \in k[H]$, $y \in G$.

$(\bar{X} \circ \phi^*(f))(y) = X(\lambda(y^{-1}).\phi^*(f)) = X(\phi^*(\lambda(\phi(y)^{-1}).f))$

$= d\phi(X)(\lambda(\phi(y)^{-1}).f) = (\overline{d\phi(X)}.f)(\phi(y)) = (\phi^* \circ \overline{d\phi(X)}(f))(y)$

□

Prop $d\phi : L(G) \longrightarrow L(H)$ is a Lie alg. hom.

Proof

$$\begin{tikzcd}
G \arrow[r, "\phi"] & \phi(G) \arrow[r, "\subseteq"] & H
\end{tikzcd}$$

$$\begin{tikzcd}
L(G) \arrow[r, "d\phi"] & L(\phi(G)) \arrow[r, "\subseteq"] & L(H)
\end{tikzcd}$$

Lie subalg.

WLOG: ϕ surjective $\Rightarrow \phi^* : k[H] \longrightarrow k[G]$ injective.

$\phi^* \circ d\phi([D, D']) = [D, D'] \circ \phi^* = (DD' - D'D) \circ \phi^*$

$= \phi^* \circ (d\phi(D) d\phi(D') - d\phi(D') d\phi(D)) = \phi^* \circ [d\phi(D), d\phi(D')]$

$\Rightarrow d\phi([D, D']) = [d\phi(D), d\phi(D')]$

□

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Lie algebra of $GL(V)$

E vector space $/k$, $\dim_k(E) < \infty$.

As variety: $k[E] = \mathrm{Sym}^\bullet(E^*)$

$E^* \subseteq k[E]$ linear fcns.

$p \in E$: $T_pE = \mathrm{Der}_k(k[E], k(p)) = \mathrm{Hom}_k(E^*, k) = E$

$X \in T_pE$, $f \in E^* \subseteq k[E]$: $X(f) = (f, X)$

Assume $E = \mathrm{End}_k(V)$, $\dim(V) < \infty$.

Perfect pairing: $E \times E \longrightarrow k$, $(A, B) \mapsto tr(AB)$

For $A \in E$, def. $f_A \in E^*$ by $(f_A, B) = tr(AB)$.

$GL(V) \subseteq \mathrm{End}(V)$ LAG.

$\mathfrak{gl}(V) = \mathrm{End}(V)$ Lie algebra: $[X, Y] = XY - YX$.

$T_e GL(V) = T_e E = \mathfrak{gl}(V)$

Prop $\mathfrak{gl}(V) \xrightarrow{\cong} L(GL(V))$, $X \mapsto \overline{X}$ iso. of Lie algebras.

Proof

$X, A \in E$, $B, C \in GL(V)$.

$\lambda(B^{-1}).f_A = f_{AB}$: $(\lambda(B^{-1}).f_A)(C) = f_A(BC) = tr(ABC) = f_{AB}(C)$

$\overline{X} f_A = f_{XA}$: $\overline{X} f_A(B) = X(\lambda(B^{-1}).f_A) = X(f_{AB}) = tr(XAB) = f_{XA}(B)$

$[\overline{X}, \overline{Y}].f_A = (\overline{X}\,\overline{Y} - \overline{Y}\,\overline{X}).f_A = f_{XYA} - f_{YXA} = \overline{[X, Y]}.f_A$.

$\square$

Lie algebra of LAG

Note: $\phi : V \longrightarrow W$ k-linear map, V, W finite dim.
Then $d\phi_v = \phi$ for all $v \in V$.

$$\begin{tikzcd}
T_v V \arrow[r, phantom, "="] \arrow[d, "d\phi_v"] & \mathrm{Der}_k(k[V], k(v)) \arrow[r, phantom, "="] \arrow[d, "d\phi_v"] & \mathrm{Hom}_k(V^*, k) \arrow[r, phantom, "="] \arrow[d, "\phi^{**}"] & V \arrow[d, "\phi"] \\
T_{\phi(v)} W \arrow[r, phantom, "="] & \mathrm{Der}_k(k[W], k(\phi(v))) \arrow[r, phantom, "="] & \mathrm{Hom}_k(W^*, k) \arrow[r, phantom, "="] & W
\end{tikzcd}$$

G LAG, $\nu : G \longrightarrow GL(V)$ rat. rep.
$d\nu : L(G) \longrightarrow \mathfrak{gl}(V)$ Lie algebra homomorphism.

Lemma $\phi : \mathrm{End}(V) \longrightarrow k$ k-linear map, $X \in T_e G$.
Then $\phi(d\nu(X)) = X(\nu^*(\phi))$.

Proof
$\phi \circ \nu : G \longrightarrow \mathbb{A}^1$ morphism, $T_{\phi(e)} \mathbb{A}^1 = k$.
$\phi(d\nu(X)) = d\phi_e(d\nu(X)) = d(\phi\circ\nu)_e(X) = X(\phi \circ \nu)$.
$\square$

Assume V has basis $\{v_1, \dots, v_n\}$.
$GL(V) = GL_n$, $\mathfrak{gl}(V) = \mathfrak{gl}_n = \mathrm{Mat}(n \times n, k)$.
$A \in \mathrm{End}(V)$: $A = (a_{ij})$, $A.v_j = \sum_i a_{ij} v_i$
$k[\mathrm{End}(V)] = k[T_{ij}]$: $T_{ij}(A) = a_{ij}$.
$\nu : G \longrightarrow GL_n$, $\nu(g) = (\nu_{ij}(g))$. $\nu_{ij} = \nu^*(T_{ij}) \in k[G]$.
$X \in T_e G$. $d\nu(X) = (b_{ij}) \in \mathfrak{gl}_n$.
$b_{ij} = T_{ij}(d\nu(X)) = X(\nu^*(T_{ij})) = X(\nu_{ij})$.

Prop G LAG, $V \subseteq k[G]$, $\dim(V) < \infty$, $\rho(x).V = V \;\forall\, x \in G$.

$\rho : G \longrightarrow GL(V)$ rat. rep., $d\rho : T_eG \longrightarrow \mathrm{End}(V)$.

Then $\overline{X}f = d\rho(X).f \;\;\forall\, X \in T_eG,\; f \in V$.

Proof

Fix $g \in G$, $f \in V$. Def. $\phi : \mathrm{End}(V) \longrightarrow k$, $\phi(Y) = (Y.f)(g)$.

$\lambda(g^{-1}).f = \rho^* \phi \in k[G]$:

$$(\lambda(g^{-1}).f)(x) = f(gx) = (\rho(x).f)(g) = \phi(\rho(x)).$$

$$(\overline{X}f)(g) = X(\lambda(g^{-1}).f) = X(\rho^*\phi) = \phi(d\rho(X)) = (d\rho(X).f)(g).$$

□

Cor $\overline{X} : k[G] \longrightarrow k[G]$ is locally finite $\forall\, X \in T_eG$.

Exer: G LAG. $\mathrm{Ad} : G \longrightarrow GL(T_eG)$, $d\mathrm{Ad} : T_eG \longrightarrow \mathrm{End}(T_eG)$.

$d\mathrm{Ad}(X)(Y) = [X, Y] \;\;\forall\, X, Y \in T_eG$.

Exer: $\nu : G \longrightarrow GL(V)$ rat. rep.

$\Lambda^n \nu : G \longrightarrow GL(\Lambda^n V)$, $d(\Lambda^n \nu) : T_eG \longrightarrow \mathrm{End}(\Lambda^n V)$.

$$d(\Lambda^n \nu)(X).(v_1 \wedge \dots \wedge v_n) = \sum_{i=1}^{n} v_1 \wedge \dots \wedge d\nu(X).v_i \wedge \dots \wedge v_n.$$

Jordan decomp in $L(G)$

G LAG, $X \in T_eG$.

$\overline{X} : k[G] \longrightarrow k[G]$ locally finite.

$\overline{X} = \overline{X}_s + \overline{X}_n$, $\overline{X}_s$ semi-simple, $\overline{X}_n$ nilpotent, $\overline{X}_s \overline{X}_n = \overline{X}_n \overline{X}_s$.

Thm (1) $\overline{X}_s, \overline{X}_n \in L(G)$ and $[\overline{X}_s, \overline{X}_n] = 0$.

(2) $\phi : G \longrightarrow G'$ hom. of LAGs $\Rightarrow$

$d\phi(X_s) = d\phi(X)_s$, $d\phi(X_n) = d\phi(X)_n$

(3) $G = GL_n \Rightarrow X = X_s + X_n$ is usual Jordan decomp. in M_n.

Fibers of morphisms

$\phi : X \longrightarrow Y$ dominant, X, Y irred. affine.

$\phi^* : k[Y] \subseteq k[X]$, $k(Y) \subseteq k(X)$.

$k[X] = k[Y][f_1, \dots, f_m] = k[Y][T_1, \dots, T_m]/I$

$X \cong Z(I) \subseteq Y \times \mathbb{A}^m$ closed subvariety.

$x \longmapsto (\phi(x), f_1(x), \dots, f_m(x))$

WLOG: $\{f_1, \dots, f_r\}$ transcendence basis of $k(X)/k(Y)$.

$r = \dim X - \dim Y$ *relative dimension*.

$k[Y] \subseteq k[Y][f_1, \dots, f_r] \subseteq k[X]$

$$\begin{tikzcd}
Y & Y \times \mathbb{A}^r \arrow[l] & X \arrow[l, "\text{gen. finite}"'] \\
\phi(x) & (\phi(x), f_1(x), \dots, f_r(x)) \arrow[l, mapsto] & x \arrow[l, mapsto]
\end{tikzcd}$$

Fact: $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq r$.

Def Assume $\phi : X \longrightarrow Y$ dominant.

ϕ is *generically finite*: $k(X)/k(Y)$ finite ext.

ϕ is *finite*: $k[X]$ f.g. $k[Y]$-module.

Finite $\Rightarrow$ generically finite.

Assume $\phi : X \longrightarrow Y$ gen. finite, $k[X] = k[Y][f]$.
$f \in k(X)$ algebraic over $k(Y)$.

$$f^d + a_{d-1} f^{d-1} + \dots + a_1 f + a_0 = 0, \quad a_i \in k(Y)$$

$d = [k(X) : k(Y)]$

Choose $0 \neq h \in k[Y]$ s.t. $a_i \in k[Y]_h \ \forall i$.
$Y_h = \{ y \in Y \mid h(y) \neq 0 \} \subseteq Y$ open affine.
$k[Y_h] = k[Y]_h$

$k[X_h] \cong k[Y_h][T] / \langle T^d + \dots + a_1 T + a_0 \rangle$
free $k[Y_h]$-module gen. by $\{1, T, \dots, T^{d-1}\}$.

$$\begin{tikzcd} X \arrow[r, "\text{gen. finite}"] & Y \\ X_h \arrow[u, phantom, "\cup"] \arrow[r, "\text{finite}"] & Y_h \arrow[u, phantom, "\cup \ \text{open}"] \end{tikzcd}$$

Note: $\phi : X_h \longrightarrow Y_h$ surjective with finite fibers:
$X_h \cong \{ (y, t) \in Y_h \times \mathbb{A}^1 \mid t^d + \dots + a_1(y) t + a_0(y) = 0 \}$.
$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d + \dots + a_1(y) t + a_0(y) = 0 \}$

Assume $f \in k(X)$ separable over $k(Y)$.
$T^d + \dots + a_1 T + a_0$ has d distinct roots in $\overline{k(Y)}$.
$\Rightarrow \forall y \in$ dense open $\subseteq Y_h$:
$T^d + \dots + a_1(y) T + a_0(y)$ has d distinct roots in k.

Assume $f \in k(X)$ purely inseparable over $k(Y)$:
$d = p^j, \quad f^d = a \in k(Y)$.
$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d = a(y) \} = \{ \sqrt[d]{a(y)} \}$.

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Thm

$\phi: X \longrightarrow Y$ dominant of irred. varieties.

$r = \dim X - \dim Y$.

$\exists$ dense open $U \subseteq X$ such that:

(1) $\phi \times 1_Z : U \times Z \longrightarrow Y \times Z$ is an open morphism $\forall Z$.

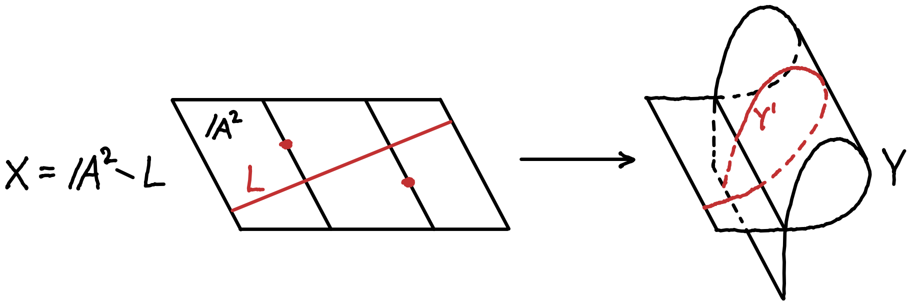
(2) $Y' \subseteq Y$ irred. closed, $X' \subseteq \phi^{-1}(Y')$ irred. comp.,
$X' \cap U \neq \emptyset \Rightarrow \dim(X') = \dim(Y') + r$.

(3) Assume $\dim X = \dim Y$.
$\forall y \in \phi(U)$: $\# \phi^{-1}(y) = [k(X)_s : k(Y)]$
$k(X)_s = \{ f \in k(X) \mid f \text{ separable } / k(Y) \}$.

Caution: $\phi: X \longrightarrow Y$ dominant, $r = \dim(X) - \dim(Y)$.

True: $y \in Y$ point, $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq r$.

False: $Y' \subseteq Y$ closed, irred, $\phi^{-1}(Y') \neq \emptyset \Rightarrow \dim \phi^{-1}(Y') \geq \dim Y' + r$



Integral extensions

A ring, B A-algebra.

$b \in B$ is <u>integral</u> over A if $\exists\ b^n + a_1 b^{n-1} + \cdots + a_n = 0$, $a_i \in A$.

B <u>integral</u> over A $\Leftrightarrow$ All elts. integral over A.

B finite over A $\Leftrightarrow$ B f.g. as A-module.

<u>Exer</u>: B finite $/A$ $\Leftrightarrow$ B integral $/A$ & f.g. as A-algebra.

$\bar{A} = \{b \in B \mid b \text{ integral } /A\} \subseteq B$ subalgebra.

<u>Def</u> A domain A is <u>normal</u> if $A = \bar{A} \subseteq K(A)$.

$\phi : X \longrightarrow Y$ morphism, X, Y affine.

ϕ is <u>finite</u> $\Leftrightarrow$ $k[X]$ is finite over $k[Y]$.

<u>Fact</u>: ϕ finite $\Leftrightarrow$ ϕ is proper with finite fibers

$\Rightarrow$ ϕ is closed with finite fibers.

Y is normal if $k[Y]$ is normal.

non-singular $\Rightarrow$ normal.

<u>Note</u>: Assume X, Y irred. affine, Y normal,

$\phi : X \longrightarrow Y$ finite, birat.

Then $\phi : X \xrightarrow{\cong} Y$ isomorphism.

$k[Y] \subseteq k[X] \subseteq \overline{k[Y]} \subseteq k(Y) = k(X)$.

Zariski's Main Theorem

$\phi : X \longrightarrow Y$ morphism of irred. varieties.

Assume ϕ is bijective and birational, Y normal.

Then ϕ is an isomorphism.

Thm G alg. group. X, Y homogeneous G-varieties.

$\phi: X \longrightarrow Y$ equivariant. $r = \dim(X) - \dim(Y)$.

(a) $\forall Z: \phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ is open.

(b) $Y' \subseteq Y$ closed, irred., $X' \subseteq \phi^{-1}(Y')$ irred. comp.

$\Rightarrow \dim(X') = \dim(Y') + r$.

(c) ϕ isomorphism $\Leftrightarrow$ ϕ bijective and $\exists p \in X$:

$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$ bijective.

Proof

WLOG G connected, X, Y irred.

(a)+(b) true for $\phi: U \to Y$, $U \subseteq X$ dense open.

Translate.

(c): $d\phi_p$ surjective $\Rightarrow$ ϕ separable.

ϕ bijective $\Rightarrow$ ϕ birational.

$\phi: U \xrightarrow{\cong} \phi(U)$ for $U \subseteq X$ dense open.

Translate.

□

Cor $\phi: G \longrightarrow G'$ surjective hom. of alg. groups.

(a) $\dim(G) = \dim(G') + \dim \operatorname{Ker}(\phi)$.

(b) ϕ isomorphism $\Leftrightarrow$ ϕ and $d\phi$ are bijective.

Semi-simple automorphisms

G connected LAG. $\mathfrak{g} = L(G) = T_eG$.

$\sigma : G \xrightarrow{\cong} G$ automorphism.

$G_\sigma = \{x \in G \mid \sigma(x) = x\} \subseteq G$ closed subgroup.

$\mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\} \subseteq \mathfrak{g}$ Lie subalg.

Def $\chi : G \longrightarrow G$, $\chi(x) = \sigma(x)x^{-1}$

$G_\sigma = \chi^{-1}(e)$.

$$\chi : G \xrightarrow{(\sigma, i)} G \times G \xrightarrow{\mu} G$$

$$d\chi_e : T_eG \longrightarrow T_eG \oplus T_eG \longrightarrow T_eG$$

$$X \longmapsto (d\sigma(X), -X) \longmapsto d\sigma(X) - X.$$

$L(G_\sigma) \subseteq \operatorname{Ker}(d\chi_e) = \mathfrak{g}_\sigma$

$G \circlearrowright G : g.x = gx$ $\qquad$ $G \circlearrowright G : g \bullet x = \sigma(g)xg^{-1}$.

$\chi : (G, .) \longrightarrow (G, \bullet)$ equivariant morphism.

$\chi(G) = G \bullet e$ is an orbit for $\bullet$ action.

Note: $e \in \overline{\chi(G)}$ non-singular point.

$\chi : G \longrightarrow \overline{\chi(G)}$ separable $\Leftrightarrow$ $d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Lemma $L(G_\sigma) = \mathfrak{g}_\sigma \Leftrightarrow d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Proof

$$\dim d\chi_e(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{g}_\sigma$$

$$\leq \dim \mathfrak{g} - \dim L(G_\sigma) = \dim G - \dim G_\sigma = \dim \overline{\chi(G)}.$$

□

Def $\sigma : G \xrightarrow{\cong} G$ is semi-simple

$\Leftrightarrow$ $\sigma^* : k[G] \longrightarrow k[G]$ is semi-simple.

Lemma $\sigma : G \xrightarrow{\cong} G$ semi-simple

$\Leftrightarrow$ $\exists\, G \subseteq GL_n$, $s \in GL_n$ semi-simple:

$$\sigma(x) = sxs^{-1} \quad \forall x \in G.$$

Proof $(\Rightarrow)$:

$k[G] = k[f_1, \dots, f_n]$

$\exists\, \mathrm{Span}_k\{f_1, \dots, f_n\} \subseteq V' \subseteq k[G]$:

$$\dim(V') < \infty, \quad \sigma^*(V') = V'.$$

$$V = \sum_{x \in G} \rho(x).V' \subseteq k[G]$$

$$\dim(V) < \infty, \quad \rho(x).V = V \quad \forall x \in G.$$

$\sigma^* \rho(x).f = \rho(\sigma^{-1}(x))\, \sigma^*.f \quad \forall f \in k[G]$:

$$(\sigma^* \rho(x).f)(y) = \rho(x).f(\sigma(y)) = f(\sigma(y)x) = f(\sigma(y\sigma^{-1}(x)))$$

$$= (\sigma^* f)(y\sigma^{-1}(x)) = (\rho(\sigma^{-1}(x))\,\sigma^*.f)(y).$$

$\sigma^*.\, \rho(x).V' = \rho(\sigma^{-1}(x)).V'$

$\therefore\ \sigma^*.V = V.$

$\rho : G \subseteq GL(V)$ closed.

$\sigma^* \rho(x)\, (\sigma^*)^{-1} = \rho(\sigma^{-1}(x))$

$s = (\sigma^*)^{-1} \in GL(V)$ semi-simple.

$\sigma(x) = sxs^{-1}.$

□

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G connected LAG.

Thm Let $\sigma : G \xrightarrow{\cong} G$ be semi-simple.

(1) $\chi(G) \subseteq G$ is closed.

(2) $d\chi_e : T_e G \twoheadrightarrow T_e \chi(G)$ is surjective.

Proof

WLOG $G \subseteq GL(V)$ closed, $s \in GL(V)$ ss.

$\sigma : GL(V) \longrightarrow GL(V)$, $\sigma(x) = sxs^{-1}$.

$\sigma : End(V) \longrightarrow End(V)$ linear extension.

$d\sigma = \sigma = Ad(s) \in GL(\mathfrak{gl}(V))$.

$G_\sigma = \{ x \in G \mid sxs^{-1} = x \}$

$\mathfrak{g}_\sigma = \{ X \in \mathfrak{g} \mid sXs^{-1} = X \} \subseteq \mathfrak{g} = L(G) \subseteq \mathfrak{gl}(V)$.

$\chi : GL(V) \longrightarrow GL(V)$, $\chi(x) = \sigma(x) x^{-1} = sxs^{-1}x^{-1}$.

Case $G = GL(V)$: $GL(V)_\sigma = \mathfrak{gl}(V)_\sigma \cap GL(V) \Rightarrow T_e(GL(V)_\sigma) = \mathfrak{gl}(V)_\sigma$

$\Rightarrow d\chi_e : T_e GL(V) \twoheadrightarrow T_e \overline{\chi(GL(V))}$ surjective.

Let $X \in T_e \overline{\chi(G)} \subseteq T_e \overline{\chi(GL(V))}$

$\exists\, Y \in \mathfrak{gl}(V) :\ X = d\chi_e(Y) = d\sigma(Y) - Y$.

$\sigma(G) = G \Rightarrow d\sigma(\mathfrak{g}) \subseteq \mathfrak{g}$

s semi-simple $\Rightarrow d\sigma \in GL(\mathfrak{gl}(V))$ semi-simple

$\Rightarrow \exists\, \mathfrak{h} \subseteq \mathfrak{gl}(V)$ $d\sigma$-stable, $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{h}$.

$Y = Y' \oplus Y'' \in \mathfrak{g} \oplus \mathfrak{h}$

$X = d\sigma(Y') - Y' = d\chi_e(Y')$.

$\therefore\ d\chi_e : T_e G \twoheadrightarrow T_e \overline{\chi(G)}$ surjective.

Show: $\mathcal{X}(G) \subseteq G$ closed.

Def $m(T) = \prod\limits_{\substack{a \text{ eigenval.} \\ \text{of } s^{-1}}} (T-a) \in k[T]$.

$$S = \left\{ y \in GL(V) \;\middle|\; \begin{array}{l} \text{(a) } yGy^{-1} = G \\ \text{(b) } m(y) = 0 \in End(V) \\ \text{(c) } ch.poly(Ad(y)|\mathfrak{g}) = ch.poly(Ad(s^{-1})|\mathfrak{g}) \end{array} \right\}$$

$S \subseteq GL(V)$ closed, $s^{-1} \in S$, all elts. of S are semi-simple.

$Y \in S$: $\quad G_Y = \{x \in G \mid YxY^{-1} = x\}$

$\qquad\qquad \mathfrak{g}_Y = \{X \in \mathfrak{g} \mid yXy^{-1} = X\}$

$$dim(G_Y) \underset{\substack{\uparrow \\ \sigma_Y : G \longrightarrow G \\ x \mapsto YxY^{-1}}}{=} dim(\mathfrak{g}_Y) \underset{\substack{\uparrow \\ (c)}}{=} dim(\mathfrak{g}_\sigma) = dim(G_\sigma)$$

$G \circlearrowright S$, $g.Y = gYg^{-1}$.

$\phi_Y : G \longrightarrow G.Y$, $\phi_Y(g) = gYg^{-1}$

$\phi_Y^{-1}(Y) = G_Y \Rightarrow dim(G.Y) = dim(G) - dim(G_Y)$

All orbits have same dimension

$\Rightarrow$ all orbits are closed.

$\therefore \mathcal{X}(G) = s(G.s^{-1}) \subseteq G$ is closed.

$\square$

$Z_G(s) = \{x \in G \mid xs = sx\} \subseteq G$ centralizer of $s \in G$.

<u>Cor</u> $s \in G$ semi-simple.

(1) $C = \{xsx^{-1} \mid x \in G\} \subseteq G$ closed.

(2) $G \longrightarrow C$, $x \mapsto xsx^{-1}$ is separable.

(3) $\mathfrak{g} = (Ad(s)-1)\mathfrak{g} \oplus L(Z_G(s))$

<u>Proof</u>

$\sigma : G \longrightarrow G$, $\sigma(x) = s^{-1}xs$ semi-simple automorphism.

$\chi : G \longrightarrow G$, $\chi(x) = \sigma(x)x^{-1} = s^{-1}xsx^{-1}$.

$\chi(G) \subseteq G$ closed, $\chi : G \longrightarrow \chi(G)$ separable.

$C = s\,\chi(G)$ closed, $x \mapsto xsx^{-1} = s\,\chi(x)$ separable.

$G_\sigma = \{s^{-1}xs = x\} = Z_G(s)$.

$$\begin{aligned} L(Z_G(s)) &= L(G_\sigma) = \mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\} \\ &= \{X \in T_eG \mid s^{-1}Xs = X\} = \{X \in T_eG \mid sXs^{-1} = X\} \\ &= \operatorname{Ker}(Ad(s)-1) \subseteq \mathfrak{g}. \end{aligned}$$

s semi-simple $\Rightarrow$ $Ad(s)-1$ semi-simple

$\Rightarrow \mathfrak{g} = \operatorname{Im}(Ad(s)-1) \oplus \operatorname{Ker}(Ad(s)-1)$

□

Action by automorphisms

D diagonalizable LAG, G connected LAG.

$D \curvearrowright G$ by automorphisms:

- G D-variety.
- $G \xrightarrow{\cong} G$, $g \mapsto d.g$ group hom. $\forall d \in D$.

Differentiate: $T_eG \longrightarrow T_eG$, $X \mapsto d.X$

$\alpha : D \longrightarrow GL(k[G])$ locally rational rep.

$(\alpha(d).f)(x) = f(d^{-1}.x)$.

D diagonalizable $\Rightarrow$ $\alpha(d) : k[G] \longrightarrow k[G]$ semi-simple

$\Rightarrow$ $g \mapsto d.g$ semi-simple automorphism.

Def $Z_G(D) = \{ g \in G \mid d.g = g \;\; \forall d \in D \} = \bigcap_{d \in D} G_d$

$Z_{\mathfrak{g}}(D) = \{ X \in \mathfrak{g} \mid d.X = X \;\; \forall d \in D \} = \bigcap_{d \in D} \mathfrak{g}_d$

Note: $L(G_d) = \mathfrak{g}_d$, $L(Z_G(D)) \subseteq Z_{\mathfrak{g}}(D)$.

Cor $L(Z_G(D)) = Z_{\mathfrak{g}}(D)$

Proof

If $D \curvearrowright \mathfrak{g}$ trivial: $L(G_d) = \mathfrak{g}_d = \mathfrak{g} \Rightarrow G_d = G$.

$Z_G(D) = G$, $Z_{\mathfrak{g}}(D) = \mathfrak{g}$.

Otherwise choose $d \in D$ such that $\mathfrak{g}_d \subsetneq \mathfrak{g}$.

D commutative $\Rightarrow$ D acts on G_d, G_d°.

$Z_G(D) = Z_{G_d}(D) \geq Z_{G_d^\circ}(D)$, $Z_{\mathfrak{g}}(D) = Z_{\mathfrak{g}_d}(D)$.

Induction on $\dim(G)$ $\Rightarrow$

$\dim Z_{\mathfrak{g}_d}(D) = \dim Z_{G_d^\circ}(D) = \dim Z_{G_d}(D)$

□

$G_s = \{x \in G \mid x \text{ semi-simple}\}$

<u>Commutator</u>: $(x,y) = xyx^{-1}y^{-1}$.

$G \neq e$ <u>nilpotent</u> $\Leftrightarrow$ $Z(G) \neq e$ and $G/Z(G)$ nilpotent

$\Leftrightarrow \exists\, n \in \mathbb{N} : \forall x_1, \dots, x_n \in G : (x_1, (x_2, (\cdots (x_{n-1}, x_n) \cdots))) = e$.

<u>Cor</u> G connected nilpotent LAG

$\Rightarrow G_s \subseteq Z(G)$ subgroup.

<u>Proof</u>

$s \in G$ semi-simple.

$\sigma = \mathrm{Int}(s) : G \xrightarrow{\cong} G$

$\chi(x) = \sigma(x) x^{-1} = sxs^{-1}x^{-1} = (s, x)$.

$\chi^n(x) = (s, (s, (\cdots, (s,x) \cdots))) = e$.

$\chi^n(G) = e$.

$d\chi_e = Ad(s) - 1$.

$(Ad(s)-1)^n = (d\chi_e)^n = 0$.

s semi-simple $\Rightarrow Ad(s)-1$ ss.

$\therefore Ad(s) = 1$.

$L(G_\sigma) = \mathfrak{g}_\sigma = \mathrm{Ker}(Ad(s)-1) = \mathfrak{g}$

$\Rightarrow G_\sigma = G \Rightarrow \sigma$ trivial $\Rightarrow s \in Z(G)$.

product of commuting ss is ss $\Rightarrow G_s \subseteq Z(G)$ subgroup.

$\square$

Ideal of a closed subgroup

G LAG, $H \subseteq G$ closed subgroup.
$I(H) \subseteq k[G]$ ideal of H.

Lemma $H = \{ g \in G \mid \rho(g).I(H) = I(H) \}$

Proof

$\subseteq$: $g \in H$, $f \in I(H)$, $h \in H \Rightarrow (\rho(g).f)(h) = f(hg) = 0$

$\supseteq$: $g \in \mathrm{RHS}$, $f \in I(H) \Rightarrow f(g) = (\rho(g).f)(e) = 0$.

□

Lemma $T_eH = \{ X \in T_eG \mid \overline{X}.I(H) \subseteq I(H) \}$

Proof

$\mathcal{D}_{G,H} = \{ D \in \mathrm{Der}_k(k[G], k[G]) \mid D.I(H) \subseteq I(H) \}$

$$\begin{tikzcd}
L(G) \cap \mathcal{D}_{G,H} \arrow[r, "\alpha_G"] \arrow[d, "\cong"'] & T_eG \\
L(H) \arrow[r, "\alpha_H", "\cong"'] & T_eH \arrow[u, "\rotatebox{90}{$\subseteq$}"']
\end{tikzcd}$$

Let $X \in T_eG$.

$X \in T_eH \Leftrightarrow \overline{X} \in \mathcal{D}_{G,H}$.

□

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G LAG, $H \subseteq G$ closed subgroup. $\mathfrak{g} = L(G)$, $\mathfrak{h} = L(H)$.

Lemma $\exists\ W \subseteq V \subseteq k[G]$:

(1) $\rho : G \longrightarrow GL(V)$ rational rep.

(2) $H = \{ g \in G \mid \rho(g).W = W \}$

(3) $\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\rho(X).W \subseteq W \}$

Proof

$I(H) = \langle f_1, \dots, f_r \rangle \subseteq k[G]$.

$\exists\ \mathrm{Span}_k \{ f_1, \dots, f_r \} \subseteq V \subset k[G]$:

$\rho : G \longrightarrow GL(V)$ rational rep.

$W = V \cap I(H)$.

$g \in G$: $g \in H \Leftrightarrow \rho(g).I(H) = I(H) \Leftrightarrow \rho(g).W = W$.

$X \in \mathfrak{g}$: $X \in \mathfrak{h} \Leftrightarrow \bar{X}.I(H) \subseteq I(H) \Leftrightarrow \bar{X}.W \subseteq W$.

$\square$

Thm $\exists$ rational rep. $\phi: G \longrightarrow GL(U)$, $0 \neq u \in U$:
$H = \{g \in G \mid \phi(g).u \in ku\}$ and
$\mathfrak{h} = \{X \in \mathfrak{g} \mid d\phi(X).u \in ku\}$.

Proof

Let $W \subseteq V \subseteq k[G]$ be as in lemma, $d = \dim(W)$.

$U = \Lambda^d V$, $0 \neq u \in \Lambda^d W \subseteq U$.

$W = \{v \in V \mid v \wedge u = 0 \in \Lambda^{d+1} V\}$ determined by u.

$\phi = \Lambda^d \rho: G \longrightarrow GL(U)$.

$x \in G$: $x \in H \Leftrightarrow \rho(x).W = W \Leftrightarrow \phi(x).u \in ku$.

$u = w_1 \wedge \cdots \wedge w_d$, $\{w_1, \dots, w_d\}$ basis of W.

$$d\phi(X).u = \sum_{i=1}^{d} w_1 \wedge \dots \wedge d\rho(X).w_i \wedge \cdots \wedge w_d$$

If $d\rho(X).W \nsubseteq W$: $d\rho(X).w_j = w + v$, $w \in W$, $v \notin W$.
$d\phi(X).u$ "contains" $w_1 \wedge \cdots \wedge w_{j-1} \wedge v \wedge w_{j+1} \wedge \cdots \wedge w_d$.

$X \in \mathfrak{h} \Leftrightarrow d\rho(X).W \subseteq W \Leftrightarrow d\phi(X).u \in ku$.
$\square$

Def $\phi: X \longrightarrow Y$ morphism of varieties.
ϕ is <u>separable</u> if $\forall$ conn. comp. $X' \subseteq X$:
X' is irred., $\overline{\phi(X')}$ is conn. comp. of Y,
$k(\overline{\phi(X')}) \subseteq k(X')$ separably generated.

Cor $\exists$ quasi-projective hom. G-variety X, $x \in X$:

(1) $H = G_x = \{g \in G \mid g.x = x\}$

(2) $\psi : G \longrightarrow X$, $g \mapsto g.x$ separable.

Proof

Let $\phi : G \to GL(U)$, $0 \neq u \in U$ be as in Theorem.

$\mathbb{P}(U) = \{[v] = kv \mid 0 \neq v \in U\}$

$G \circlearrowright \mathbb{P}(U)$, $g.[v] = [\phi(g).v]$

$x = [u]$, $X = G.x \subseteq \mathbb{P}(U)$. $H = G_x$ is clear.

$$\begin{tikzcd}
\psi : G \arrow[r, "\phi"] & GL(U) \arrow[r, "A \mapsto A.u"] \arrow[d, phantom, "\cap"] & U \smallsetminus \{0\} \arrow[r, "\pi"] \arrow[d, phantom, "\cap"] & \mathbb{P}(U) \\
 & End(U) \arrow[r, "\text{linear}"] & U &
\end{tikzcd}$$

$$\begin{tikzcd}
d\psi_e : T_eG \arrow[r, "d\phi"] & End(U) \arrow[r, "A \mapsto A.u"] & U \arrow[r] & U/ku. \\
X \arrow[r, mapsto] & d\phi(X) \arrow[rr, mapsto] & & d\phi(X).u + ku
\end{tikzcd}$$

$Ker(d\psi_e) = \mathfrak{h} \Rightarrow$

$\dim d\psi_e(\mathfrak{g}) = \dim G - \dim H = \dim X$.

$d\psi_e : T_eG \twoheadrightarrow T_xX$ surjective.

$\therefore$ $\psi : G \longrightarrow X$ separable.

$\square$

Lemma $h : X \longrightarrow Y$ surjective open map of top. spaces.
$Y' \subseteq Y$ subset. $h^{-1}(Y') \subseteq X$ closed $\Rightarrow$ $Y' \subseteq Y$ closed.

Proof: $Y - Y' = h(X - h^{-1}(Y'))$ is open. $\square$

Lemma $F \subseteq E$ separably gen. extension.
$a \in E$ alg. over F $\Rightarrow$ a separable over F.

Proof

Choose tr. basis $\{b_1, \dots, b_n\}$ of E/F s.t.
E/E' separable, $E' = F(b_1, \dots, b_n)$.

$p(T) \in E'[T]$ min. poly of a / E'.
Then $p(T)$ has distinct roots.
$q(T) \in F[T]$ min. poly of a / F.
$p(T) \mid q(T)$ in $E'[T]$ $\Rightarrow$ $p(T) \in \overline{F}[T] \cap E'[T] = F[T]$.
$\therefore$ $q(T) = p(T)$ has distinct roots.
$\square$

Prop X hom. G-variety, $x \in X$.

Assume $\psi: G \longrightarrow X$, $\psi(g) = g.x$ separable.

$U \subseteq X$ open, $f: U \longrightarrow k$ any function.

Then $f \in \mathcal{O}_X(U) \Leftrightarrow f\psi \in \mathcal{O}_G(\psi^{-1}(U))$.

Proof of $\Leftarrow$:

WLOG G connected.

$\Gamma = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{A}^1$ subset.

$\psi: G \longrightarrow X$ equivariant of hom. G-varieties

$\Rightarrow \psi \times 1 : G \times \mathbb{A}^1 \longrightarrow U \times \mathbb{A}^1$ is open.

$f\psi$ regular fcn $\Rightarrow$

$(\psi \times 1)^{-1}(\Gamma) = \{(g, f\psi(g)) \mid g \in \psi^{-1}(U)\} \subseteq \psi^{-1}(U) \times \mathbb{A}^1$ closed

$\Rightarrow \Gamma \subseteq U \times \mathbb{A}^1$ closed. (Lemma)

$\therefore$ Γ is a variety.

$$\begin{tikzcd}
G \arrow[r, "(\psi, f\psi)"] & \Gamma \arrow[r, "pr_1"] & U \\
k(G) \arrow[r, phantom, "\supseteq"] & k(\Gamma) \arrow[r, phantom, "\supseteq"] & k(X)
\end{tikzcd}$$

$k(G)/k(X)$ separably gen. $\Rightarrow$ $k(\Gamma)/k(X)$ separable.
(Lemma)

$\Gamma \longrightarrow U$ bijective & separable $\Rightarrow$ birational.

U non-singular.

Zariski's Main Thm. $\Rightarrow$ $pr_1 : \Gamma \xrightarrow{\cong} U$ iso.

$\therefore$ $f: U \xrightarrow{\cong} \Gamma \xrightarrow{pr_2} \mathbb{A}^1$ regular.

$\square$

Quotients

X SWF. $\sim$ equiv. rel. on X.

$\pi : X \longrightarrow X'$ morphism.

Def π *respects* $\sim$ if $x_1 \sim x_2 \Rightarrow \pi(x_1) = \pi(x_2)$.

π is a *universal morphism respecting* $\sim$ if

$\forall$ morphism of SWF $f : X \longrightarrow Y$ respecting $\sim$

$\exists!$ morphism $\tilde{f} : X' \longrightarrow Y$ s.t. $f = \tilde{f}\pi$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\
& X' \arrow[ur, "\exists!", "\tilde{f}"'] &
\end{tikzcd}$$

Construction

$X' = X/\sim$ as set. $\pi : X \longrightarrow X/\sim$

$U \subseteq X'$ open $\Leftrightarrow \pi^{-1}(U) \subseteq X$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow f\pi : \pi^{-1}(U) \longrightarrow k$ regular.

Exer $\pi : X \longrightarrow X/\sim$ univ. morphism respecting $\sim$.

X SWF, $X \circlearrowleft G$ right action.

$X/G = X/\sim$, $x_1 \sim x_2 \Leftrightarrow x_1.G = x_2.G$.

Example $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$.

Example $\mathbb{A}^1/\mathbb{G}_m = \{0, *\}$ SWF.

$\{0\}$ closed, $\{*\}$ open. $\mathcal{O}(\mathbb{A}^1/\mathbb{G}_m) = k$.

Def G alg. group, X G-variety.

The quotient X/G is *separable* if

(1) X/G alg. variety

(2) $\pi : X \longrightarrow X/G$ is separable.

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Thm G LAG, $H \subseteq G$ closed subgroup.

(1) G/H is quasi-projective.

(2) $G \longrightarrow G/H$ is separable.

(3) $\dim(G/H) = \dim(G) - \dim(H)$.

Proof

Let (X, x) be as in Corollary:

- X quasi-projective homogeneous G-variety.
- $G_x = H$
- $\psi : G \longrightarrow X$, $g \longmapsto g.x$ is separable.

$H = G_x \Rightarrow X = G/H$ as set.

ψ open: $U \subseteq X$ open $\Leftrightarrow$ $\psi^{-1}(U) \subseteq G$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow$ $f\psi : \psi^{-1}(U) \longrightarrow k$ regular.

$\therefore$ $X = G/H$ as SWF.

□

Notation:

$G/H = \{g.H \mid g \in G\}$, $\pi : G \longrightarrow G/H$, $\pi(g) = g.H$.

Fiber bundles

G LAG, $H \subseteq G$ closed subgp, $H \circlearrowright X$.

$G \times X \circlearrowleft H$, $(g,x).h = (gh, h^{-1}.x)$.

$$G \times^H X = (G \times X)/H \quad \text{SWF}$$
$$= \{[g,x] \mid g \in G, x \in X\} / \{[gh,x] = [g,h.x] \ \forall h \in H\}$$

$p : G \times^H X \longrightarrow G/H$ morphism
$[g,x] \longmapsto g.H$

$$\begin{tikzcd}
G \times X \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\
G \times^H X \arrow[r, dashed, "\exists !", "p"'] & G/H
\end{tikzcd}$$

<u>Note</u>: $y = g.H \in G/H$

$\Rightarrow X \longrightarrow p^{-1}(y)$, $x \longmapsto [g,x]$ bijective morphism.

<u>Local section of π</u>:

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/H \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
 & \mathcal{U} \neq \emptyset \text{ open} \arrow[ul, "\sigma"]
\end{tikzcd}
\qquad \pi\sigma = 1_{\mathcal{U}}.$$

<u>Note</u>: $\exists$ local section $\Rightarrow G/H$ "covered" by sections.

$\sigma^g : g.\mathcal{U} \longrightarrow G$, $\sigma^g(y) = g\,\sigma(g^{-1}.y)$.

$\pi\sigma^g = 1_{g.\mathcal{U}}$

<u>Example</u> G/H projective $\Rightarrow \exists$ local sections.

Prop Assume $\pi : G \longrightarrow G/H$ has local sections. Then $G \times^H X$ is a variety and $p : G \times^H X \longrightarrow G/H$ is locally trivial with fiber X.

$$\begin{tikzcd}
G \times^H X \arrow[r, "p"] & G/H \\
p^{-1}(U) \cong U \times X \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "pr_1"] & U \text{ open} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Proof

Assume $\sigma : U \longrightarrow G$ local section of π.

$$\begin{aligned}
U \times X &\xrightarrow{\cong} p^{-1}(U) \\
(y, x) &\longmapsto [\sigma(y), x] \\
(\pi(g), \underbrace{\sigma(\pi(g))^{-1} g}_{\in H} . x) &\longleftarrow\!\shortmid [g, x]
\end{aligned}$$

□

Cor π has local sections $\Leftrightarrow$ π locally trivial (fiber H)

Proof: $G = G \times^H H \xrightarrow{\pi = p} G/H$. □

Example $H \longrightarrow GL(V)$ rational rep. $\Rightarrow$

$G \times^H V \longrightarrow G/H$ vector bundle.

Thm $\phi : X \longrightarrow Y$ morphism of irred. vars.

ϕ is separable (and dominant) $\Leftrightarrow$

$\exists$ dense open $U \subseteq X$: $\forall x \in U$: $d\phi_x : T_x X \twoheadrightarrow T_{\phi(x)} Y$ surj.

Cor $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$, X, Y, Z irred.

ϕ and ψ separable $\Rightarrow$ $\psi\phi$ separable $\Rightarrow$ ψ separable.

Cor $\phi : X \longrightarrow X'$, $\psi : Y \longrightarrow Y'$, $\phi \times \psi : X \times Y \longrightarrow X' \times Y'$.

ϕ and ψ separable $\Leftrightarrow$ $\phi \times \psi$ separable.

Quotients of products

X, Y irred. varieties, $\sim_X$, $\sim_Y$ equiv rels.

$\sim \;=\; (\sim_X, \sim_Y)$ on $X \times Y$.

Bijective morphism of SWF:

$$(X \times Y)/_{\sim} \longrightarrow (X/_{\sim_X}) \times (Y/_{\sim_Y})$$

Prop Assume $X/_{\sim_X}$, $Y/_{\sim_Y}$, $(X\times Y)/_{\sim}$ are varieties,
$X/_{\sim_X}$ and $Y/_{\sim_Y}$ are normal,
and $X \longrightarrow X/_{\sim_X}$ and $Y \longrightarrow Y/_{\sim_Y}$ separable.
Then $(X \times Y)/_{\sim} \cong (X/_{\sim_X}) \times (Y/_{\sim_Y})$.

Proof

$$\begin{tikzcd}
X \times Y \arrow[d] \arrow[dr, "\text{separable}"] & \\
(X \times Y)/_{\sim} \arrow[r, "\text{bijective}"'] & (X/_{\sim_X}) \times (Y/_{\sim_Y})
\end{tikzcd}$$

separable,
normal target. Zariski $\Rightarrow$ iso.

□

Prop G LAG, $H \subseteq G$ closed normal subgroup.
Then G/H is a LAG.

Proof

$$\begin{tikzcd}
G \times G \arrow[r, "(x,y) \mapsto xy^{-1}"] \arrow[d] & G \arrow[d] & \\
G \times G/H \times H \arrow[r, dashed, "\exists!"] \arrow[d, equal] & G/H & xy^{-1}.H \\
G/H \times G/H \arrow[ru] & (x.H, y.H) \arrow[ru, mapsto] &
\end{tikzcd}$$

$\therefore$ G/H alg. group.

Choose $\phi : G \longrightarrow GL(V)$, $0 \neq v \in V$:

$H = \{ g \in G \mid \phi(g).v \in kv \}$

$\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\phi(X).v \in kv \}$

Given character $\chi : H \longrightarrow \mathbb{G}_m$:

$V_\chi = \{ u \in V \mid \phi(h).u = \chi(h)u \ \forall h \in H \}$

$g \in G \Rightarrow g.V_\chi = V_{g.\chi}$ where $(g.\chi)(h) = \chi(g^{-1}hg)$:

$g.u \in g.V_\chi$: $h.(g.u) = gg^{-1}hg.u = \chi(g^{-1}hg)\, g.u$

Note: $v \in V_\chi$ for some χ.

Note: $\sum V_\chi = \bigoplus V_\chi$.

WLOG: $V = \bigoplus_\chi V_\chi$.

Def: $W = \bigoplus_\chi \mathrm{End}_k(V_\chi) \subseteq \mathrm{End}_k(V)$.

Note: Given $\alpha : V \longrightarrow V$ linear:

$\alpha f = f \alpha \ \forall f \in W \Leftrightarrow$

$\alpha : V_\chi \longrightarrow V_\chi$ mult. by scalar $\forall \chi$

Def: $\psi : G \longrightarrow GL(W)$, $\psi(g).f = \phi(g) f \phi(g)^{-1}$, $f \in W$.

$\mathrm{Ker}(\psi) = H$: $\psi(g) = 1_W \Leftrightarrow \phi(g) f = f \phi(g) \ \forall f \in W$

$\Rightarrow \phi(g).v \in kv \Rightarrow g \in H$

$$\begin{tikzcd}
G \arrow[rr, "\psi"] \arrow[dr, twoheadrightarrow, "\pi"'] & & GL(W) \\
& G/H \arrow[ur, dashed, "\exists !", "\lambda"'] &
\end{tikzcd}$$

λ injective hom. of alg. groups.

$\lambda(G/H) = \psi(G) \subseteq GL(W)$ closed subgroup.

$$\begin{tikzcd}
T_eG \arrow[rr, "d\psi"] \arrow[dr, "d\pi_e"'] & & T_e(\psi(G)) \\
& T_e(G/H) \arrow[ur, "d\lambda_e"'] &
\end{tikzcd}$$

Exer: $X \in \mathfrak{g}$, $f \in W \Rightarrow d\psi(X).f = d\phi(X) f - f d\phi(X)$

$\mathrm{Ker}(d\psi) \subseteq T_eH$:

$X \in \mathrm{Ker}(d\psi) \Leftrightarrow d\phi(X) f = f d\phi(X) \quad \forall f \in W$

$\Rightarrow d\phi(X).v \in kv \Rightarrow X \in T_eH$

$$\begin{aligned}
\dim d\psi(T_eG) &= \dim G - \dim \mathrm{Ker}(d\psi) \\
&\geq \dim G - \dim H = \dim G/H = \dim \psi(G).
\end{aligned}$$

$\lambda : G/H \longrightarrow \psi(G)$ bijective separable group hom.

$\Rightarrow$ isomorphism.

$\square$

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Def X variety. X is <u>complete</u> if $\forall$ varieties Z:

$\pi_2 : X \times Z \longrightarrow Z$ is a closed map.

Example: $\mathbb{A}^1$ is not complete. $\pi_2 : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$.

$\pi_2(Z(xy-1)) = \mathbb{A}^1 \setminus \{0\}$ not closed.

Properties: Assume X complete.

(1) $Y \subseteq X$ closed $\Rightarrow$ Y is complete.

(2) Y complete $\Rightarrow$ $X \times Y$ is complete.

(3) $\phi : X \longrightarrow Y$ morphism $\Rightarrow$ $\phi(X)$ closed & complete.

Pf: $X \xrightarrow{\gamma} X \times Y \xrightarrow{\pi_Y} Y$, $\gamma(X) \subseteq X \times Y$ closed.

(4) X connected $\Rightarrow$ $\mathcal{O}_X(X) = k$.

Pf: $f : X \longrightarrow \mathbb{A}^1$, $f(X)$ closed, complete, connected.

(5) X affine $\Rightarrow$ X is finite.

Thm $\mathbb{P}^n$ is complete.

Thm X complete, C non-singular curve, $P \in C$.
Any morphism $\phi : C \setminus \{P\} \longrightarrow X$ extends to $\phi : C \longrightarrow X$.

Lemma $\phi : X \times Y \longrightarrow Z$ morphism, X, Y, Z irred., X complete. For $y \in Y$ set $\phi_y(x) = \phi(x,y)$.
Assume $\exists\, a \in Y$: $\phi_a : X \longrightarrow Z$ constant.
Then ϕ_y constant $\forall y \in Y$.

Proof

$\Gamma = \{(x, y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq X \times Y \times Z$ closed & irred.
(graph)

$C = \{(y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq Y \times Z$ closed & irred.
(image of Γ, X complete)

$\therefore$ C irred. variety.

$\pi_Y : C \longrightarrow Y$ surjective morphism.

ϕ_a constant $\Leftrightarrow$ $\pi_Y^{-1}(a) =$ point.

$\therefore$ $\dim(C) = \dim(Y)$.

$x \in X$: $C_x = \{(y, \phi(x,y)) \mid y \in Y\} \subseteq C$ closed & irred.
(graph)

$C_x \cong Y \Rightarrow \dim(C_x) = \dim(C) \Rightarrow C_x = C$.

$\pi_Y : C \xrightarrow{\cong} Y$ isomorphism.

$\pi_Y^{-1}(y) =$ point $\Rightarrow$ ϕ_y constant $\forall y \in Y$.

□

Cor G complete alg. group $\Rightarrow$ G is commutative.

Proof $\phi : G \times G \longrightarrow G$, $\phi(x,y) = x y x^{-1}$.

ϕ_e constant $\Rightarrow$ ϕ_y const. $\forall y \in G$.

□

Exer $\mathbb{P}^n$ is not an alg. group for $n \geq 1$.

Lemma $\phi : X \longrightarrow Y$ bijective equivariant morphism of homogeneous G-varieties.

Then X complete $\Leftrightarrow$ Y complete.

Proof: $\phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ homeomorphism $\forall Z$. $\square$

Def G LAG, $P \subseteq G$ closed subgroup.

P is parabolic if G/p is complete.

Def $P \subseteq G$ subgroup, Z set. $A \subseteq G \times Z$ is P-stable if $(g,z) \in A$, $p \in P \Rightarrow (gp, z) \in A$.

Lemma $P \subseteq G$ is parabolic $\Leftrightarrow$

$\forall$ var. Z $\forall A \subseteq G \times Z$ closed P-stable: $\pi_Z(A) \subseteq Z$ closed.

Proof

$A \subseteq G/p \times Z$ closed

$\updownarrow$

$\pi^{-1}(A) \subseteq G \times Z$ closed, P-stable.

$$\begin{tikzcd}
G \times Z \arrow[r] \arrow[d, "\pi"'] & Z \\
G/p \times Z \arrow[ur] &
\end{tikzcd}$$

Same image in Z.

$\square$

<u>Lemma</u> G LAG, $Q \subseteq P \subseteq G$ closed subgroups.

$Q \subseteq G$ parabolic $\Leftrightarrow$ $Q \subseteq P$ and $P \subseteq G$ parabolic.

<u>Proof</u>

$\Rightarrow$: $P/Q \subseteq G/Q$ closed and $G/Q \twoheadrightarrow G/P$.

$\Leftarrow$:
$$\begin{tikzcd} P \times G \times Z \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\pi"'] & G \times Z \arrow[r, "\pi_Z"] & Z \end{tikzcd}$$

$$\alpha(p,g,z) = (gp, z), \quad \pi(p,g,z) = (g,z).$$

Let $A \subseteq G \times Z$ be closed, Q-stable.

$\alpha^{-1}(A) \subseteq P \times G \times Z$ closed, Q-stable.

$Q \subseteq P$ parab. $\Rightarrow$ $\pi(\alpha^{-1}(A)) \subseteq G \times Z$ closed, P-stable.

$P \subseteq G$ parab. $\Rightarrow$ $\pi_Z(A) = \pi_Z(\pi(\alpha^{-1}(A))) \subseteq Z$ closed.

□

<u>Cor</u> $P \subseteq G$ parabolic $\Leftrightarrow$ $P^\circ \subseteq G^\circ$ parabolic.

<u>Proof</u>

$$\begin{array}{ccc} P & \subset & G \\ \cup & & \cup \\ P^\circ & \subset & G^\circ \end{array}$$

<u>Note:</u>
$G^\circ \subseteq G$ parabolic.

□

Thm G connected LAG. TFAE:

(a) G has no proper parabolic subgroups.

(b) $G \circlearrowright X$, X complete $\Rightarrow X^G \neq \emptyset$.

(c) $G \subseteq GL_n \Rightarrow \exists x \in GL_n : xGx^{-1} \subseteq B_n$ (upper Δ)

(d) G is solvable.

Proof

(a)$\Rightarrow$(b): Choose closed orbit $\Omega \subseteq X$.

$x \in \Omega$, $G_x \subseteq G$ isotropy group.

$G/G_x \longrightarrow \Omega$, $g.G_x \longmapsto g.x$

bijective morphism of hom. G-varieties.

X complete $\Rightarrow \Omega$ complete $\Rightarrow G/G_x$ complete

$\Rightarrow G_x \subseteq G$ parabolic $\Rightarrow G_x = G \Rightarrow x \in X^G$.

(b)$\Rightarrow$(c): $G \subseteq GL(V)$ closed subgroup.

$Fl(V) = \{ V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V) \mid \dim(V_i) = i \}$

Exer: $Fl(V)$ projective variety.

(b) $\Rightarrow \exists$ G-stable flag $V_\bullet \subseteq V \Rightarrow$ (c).

(c)$\Rightarrow$(d): B_n is solvable.

(d)$\Rightarrow$(a): Choose minimal parabolic $P \subseteq G$.

$P \subsetneq G \Rightarrow G/P$ not affine $\Rightarrow P \subseteq G$ not normal $\Rightarrow (G,G) \not\subseteq P$.

$(G,G)/(G,G)\cap P \longrightarrow (G,G)P/P$

bijective equiv. morphism of homogeneous (G,G)-varieties.

$P \subsetneq (G,G)P$ parab. $\Rightarrow (G,G)\cap P \subsetneq (G,G)$ parab.

Induction on $\dim(G) \Rightarrow \lightning$

$\square$

Lemma $H \subseteq G$ connected solvable, $P \subseteq G$ parabolic.

$\exists g \in G : gHg^{-1} \subseteq P$.

Proof

$H \circlearrowright G/P$. Let $g.P \in (G/P)^H$ be a fixed point.

$\forall h \in H : hg.P = g.P \Rightarrow g^{-1}Hg \subseteq P$.

□

Def G LAG. A Borel subgroup of G is a maximal closed connected solvable subgroup.

Thm G LAG, $B \subseteq G$ closed subgroup. TFAE:

(1) $B \subseteq G$ Borel

(2) $B \subseteq G$ min. parabolic.

(3) $B \subseteq G$ connected solvable parabolic.

Proof

(3) $\Rightarrow$ (1) + (2) : Lemma.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) :

Choose $B \subseteq G$ Borel, $P \subseteq G$ min. parabolic.

WLOG $B \subseteq P$.

$P = P^\circ$ is connected, contains no proper parabolic.

$\Rightarrow$ P closed connected solvable.

$\therefore$ $B = P$ satisfy (3).

□

Cor All Borel subgroups are conjugate.

Cor $\phi : G \twoheadrightarrow G'$ surjective homomorphism of LAGs.

$P \subseteq G$ Borel/parabolic $\Rightarrow$ $\phi(P) \subseteq G'$ Borel/parabolic.

Proof

$G/P \twoheadrightarrow G'/\phi(P)$. $P \subseteq G$ parab. $\Rightarrow$ $\phi(P) \subseteq G'$ parab.

P Borel $\Rightarrow$ $\phi(P)$ connected solvable parabolic.

□

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Cor G connected, $B \subseteq G$ Borel $\Rightarrow Z(G)^\circ \subseteq Z(B) \subseteq Z(G)$.

Proof

$Z(G)^\circ \subseteq G$ closed connected solvable

$\Rightarrow Z(G)^\circ \subseteq B'$, B' Borel

$\Rightarrow Z(G)^\circ \subseteq gBg^{-1}$, $g \in G$

$\Rightarrow Z(G)^\circ = g^{-1} Z(G)^\circ g \subseteq B$.

$\therefore Z(G)^\circ \subseteq Z(B)$.

Let $g \in Z(B)$.

Well defined morphism:

$$G/B \longrightarrow G, \quad x \longmapsto xgx^{-1}$$

G/B irred. and complete, G affine

$\Rightarrow$ morphism is constant.

$\therefore xgx^{-1} = g \ \forall x \in G \quad \Rightarrow \quad g \in Z(G)$.

$\square$

Lemma $G \neq e$ connected nilpotent LAG $\Rightarrow Z(G)^\circ \neq e$.

Proof

$G_0 = G$, $G_{i+1} = (G, G_i)$ lower central series.

G nilpotent $\Leftrightarrow G_n = e$ for some n.

Choose $n \geq 1$ such that $G_{n-1} \neq e$, $G_n = e$.

$(G, G_{n-1}) = e \Rightarrow G_{n-1} \subseteq Z(G)$.

G_{n-1} connected $\Rightarrow G_{n-1} \subseteq Z(G)^\circ$.

$\square$

Cor G LAG, $B \subseteq G$ Borel. B nilpotent $\Rightarrow B = G^\circ$

Proof

WLOG: G connected.

Assume $B \subsetneq G$.

G/B not affine $\Rightarrow B \neq e \Rightarrow e \neq Z(B)^\circ \subseteq Z(G)$.

$\therefore Z(B)^\circ \trianglelefteq G$ normal.

$B/Z(B)^\circ \subsetneq G/Z(B)^\circ$ proper nilpotent Borel subgp.

Induction on $\dim(G) \Rightarrow$ ↯

□

Cor G connected nilpotent LAG.

(1) G_s and G_u are closed connected subgroups.

(2) $G_s \subseteq G$ is a central torus.

(3) $\mu : G_s \times G_u \xrightarrow{\cong} G$ iso. of alg. groups.

Proof

Already proved: $G_s \subseteq Z(G)$ abstract subgroup.

$G \subseteq GL(V)$ closed subgroup.

$V = \bigoplus_{\chi} V_\chi$, $\chi : G_s \longrightarrow \mathbb{G}_m$ char. of abstract groups.

$G.V_\chi = V_\chi \quad \forall \chi$.

$G \circlearrowright Fl(V_\chi)$, $Fl(V_\chi)^G \neq \emptyset$.

$\therefore \exists\, G \subseteq GL_n$ such that $G \subseteq B_n$ and $G_s = G \cap D_n$.

Note: $G_u = G \cap U_n$.

$$G_s \times G_u \underset{\text{project}}{\overset{\mu}{\rightleftarrows}} G$$ iso. of alg. groups (since $G_s \subseteq Z(G)$.)

□

__Note__: G LAG. G diagonalizable $\Leftrightarrow$
G commutative & all elts. semi-simple.

__Proof__ $\Leftarrow$: $G \subseteq GL_n$ closed subgroup.
$(2.4.2) \Rightarrow \exists x \in GL_n : xGx^{-1} \subseteq D_n$.

__Cor__ G connected solvable LAG.

(1) $(G,G) \subseteq G$ closed connected unipotent normal subgp.

(2) $G_u \subseteq G$ closed connected unipotent normal subgp.

(3) G/G_u is a torus.

__Proof__

$(G,G) \subseteq G$ closed connected normal.

WLOG: $G \subseteq B_n \subseteq GL_n$

$(G,G) \subseteq (B_n, B_n) = U_n \Rightarrow (G,G)$ unipotent.

$G_u = G \cap U_n \Rightarrow G_u \subseteq G$ closed unipotent normal.

$G/G_u \xrightarrow{\phi} B_n/U_n \cong D_n$ injective $\Rightarrow G/G_u$ commutative.

All $x \in G/G_u$ semi-simple:
$\phi(x_u) = \phi(x)_u = e \Rightarrow x_u = e \Rightarrow x = x_s$.

G/G_u connected $\Rightarrow G/G_u$ torus (by Note).

__G_u connected:__

$G_u^\circ \trianglelefteq G$ normal, $G_u/G_u^\circ \xrightarrow{\cong} (G/G_u^\circ)_u$ iso. of finite groups.

__Show:__ G connected solvable, G_u finite $\Rightarrow G_u = e$.

$G_u \trianglelefteq G$ normal & finite $\Rightarrow G_u \subseteq Z(G)$.

($y \in G_u$. $G \longrightarrow G_u$, $x \longmapsto xyx^{-1}$ must be constant.)

G/G_u commutative $\Rightarrow G/Z(G)$ commutative
$\Rightarrow G$ nilpotent $\Rightarrow G_u$ connected.

□

Def G connected solvable LAG. A maximal torus of G is a subtorus $T \subseteq G$ with $\dim(T) = \dim(G/G_u)$.

Lemma G connected solvable LAG, $T \subseteq G$ max torus. Then $\mu : T \times G_u \xrightarrow{\cong} G$ isomorphism of varieties.

Proof

$T \times G_u \circlearrowright G$, $(t,u).x = txu^{-1}$.

Isotropy group of $e \in G$: $(T \times G_u)_e = T \cap G_u = e$.

$\pi : T \times G_u \longrightarrow G$, $\pi(t,u) = tu^{-1}$

bijective equiv. morphism of hom. varieties.

$d\pi_{(e,e)} : L(T) \oplus L(G_u) \longrightarrow L(G)$, $(X,Y) \longmapsto X - Y$

X semi-simple and Y nilpotent (in $\mathrm{End}_k(k[G])$).

$\Rightarrow d\pi_{(e,e)}$ injective $\Rightarrow d\pi_{(e,e)}$ bijective.

$\square$

Cor G connected solvable, $T \subseteq G$ max. torus $\Rightarrow T \xrightarrow{\cong} G/G_u$ isomorphism.

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Lemma G connected solvable LAG, G not a torus.
$\exists$ closed normal $N \lhd G$: $N \cong \mathbb{G}_a$ and $N \subseteq Z(G_u)$.

Proof

G not torus $\Rightarrow$ $G_u \neq e$.

$G_u \lhd G$ normal $\Rightarrow$ $Z(G_u)^\circ \lhd G$ normal.

$G_u \neq e$ connected nilpotent $\Rightarrow$ $Z(G_u)^\circ \neq e$.

$p = \mathrm{char}(k)$.

$p > 0$: $L(G_u)$ unipotent $\Rightarrow$ $L(G_u)^{p^r} = e$ for some $r > 0$.

$\therefore$ $\exists$ $H \lhd G$ closed normal connected, $e \neq H \subseteq Z(G_u)$.
$p > 0 \Rightarrow H^p = e$.

H connected elementary unipotent.

$H \cong \mathbb{G}_a^m$ for some $m \geq 1$.

$m = 1$: Take $N = H$. Assume $m > 1$.

$G \circlearrowright H$, $g.h = ghg^{-1}$

$G \longrightarrow GL(k[H])$ locally rational. $(g.f)(h) = f(g^{-1}hg)$.

$\mathcal{A} \subseteq k[H]$ additive functions.

$G \longrightarrow GL(\mathcal{A})$ locally rational.

G_u acts trivially on H: $G \longrightarrow G/G_u \longrightarrow GL(\mathcal{A})$

G/G_u torus $\Rightarrow$ $\exists$ $0 \neq f \in \mathcal{A}$: $g.f \in kf$ $\forall g \in G$.

$f: H \longrightarrow \mathbb{G}_a$ group hom.

$H' = (\mathrm{Ker}\, f)^\circ \cong \mathbb{G}_a^{m-1}$.

$H' \lhd G$ normal: $g \in G$, $h \in \mathrm{Ker}(f)$ $\Rightarrow$
$f(ghg^{-1}) = (g^{-1}.f)(h) = 0$

Induction on m $\Rightarrow$ $\exists N$.

$\square$

Note $\mathbb{G}_m = k^\times \subseteq \mathbb{G}_a = k.$

Standard action: $\mathbb{G}_m \times \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad t.x = tx$

Exer

- $\mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$
- G LAG, $G \times \mathbb{G}_a \longrightarrow \mathbb{G}_a$ action by automorphisms.
  $\exists$ character $\alpha : G \longrightarrow \mathbb{G}_m : \quad g \cdot x = \alpha(g) x$

Note $\mathrm{char}(k) = p > 0$:

$GL_2 \subsetneq \mathrm{Aut}(\mathbb{G}_a^2)$

Action by automorphisms:

$$\mathbb{G}_a \times \mathbb{G}_a^2 \longrightarrow \mathbb{G}_a^2, \quad a.(x,y) = (x + a y^p, y)$$

Not given by group hom. $\mathbb{G}_a \longrightarrow GL_2$.

Note:

G unipotent LAG, $\mathrm{char}(k) = 0 \Rightarrow G$ connected.

G/G° finite unipotent $\Rightarrow G/G^\circ = e$.

Thm G connected solvable LAG.

(1) $s \in G$ semi-simple $\Rightarrow$ $s \in$ max. torus of G.

(2) $s \in G$ semi-simple $\Rightarrow$ $Z_G(s)$ is connected.

(3) All max. tori in G are conjugate.

Proof

Assume first $\dim(G_u) = 1$.

Then $G_u \cong \mathbb{G}_a$. Fix isomorphism $\phi : \mathbb{G}_a \xrightarrow{\cong} G_u$.

$\psi : G \longrightarrow G/G_u$ projection.

$G/G_u \circlearrowright G_u$, $\psi(g).u = gug^{-1}$.

$\exists$ character $\alpha : G/G_u \longrightarrow \mathbb{G}_m$:

$$\circledast \quad g\phi(a)g^{-1} = \phi(\alpha(\psi(g))\, a) \qquad \text{for } g \in G,\ a \in \mathbb{G}_a.$$

Assume α trivial.

Then $G_u \subseteq Z(G)$ $\Rightarrow$ $G/Z(G)$ is commutative

$\Rightarrow$ G nilpotent $\Rightarrow$ $G \cong G_s \times G_u$.

WLOG: α not trivial.

Let $s \in G$ be semi-simple. $Z = Z_G(s)$.

(5.4.2): $L(Z) = \operatorname{Ker}(\operatorname{Ad}(s) - 1) \subseteq L(G)$.

$$L(G) = (\operatorname{Ad}(s)-1)L(G) \oplus L(Z).$$

$\psi(sgs^{-1}) = \psi(g) \Rightarrow \psi \circ \operatorname{Int}(s) = \psi$

$\Rightarrow d\psi \circ \operatorname{Ad}(s) = d\psi \Rightarrow d\psi \circ (\operatorname{Ad}(s) - 1) = 0$

$\Rightarrow (\operatorname{Ad}(s)-1)L(G) \subseteq \operatorname{Ker}(d\psi) = L(G_u)$.

$\dim (\operatorname{Ad}(s)-1)L(G) \leq 1$

$\dim(Z) = \dim L(Z) \geq \dim(G) - 1$.

Assume $\alpha(\psi(s)) \neq 1$:

$Z \cap G_u = e$: $\phi(a) = s\phi(a)s^{-1} = \phi(\alpha(\psi(s))a) \Leftrightarrow a = 0$.

$\therefore \dim(Z) = \dim(G) - 1$.

$Z^\circ = Z^\circ/(Z^\circ)_u$ max. torus in G.

$G = Z^\circ \ltimes G_u$

$Z = Z_G(s) = Z^\circ \ltimes Z_{G_u}(s) = Z^\circ$

Conclude: $s \in G$ semi-simple, $\alpha(\psi(s)) \neq 1$
$\Rightarrow Z_G(s) \subseteq G$ max. torus.

Note: Any max. torus $T \subseteq G$ has this form:
Choose $t \in T$ s.t. $\alpha(\psi(t)) \neq 1$.
Then $T \subseteq Z_G(t) \subseteq G$, $Z_G(t)$ max. torus.

Assume $\alpha(\psi(s)) = 1$:

$G_u \subseteq Z = Z_G(s)$.

$(\mathrm{Ad}(s)-1)L(G) \subseteq L(G_u) \subseteq L(Z) = \mathrm{Ker}(\mathrm{Ad}(s)-1)$

$\mathrm{Ad}(s)-1$ semi-simple $\Rightarrow \mathrm{Ad}(s)-1 = 0$.

$L(Z) = L(G) \Rightarrow Z_G(s) = G$ is connected.

Choose $t \in G$ s.t. $\alpha(\psi(t)) \neq 1$.

$s \in Z(G) \Rightarrow s \in Z_G(t) \subseteq G$ max. torus.

Let $T, T' \subseteq G$ be max. tori, $T = Z_G(t)$, $\alpha(\psi(t)) \neq 1$.

$G = T' \ltimes G_u$. $t = t'\phi(a)$, $t' \in T'$, $a \in \mathbb{G}_a$.

$$\phi(b)t\phi(b)^{-1} = t\,t^{-1}\phi(b)t\phi(b)^{-1} = t'\phi(a)\,\phi(\alpha(\psi(t))^{-1}b)\,\phi(-b)$$
$$= t'\phi(a + (\alpha(\psi(t))^{-1}-1)b)$$

$\exists\, b \in \mathbb{G}_a$: $\phi(b)t\phi(b)^{-1} = t'$.

$\Rightarrow \phi(b)T\phi(b)^{-1} = Z_G(t') = T'$.

Assume $\dim(G_u) \geq 2$:

Choose $N \lhd G$ closed normal, $N \cong \mathbb{G}_a$, $N \subseteq Z(G_u)$.

$\bar{G} = G/N$. $\dim(G/G_u) = \dim(\bar{G}/\bar{G}_u)$

$s \in G$ semi-simple. $\bar{s}$ = image in $\bar{G}$.

Induction on $\dim(G)$ $\Rightarrow$ $\exists$ max. torus $\bar{T} \subseteq \bar{G}$, $\bar{s} \in \bar{T}$.

$$\begin{tikzcd} G \arrow[r] & \bar{G} \\ H \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & \bar{T} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

H = inverse image in G.

H connected solvable, $H_u = N$, $s \in H$.

$\exists$ max. torus $T \subseteq H$, $s \in T$.

$T \subseteq G$ also max. torus.

Let $T, T' \subseteq G$ be max. tori.

$\bar{T}, \bar{T}' \subseteq \bar{G}$ max. tori.

$\exists\, g \in G$: $\overline{gTg^{-1}} = \bar{T}'$ $\Rightarrow$ $(gTg^{-1})N = T'N$.

$T'N$ connected solvable, $(T'N)_u = N$

$\Rightarrow$ gTg^{-1} and T' are conjugate in $T'N$.

<u>Show</u>: $Z_G(s)$ is connected.

Choose max. torus $T \subseteq G$ with $s \in T$.

$G = T \ltimes G_u$ $\Rightarrow$ $Z_G(s) = T \ltimes Z_{G_u}(s)$.

<u>Enough</u>: $Z_{G_u}(s)$ is connected.

<u>Note</u>: Clear if $\mathrm{char}(k) = 0$ since $Z_{G_u}(s)$ is unipotent.

WLOG: $s \notin Z(G)$ $\Rightarrow$ $G_u \not\subseteq Z_G(s)$.

$G_1 = \{g \in G \mid sgs^{-1}g^{-1} \in N\} \subseteq G$ closed subgroup.

$Z_G(s) \subseteq G_1$ and $G_1/N = Z_{\bar{G}}(\bar{s})$.

Induction on $\dim(G) \Rightarrow Z_{\bar{G}}(\bar{s})$ connected.

G_1/N and N connected $\Rightarrow G_1$ connected.

If $G_1 \neq G$ then $Z_G(s) = Z_{G_1}(s)$ is connected
by induction on $\dim(G)$.

WLOG: $G_1 = G$.

$\{sus^{-1}u^{-1} \mid u \in G_u\} \subseteq N$.

LHS is closed (5.4.4 (i)) and connected.

LHS $\neq e$ since $G_u \not\subseteq Z_G(s)$.

$\therefore\ N = \{sus^{-1}u^{-1} \mid u \in G_u\}$.

<u>Enough</u>: $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is bijective.

Assume $z \in Z_{G_u}(s)$, $x, y \in N$, $zx = y \in G_u$.

$N \subseteq Z(G_u) \Rightarrow zx = xz$; $z \in Z_G(s) \Rightarrow zs = sz$.

Write $x = usu^{-1}s^{-1}$, $y = vsv^{-1}s^{-1}$, $u, v \in G_u$.

$$\underset{\substack{\uparrow \\ \text{unipotent}}}{z}\ \underset{\nwarrow}{usu^{-1}} = \underset{\substack{\uparrow \\ \text{semi-simple}}}{vsv^{-1}} \qquad \Rightarrow z = e,\ x = y.$$

This shows $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is injective.

$Z_{G_u}(s) =$ Fiber of morphism $G_u \longrightarrow N$.

$\dim Z_{G_u}(s) \geq \dim(G_u) - 1$.

$\therefore\ Z_{G_u}(s) \times N \longrightarrow G_u$ surjective.

□

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Cor G connected solvable.
$H \subseteq G$ (any) subgroup whose elts. are semi-simple.

(1) $H \subseteq T$ for some max. torus $T \subseteq G$.

(2) $Z_G(H) = N_G(H)$ is connected.

Proof

$H \longrightarrow G/G_u$ injective $\Rightarrow$ H commutative.

If $H \subseteq Z(G)$ then clear (since all max. tori conjugate).

Assume $s \in H$, $s \notin Z(G)$.

$Z_G(s)$ is connected, $H \subseteq Z_G(s) \subsetneq G$.

Induction on $\dim(G) \Rightarrow$

$\exists$ max. torus $T \subseteq Z_G(s)$ with $H \subseteq T$.

$Z_G(H) = Z_{Z_G(s)}(H)$ is connected.

Let $x \in N_G(H)$, $h \in H$.

$xhx^{-1}h^{-1} \in H \cap (G,G) \subseteq H \cap G_u = e$

$\therefore x \in Z_G(H)$.

$\square$

G LAG.

Max. torus in G: subtorus not contained in strictly larger subtorus.

Cor $\Rightarrow$ same def. when G is connected solvable.

Thm G LAG. All max. tori in G are conjugate.

Proof

$T, T' \subseteq G$ max. tori.

Choose Borel subgps. $B, B' \subseteq G$ with $T \subseteq B$, $T' \subseteq B'$.

$\exists g \in G: \quad g B' g^{-1} = B$.

$T, \; g T' g^{-1} \subseteq B$ max. tori.

$\exists b \in B: \quad b g T' g^{-1} b^{-1} = T$.

□

Lemma $T \subseteq G$ max. torus, $H, H' \subseteq G$ conjugate subgps., $T \subseteq H \cap H'$. Then $\exists n \in N_G(T): H' = nHn^{-1}$.

Proof

$\exists g \in G: H' = gHg^{-1}$.

$g^{-1}Tg, \; T \subseteq H$ max. tori.

$\exists h \in H: \; h^{-1}g^{-1}Tgh = T$.

$n = gh \in N_G(T), \quad nHn^{-1} = H'$.

□

Cartan subgroup of LAG G:

$C = Z_G(T)^\circ$ where $T \subseteq G$ max. torus.

Properties

(1) C is nilpotent:

$B \subseteq C$ Borel subgp., $T \subseteq B$.

$B = T \ltimes B_u = T \times B_u$ since $T \subseteq Z(B)$.

B nilpotent $\Rightarrow B = C^\circ = C$.

(2) $T = C_s$ is the only max. torus of C.

(3) $N_G(T) = N_G(C)$.

(4) $N_G(C)/C$ is finite.

Since $N_G(T)/Z_G(T)$ and $Z_G(T)/C$ are finite.

(5) $T \subseteq B \subseteq G$, B Borel $\Rightarrow C \subseteq B$:

$\exists$ Borel subgp. $B' \subseteq G$ with $C \subseteq B'$.

$\exists\, n \in N_G(T): \; B = nB'n^{-1}$.

$C = nCn^{-1} \subseteq B$.

Lemma G LAG, $S \subseteq G$ subtorus. $\exists\, s \in S: Z_G(s) = Z_G(S)$.

Proof

$G \subseteq GL(V)$, $V = \bigoplus V_\chi$, $\chi: S \longrightarrow \mathbb{G}_m$.

Choose $s \in S$ such that

$$0 \neq V_\chi \neq V_{\chi'} \neq 0 \Rightarrow \chi(s) \neq \chi'(s)$$

$Z_G(s) = \{g \in G \mid \forall \chi: \; g.V_\chi = V_\chi\} = Z_G(S)$.

□

Lemma G LAG, $T \subseteq G$ max. torus, $C = Z_G(T)^\circ$.

$\exists t \in T \ \forall g \in G: \ t \in gCg^{-1} \Rightarrow gCg^{-1} = C$.

Proof

Choose $t \in T$ such that $Z_G(t) = Z_G(T)$.

$t \in gCg^{-1} \Rightarrow g^{-1}tg \in C_s = T$.

$C = Z_G(T)^\circ \subseteq Z_G(g^{-1}tg)^\circ = g^{-1}Cg$.

□

Lemma G LAG, $H \subseteq G$ closed subgp., $X = \bigcup_{g \in G} gHg^{-1} \subseteq G$.

(1) X contains dense open $\subseteq \overline{X}$.

(2) H parabolic $\Rightarrow$ $X = \overline{X} \subseteq G$ is closed.

(3) Assume $N_G(H)/H$ is finite and $\exists h \in H$:

h is in finitely many conjugates of H.

Then $\dim(\overline{X}) = \dim(G)$.

Proof

$A = \{(g,x) \in G \times G \mid x \in gHg^{-1}\} \subseteq G \times G$ closed.

A is H-stable: $(g,x) \in A$, $h \in H \Rightarrow (gh,x) \in A$.

$X = \pi_2(A)$, $\pi_2 : G \times G \longrightarrow G$. (1) + (2) follow from this.

$\dim(A) = \dim(G \times H)$: $G \times H \xrightarrow{\cong} A$, $(g,h) \longmapsto (g, ghg^{-1})$

Assume $N_G(H)/H$ and $\mathcal{H} = \{gHg^{-1} \mid g \in G, h \in gHg^{-1}\}$ finite.

$\mathcal{H} = \{y_1 H y_1^{-1}, \dots, y_n H y_n^{-1}\}$, $y_1, \dots, y_n \in G$.

$$\pi_2^{-1}(h) \cong \{g \in G \mid h \in gHg^{-1}\} = \bigcup_{i=1}^{n} \{g \in G \mid gHg^{-1} = y_i H y_i^{-1}\}$$
$$= \bigcup_{i=1}^{n} y_i N_G(H).$$

$\Rightarrow \dim \pi_2^{-1}(h) = \dim(H)$

$\Rightarrow \dim \pi_2(A) \geq \dim(A) - \dim(H) = \dim(G)$.

□

Thm G connected LAG.

(1) $\forall g \in G \; \exists B \subseteq G$ Borel : $g \in B$.

(2) $\forall s \in G_s \; \exists T \subseteq G$ max. torus : $s \in T$.

(3) The union of all Cartan subgps contains dense open $\subseteq G$.

Proof

$T \subseteq G$ max. torus, $C = Z_G(T)^0$.

$N_G(C)/C$ is finite.

$\exists t \in T$: t is in finitely many conjugates of C.

Lemma $\Rightarrow \bigcup_{g \in G} gCg^{-1}$ contains dense open $\subseteq G$.

$\Rightarrow \bigcup_{g \in G} gBg^{-1} = G$ (since larger and closed.)

Let $s \in G$ be semi-simple.

$\exists B \subseteq G$ Borel, $s \in B$.

$\exists T \subseteq B$ max. torus, $s \in T$.

□

Cor G connected LAG, $B \subseteq G$ Borel $\Rightarrow Z(B) = Z(G)$.

Proof

Already proved: $Z(B) \subseteq Z(G)$.

Let $z \in Z(G)$.

$\exists B' \subseteq G$ Borel, $z \in B'$.

$\exists g \in G$: $B = gB'g^{-1}$

$z = gzg^{-1} \in B$.

□

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Thm G connected LAG, $S \subseteq G$ subtorus.

(1) $Z_G(S)$ is connected.

(2) $B \subseteq G$ Borel, $S \subseteq B \Rightarrow Z_G(S) \cap B \subseteq Z_G(S)$ Borel.

All Borel subgps. of $Z_G(S)$ are obtained this way.

Proof

Let $z \in Z = Z_G(S)$. Show: $z \in Z^\circ$.

Choose $B \subseteq G$ Borel, $z \in B$.

$X = \{x.B \in G/B \mid z \in xBx^{-1}\} \subseteq G/B$ closed.

$S \circlearrowright X$, $s.(x.B) = sx.B$ (well def. since $sz = zs$).

Choose $x.B \in X^S \neq \emptyset$.

$\forall s \in S : sx.B = x.B \Leftrightarrow S \subseteq xBx^{-1}$.

Replace $B \mapsto xBx^{-1}$. WLOG $z \in B$ and $S \subseteq B$.

$Z_B(S)$ is connected $\Rightarrow z \in Z_B(S) \subseteq Z^\circ$.

$\therefore Z = Z^\circ$ is connected.

Assume $B \subseteq G$ Borel, $S \subseteq B$.

$Z \cap B = Z_B(S)$ is connected & solvable.

Show: $Z \cap B \subseteq Z$ parabolic.

$Y = \{y \in G \mid y^{-1}Sy \subseteq B\} \subseteq G$ is closed.

$ZB \subseteq Y \Rightarrow \overline{ZB} \subseteq Y$.

$\phi : \overline{ZB} \times S \longrightarrow B/B_u$, $\phi(y,s) = y^{-1}sy.B_u$

Rigidity of diagonalizable groups $\Rightarrow$

$\phi(y,s)$ is independent of y.

$\therefore y^{-1}sy.B_u = s.B_u \quad \forall y \in \overline{ZB},\ s \in S.$

$\Rightarrow y^{-1}Sy \subseteq SB_u \quad \forall y \in \overline{ZB}.$

$B_u \lhd B$ closed connected unipotent normal.

$SB_u \subseteq B$ closed subgp., $S \subseteq SB_u$ max. torus.

Let $y \in \overline{ZB}$

$\exists\, b \in B_u : \ b^{-1}Sb = y^{-1}Sy$.

$yb^{-1} \in N = N_G(S) \Rightarrow y = (yb^{-1})b \in NB$.

$\therefore\ \overline{ZB} \subseteq NB$.

N/Z finite $\Rightarrow N = n_1 Z \cup \cdots \cup n_\ell Z$, $n_1, \dots, n_\ell \in N$.

$\overline{NB} = n_1 \overline{ZB} \cup \cdots \cup n_\ell \overline{ZB} \subseteq NB$.

$\Rightarrow NB \subseteq G$ closed $\Rightarrow NB/B \subseteq G/B$ complete.

$N/N\cap B \longrightarrow NB/B$ bijective equiv. map of hom. N-vars.

$\therefore\ N \cap B \subseteq N$ parabolic.

$(N\cap B)/(Z \cap B)$ finite

$\Rightarrow Z \cap B \subseteq N \cap B$ parabolic

$\Rightarrow Z \cap B \subseteq Z$ parabolic.

$$\begin{tikzcd} N\cap B \arrow[r, phantom, "\subseteq"] & N \\ Z\cap B \arrow[u, phantom, "\cup|" description] \arrow[r, phantom, "\subseteq"] & Z \arrow[u, phantom, "\cup|" description] \end{tikzcd}$$

$\therefore\ Z \cap B \subseteq Z$ Borel.

Let $B' \subseteq Z$ be any parabolic subgroup.

$\exists\, z \in Z : \ B' = z(Z \cap B)z^{-1} = Z \cap zBz^{-1}$!

□

<u>Cor</u> G connected LAG, $T \subseteq B \subseteq G$, T max. torus, B Borel.
Then $Z_G(T) = Z_B(T) = N_B(T)$ is connected.

<u>Proof</u>: $Z_G(T) = Z_G(T)^\circ = C \subseteq B$. □

Thm G connected LAG, $B \subseteq G$ Borel $\Rightarrow N_G(B) = B$.

Proof

Induction on $\dim(G)$.

$H = N_G(B)$. Let $x \in H$. Show: $x \in B$.

$T \subseteq B$ max. torus.

$xTx^{-1} \subseteq B$ also max. torus.

$\exists\, b \in B: \quad bxTx^{-1}b^{-1} = T$.

Replace $x \mapsto bx$: WLOG $xTx^{-1} = T$ and $xBx^{-1} = B$.

$\psi: T \longrightarrow T, \quad t \mapsto xtx^{-1}t^{-1}$ group hom.

Assume $\psi(T) \neq T$:

$S = (\operatorname{Ker}\psi)^\circ \subseteq T$ non-trivial torus.

$x \in Z = Z_G(S)$.

$Z = G$: $\quad \bar{G} = G/S$. $\quad \bar{x} \in N_{\bar{G}}(\bar{B}) = \bar{B} \Rightarrow x \in B$

$Z \neq G$: $\quad x \in N_Z(Z \cap B) = Z \cap B$ since $Z \cap B \subseteq Z$ Borel.

Assume $\psi(T) = T$:

Choose rat. rep. $\phi: G \longrightarrow GL(V)$, $0 \neq v \in V$ such that

$H = \{g \in G \mid \phi(g).v \in kv\}$.

$\exists$ character $\chi: H \longrightarrow \mathbb{G}_m$, $\phi(h).v = \chi(h)v$, for $h \in H$.

$\chi(B_u) = 1$, $\quad \chi(T) = 1$ since $T \subseteq (H, H)$.

$\Rightarrow \chi(B) = 1$.

$G/B \longrightarrow V, \quad g.B \mapsto \phi(g).v$.

Image is complete, affine, connected.

$\therefore \phi(g).v = v \quad \forall g \in G$.

$H = G \Rightarrow B \trianglelefteq G$ normal.

$\Rightarrow G/B$ complete, affine, connected $\Rightarrow B = G$.

$\square$

G connected LAG, $T \subseteq G$ max. torus.

Weyl group: $W = W(G,T) = N_G(T)/Z_G(T)$.

Notation: Given $w \in W$, $\dot{w} \in N_G(T)$ is a representative.

Flag variety: $\mathcal{B} = \{B \subseteq G \text{ Borel}\}$

$G \circlearrowright \mathcal{B}$, $g.B = gBg^{-1}$ (transitive action.)

Isotropy group of $B_0 \in \mathcal{B}$: $G_{B_0} = N_G(B_0) = B_0$.

Identify: $G/B_0 = \mathcal{B}$, $g.B_0 \longleftrightarrow gB_0g^{-1}$.

Note: $\mathcal{B}^T = \{B \in \mathcal{B} \mid T \subseteq B\}$

Cor $W \circlearrowright \mathcal{B}^T$, $w.B = \dot{w}B\dot{w}^{-1}$ simply transitive action.

Proof

$N_G(T) \circlearrowright \mathcal{B}^T$ is transitive.

Isotropy group of B: $B \cap N_G(T) = N_B(T) = Z_G(T)$.

$\square$

Example $G = GL_n$. Borel: $B = \boxed{\begin{matrix} * & * & * \\ 0 & * & * \\ & & * \end{matrix}}$

$F\ell(k^n) = \{(V_1 \subset V_2 \subset \cdots \subset V_n = k^n) \mid \dim(V_i) = i\}$

$GL_n \circlearrowright F\ell(k^n)$ transitive.

$E = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset k^n)$ standard flag.

Isotropy group: $G_E = B$.

$GL_n/B \cong F\ell(k^n)$.

Cor G connected LAG, $P \subseteq G$ parabolic.
Then P is connected and $N_G(P) = P$.

Proof

Let $x \in N_G(P)$. Enough to show $x \in P^\circ$.

$B \subseteq P^\circ$ Borel subgroup. $xBx^{-1} \subseteq P^\circ$ also Borel.

$\exists y \in P^\circ : \; xBx^{-1} = yBy^{-1}$.

$y^{-1}x \in N_G(B) = B \Rightarrow x = y(y^{-1}x) \in P^\circ$.

□

Let $T \subseteq B \subseteq P \subseteq G$, T max. torus, B Borel, P parabolic.

Note: $Z_B(T) = Z_P(T) = Z_G(T)$.

$$\Rightarrow W(P,T) = N_P(T)/Z_P(T) \subseteq N_G(T)/Z_G(T) = W(G,T).$$

Flag variety: $\mathcal{P} = \{ gPg^{-1} \mid g \in G \} \cong G/P$.

Cor: $\mathcal{P}^B = \{ P' \in \mathcal{P} \mid B \subseteq P' \} = \{P\}$.

Proof:

Assume $B \subseteq gPg^{-1}$. Then $g^{-1}Bg \subseteq P$.

$\exists p \in P : \; p^{-1}g^{-1}Bgp = B$.

$gp \in N_G(B) = B \Rightarrow gPg^{-1} = (gp)P(gp)^{-1} = P$.

□

Note:
$$\begin{tikzcd} G/B \arrow[r] \arrow[d, equal] & G/P \arrow[d, equal] & , \; g.B \mapsto g.P \\ \mathcal{B} \arrow[r] & \mathcal{P} & , \; gBg^{-1} \mapsto gPg^{-1} \end{tikzcd}$$

gPg^{-1} = unique conjugate of P containing gBg^{-1}.

Exer: $\mathcal{P}^T = \{ P' \in \mathcal{P} \mid T \subseteq P' \}$.

$W(G,T) \circlearrowright \mathcal{P}^T$, $w.P' = \dot{w}P'\dot{w}^{-1}$ transitive action.

Isotropy group: $W_P = W(P,T)$.

$\therefore \; \mathcal{P}^T \cong W/W_P$ as W-sets.

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Semi-simple & Reductive

G connected LAG, $T \subseteq G$ max. torus.

$N, N' \trianglelefteq G$ normal $\Rightarrow$ $NN' \trianglelefteq G$ normal.

N, N' also solvable $\Rightarrow$ NN' solvable.

$\therefore \exists!$ max. closed connected *solvable* *normal* subgp.

$R(G) \trianglelefteq G$ (*radical* of G)

Unipotent radical: $R_u(G) = R(G)_u$

$R_u(G)$ = unique max. closed connected *unipotent* *normal* subgroup of G.

Def G is *semi-simple* $\Leftrightarrow$ $R(G) = e$.

G is *reductive* $\Leftrightarrow$ $R_u(G) = e$.

Note: $G/R(G)$ is semi-simple.

$G/R_u(G)$ is reductive.

Note: $R(G) = \left(\bigcap_{B \in \mathcal{B}} B \right)^\circ$

Soon: $R_u(G) = \left(\bigcap_{B \in \mathcal{B}^T} B_u \right)^\circ$.

$\pi : G \longrightarrow \bar{G} = G/R(G)$ semi-simple quotient.

Claim: $\bar{T} = \pi(T) \subseteq \bar{G}$ max. torus.

Proof: $\exists$ max. torus $\bar{S} \subseteq \bar{G}$ with $\bar{T} \subseteq \bar{S}$.

$H \subseteq \pi^{-1}(\bar{S})$ connected closed subgroup.

$H/R(G) \subseteq \bar{S}$ commutative $\Rightarrow$ H solvable.

$H = T \ltimes H_u \Rightarrow \bar{S} = \pi(H) = \pi(T) = \bar{T}$.

□

Weyl groups

$$W = N_G(T)/Z_G(T) \xrightarrow{\pi} N_{\bar G}(\bar T)/Z_{\bar G}(\bar T) = \bar W$$

$$\begin{tikzcd}
\mathcal{B} = \{B \subseteq G \text{ Borel}\} \arrow[r, "\cong", "\pi"'] & \bar{\mathcal{B}} = \{\bar B \subseteq \bar G \text{ Borel}\} \\
B \arrow[r, mapsto] & \pi(B) \\
\pi^{-1}(B) & \bar B \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}^T \arrow[r, "\cong", "\pi"'] & \bar{\mathcal{B}}^{\bar T} \\
\circlearrowleft & \circlearrowleft \\
W \arrow[r, "\pi"] & \bar W
\end{tikzcd}$$

Simply transitive actions

$\Rightarrow \pi : W(G,T) \xrightarrow{\cong} W(\bar G, \bar T)$

Def $\operatorname{rank}(G) = \dim(T)$.

$\operatorname{ssrank}(G) = \operatorname{rank}(G/R(G))$. (semi-simple rank).

- $\operatorname{rank}(G) = 0 \Leftrightarrow G$ unipotent.

Note: $W \subseteq \operatorname{Aut}(T) = \operatorname{Aut}(\mathbb{Z}^n)$, $n = \dim(T)$.

- $\operatorname{ssrank}(G) = 0 \Leftrightarrow G$ solvable $\Rightarrow W = e$.
- $\operatorname{ssrank}(G) = 1 \Rightarrow |W| \leq 2$.

Example: $G = GL_n$.

Borel: $B = \begin{bmatrix} * & & * \\ & * & \\ & & * \end{bmatrix}$. $B_u = \begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{bmatrix}$ $B/B_u \cong \mathbb{G}_m^n$.

Max torus: $T = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$. Opposite Borel: $B^- = \begin{bmatrix} * & & \\ & * & \\ * & & * \end{bmatrix}$

$R(G) \subseteq B \cap B^- = T$. $R_u(G) = e$.

Exer: $R(G) = Z(G) \cong \mathbb{G}_m$.

GL_n is reductive, not semi-simple.

$SL_n \subseteq GL_n$ and $PGL_n = GL_n/Z(GL_n)$ are semi-simple.

Exer: $SL_n = Z(\det - 1) \subseteq GL_n$ is connected.

$PGL_n = \operatorname{Aut}(\mathbb{P}^{n-1})$.

Example $G = \left[\begin{array}{c|c} * & * \\ \hline & * \end{array}\right] \subseteq GL_n.$ $\quad R_u(G) = \left[\begin{array}{c|c} I & * \\ \hline & I \end{array}\right].$

$$R(G) = Z(G) \ltimes R_u(G) \cong \mathbb{G}_m \ltimes R_u(G)$$

Notes

(1) X SWF. $X \circlearrowleft G$ action by automorphisms.

$G \circlearrowright \mathcal{O}(X)$, $\quad (g \cdot f)(x) = f(x.g)$.

X/G SWF. $\mathcal{O}(X/G) = \mathcal{O}(X)^G = \{ f \in \mathcal{O}(X) \mid g.f = f \ \forall g \in G \}$.

(2) X affine variety, $X \circlearrowleft \mathbb{G}_m$.

$\mathbb{G}_m \circlearrowright k[X]$ locally rational rep.

$k[X] = \bigoplus_{d \in \mathbb{Z}} k[X]_d$ graded ring.

$k[X]_d = \{ f \in k[X] \mid t.f = t^d f \}$

$\mathcal{O}(X/\mathbb{G}_m) = k[X]_0$

(3) $k[GL_n] = k[x_{ij} \mid 1 \le i,j \le n]_{\det}$.

$k[PGL_n] = k[GL_n]_0$

$\quad = k\left[\frac{x_{i_1 j_1} \cdots x_{i_n j_n}}{\det} \,\middle|\, i_\ell, j_\ell \in \{1, 2, \dots, n\},\ 1 \le \ell \le n \right]$

Example

$\phi : SL_2 \xrightarrow{\subseteq} GL_2 \twoheadrightarrow PGL_2$ group hom.

$\mathrm{Ker}(\phi) = Z(SL_2) = \{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x^2 = 1 \}$.

$\mathrm{Char}(k) \neq 2$: $|Z(SL_2)| = 2$, $PGL_2 = SL_2/Z(SL_2)$.

$\mathrm{Char}(k) = 2$: $\phi : SL_2 \longrightarrow PGL_2$ bijective,
purely inseparable.

__Def__ G LAG, X G-variety.
An __equivariant projective embedding__ of X is an embedding $\psi : X \xrightarrow{\subseteq} \mathbb{P}^n$ together with a alg. group hom. $\rho : G \longrightarrow GL_{n+1}$ such that $\psi(g.x) = \rho(g).\psi(x) \quad \forall x \in X, g \in G.$

__Thm__ (Sumihiro 1973)
G connected LAG, X quasi-projective normal G-variety.
Then $\exists$ equiv. proj. embedding of X.

Proof for $X = G/H$, $H \subseteq G$ closed subgroup: LAG 14.

__Prop__ $T \neq e$ torus, X irred. projective T-variety with equiv. proj. embedding.

(1) $\exists\ \mathbb{G}_m \subseteq T : \ X^{\mathbb{G}_m} = X^T$

(2) $|X^T| = 1 \Leftrightarrow X = \{\text{point}\}$.

(3) $|X^T| = 2 \Leftrightarrow \dim(X) = 1$ and $X^T \neq X$.

__Cor__ G connected LAG, $B \subseteq G$ Borel, W Weyl group.

(1) $W = e \Leftrightarrow G$ solvable $\Leftrightarrow \operatorname{ssrank}(G) = 0$.

(2) $|W| = 2 \Leftrightarrow \dim(G/B) = 1 \Leftrightarrow G/B = \mathbb{P}^1 \Leftarrow \operatorname{ssrank}(G) = 1$.

__Proof__ : $W \longleftrightarrow (G/B)^T$.

__Proof of Prop__:
$X \subseteq \mathbb{P}(V)$ equiv. embedding, $T \longrightarrow GL(V)$ rat. rep.

$V = \bigoplus V_\chi$, $\quad \chi \in X^*(T) = \{T \longrightarrow \mathbb{G}_m\}$

Choose $\lambda \in X_*(T) = \{\mathbb{G}_m \longrightarrow T\}$:

$$0 \neq V_\chi \neq V_{\chi'} \neq 0 \Rightarrow \langle \lambda, \chi - \chi' \rangle \neq 0.$$

$$\mathbb{P}(V)^T = \coprod \mathbb{P}(V_\chi) = \mathbb{P}(V)^{\lambda(\mathbb{G}_m)} \Rightarrow X^T = X^{\lambda(\mathbb{G}_m)}.$$

<u>WLOG</u>: $T = \mathbb{G}_m$, $V = \bigoplus_{d \in \mathbb{Z}} V_d$, $V_d = \{v \in V \mid t.v = t^d v \ \forall t \in T\}$.

Given $x \in X$, set

$$x_0 = \lim_{t \to 0} t.x \quad , \quad x_\infty = \lim_{t \to \infty} t.x.$$

<u>Note</u>: (a) $x_0, x_\infty \in X^T$, (b) $x_0 = x_\infty \Leftrightarrow x \in X^T$.

$x = [u]$, $u = \sum u_d \in V$. $t.x = \left[\sum t^d u_d\right]$

$m = \min\{d : u_d \neq 0\}$, $M = \max\{d : u_d \neq 0\}$.

Then $x_0 = [u_m]$, $x_\infty = [u_M]$.

$\therefore$ $|X^T| = 1 \Leftrightarrow X = \{\text{point}\}$.

Assume $|X^T| = 2$.

Choose $x = [u] \in X \smallsetminus X^T$.

$\ell_m : V \longrightarrow k$ linear, $\ell_m(u_m) \neq 0$, $\ell_m(V_d) = 0$ for $d \neq m$.

$\ell_M : V \longrightarrow k$ linear, $\ell_M(u_M) \neq 0$, $\ell_M(V_d) = 0$ for $d \neq M$.

If $y = [v] \in X \smallsetminus X^T$, then

$y_0 = x_0 \Rightarrow \ell_m(v) \neq 0$ and $y_\infty = x_\infty \Rightarrow \ell_M(v) \neq 0$.

$\phi : X \longrightarrow \mathbb{P}^1$, $\phi([u]) = [\ell_m(u) : \ell_M(u)]$.

T-equiv. morphism: $t.[a:b] = [t^m a : t^M b]$.

$\phi^{-1}(0) \subseteq X$ closed and T-stable.

$(\phi^{-1}(0))^T = \{x_0\} \Rightarrow \phi^{-1}(0) = \{x_0\}$.

$\therefore$ $\dim(X) = 1$.

If $\dim(X) = 1$ and $x \in X \smallsetminus X^T$, then $X = T.x \cup \{x_0, x_\infty\}$.

$\square$

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From now: G connected LAG, $T \subseteq G$ max. torus.

$W = W(G,T) = N_G(T)/Z_G(T)$ Weyl group.

$W \circlearrowright T$: $w.t = \dot{w} t \dot{w}^{-1}$.

$W \circlearrowright X_*(T) = \{\mathbb{G}_m \longrightarrow T\}$: $(w.\lambda)(t) = w.\lambda(t)$.

$W \circlearrowright X^*(T) = \{T \longrightarrow \mathbb{G}_m\}$: $(w.\alpha)(t) = \alpha(w^{-1}.t)$.

Note: $\langle w.\alpha, w.\lambda\rangle = \langle \alpha, \lambda\rangle$.

$\mathfrak{g} = L(G) = T_e G$.

Adjoint action: $G \circlearrowright \mathfrak{g}$, $g.X = Ad(g).X$

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha , \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid t.X = \alpha(t)X \ \forall t \in T\}.$$

Note: $S \subseteq T$ subtorus. $Z_G(S) \subseteq G$ closed conn. subgp.

$$L(Z_G(S)) = \{X \in \mathfrak{g} \mid s.X = X \ \forall s \in S\} = \bigoplus_{\alpha(S)=1} \mathfrak{g}_\alpha .$$

Example: $L(Z_G(T)) = \mathfrak{g}_0$.

Def For $\alpha \in X^*(T)$ define

$T_\alpha = \mathrm{Ker}(\alpha)^\circ \subseteq T$ subtorus.

$G_\alpha = Z_G(T_\alpha) \subseteq G$ closed connected.

$$L(G_\alpha) = \bigoplus_{\beta \in \mathbb{Q}\alpha} \mathfrak{g}_\beta$$

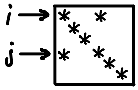
Example $G = GL_n$. $T = \boxed{\begin{smallmatrix} * & & \\ & \ddots & \\ & & * \end{smallmatrix}}$. $\mathfrak{g} = Mat(n \times n)$

$\alpha = \alpha_{ij} \in X^*(T)$: $\alpha(t) = t_i t_j^{-1}$. $(i \neq j)$.

$\mathfrak{g}_\alpha = k E_{ij}$, $E_{ij} \in \mathfrak{g}$ has (i,j)-entry 1, all others 0.

$T_\alpha = \{t \in T \mid t_i = t_j\}$

$G_\alpha = \{g \in GL_n \mid g_{pq} \neq 0 \Rightarrow p = q \text{ or } \{p,q\} = \{i,j\}\}$



Note: $w \in W$, $\alpha \in X^*(T)$.

$$T_{w.\alpha} = w.T_\alpha = \dot{w}\, T_\alpha\, \dot{w}^{-1}$$

$$G_{w.\alpha} = Z_G(\dot{w}\, T_\alpha\, \dot{w}^{-1}) = \dot{w}\, G_\alpha\, \dot{w}^{-1}$$

$$\mathfrak{g}_{w.\alpha} = \dot{w}.\mathfrak{g}_\alpha :$$

$$X \in \mathfrak{g}_\alpha \Rightarrow t.(\dot{w}.X) = \dot{w}\,\dot{w}^{-1} t\,\dot{w}.X = \dot{w}.(\alpha(w^{-1}.t)X)$$
$$= (w.\alpha)(t)\ \dot{w}.X.$$

$P = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$.

Lemma (1) $G = \langle G_\alpha \mid \alpha \in P \rangle$

(2) G solvable $\Leftrightarrow$ G_α solvable $\forall \alpha \in P$.

Proof:

(1) $L(G) = L(\langle G_\alpha \mid \alpha \in P \rangle)$.

(2) $B \subseteq G$ Borel, $T \subseteq B$.

$B \cap G_\alpha \subseteq G_\alpha$ Borel.

G_α solvable $\Leftrightarrow$ $G_\alpha \subseteq B$.

□

Note: $T_\alpha \subseteq Z(G_\alpha)^\circ \subseteq R(G_\alpha) \Rightarrow \operatorname{ssrank}(G_\alpha) \leq 1$.

$\operatorname{ssrank}(G_\alpha) = 1 \Leftrightarrow G_\alpha$ not solvable $\Leftrightarrow |W(G_\alpha, T)| = 2$.

Assume G_α not solvable.

$T \subseteq G_\alpha \subseteq G \Rightarrow W(G_\alpha, T) \subseteq W(G, T)$.

Def (Reflection along α): $s_\alpha \in W$: $W(G_\alpha, T) = \{1, s_\alpha\}$.

Note: $s_\alpha^2 = 1$.

$X_*(T)_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}, \quad X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$

<u>Lemma</u> Assume G_α not solvable.

(1) $\exists!\, \alpha^\vee \in X_*(T)_{\mathbb{R}} : \ s_\alpha . \alpha^\vee = -\alpha^\vee$ and $\langle \alpha, \alpha^\vee \rangle = 2.$

(2) $\lambda \in X_*(T) \Rightarrow s_\alpha . \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee.$

(3) $\beta \in X^*(T) \Rightarrow s_\alpha . \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$

<u>Proof</u>

$(1 - s_\alpha) + (1 + s_\alpha) = 2 \in \mathrm{End}_{\mathbb{R}}(X_*(T)_{\mathbb{R}}).$

$(1 - s_\alpha)(1 + s_\alpha) = 1 - s_\alpha^2 = 0.$

$\therefore X_*(T)_{\mathbb{R}} = \mathrm{Ker}(s_\alpha - 1) \oplus \mathrm{Ker}(s_\alpha + 1).$

$\dot{s}_\alpha \in G_\alpha \Rightarrow s_\alpha . t = t \ \forall t \in T_\alpha \Rightarrow X_*(T_\alpha) \subseteq \mathrm{Ker}(s_\alpha - 1).$

$s_\alpha \neq 1 \Rightarrow \dim_{\mathbb{R}} \mathrm{Ker}(s_\alpha + 1) = 1.$

$X_*(T_\alpha) = \alpha^\perp = \{ \lambda \in X_*(T) \mid \langle \alpha, \lambda \rangle = 0 \}.$

$\therefore \exists!\, \alpha^\vee \in \mathrm{Ker}(s_\alpha + 1) : \langle \alpha, \alpha^\vee \rangle = 2.$

$s_\alpha . \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee :$

true for $\lambda = \alpha^\vee$, true for $\lambda \in X_*(T_\alpha).$

$\beta \in X^*(T), \ \lambda \in X_*(T) \Rightarrow$

$$\langle s_\alpha . \beta, \lambda \rangle = \langle \beta, s_\alpha . \lambda \rangle = \langle \beta, \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \rangle$$

$$= \langle \beta, \lambda \rangle - \langle \alpha, \lambda \rangle \langle \beta, \alpha^\vee \rangle = \langle \beta - \langle \beta, \alpha^\vee \rangle \alpha, \lambda \rangle$$

$\therefore s_\alpha . \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$

□

W-invariant form

$f : X^*(T)_{\mathbb{R}} \times X^*(T)_{\mathbb{R}} \longrightarrow \mathbb{R}$ any positive definite symmetric bilinear form.

For $\alpha, \beta \in X^*(T)_{\mathbb{R}}$, set $(\alpha, \beta) = \sum_{w \in W} f(w.\alpha, w.\beta)$.

Then $(\cdot, \cdot)$ is W-invariant, but not unique (unless G is semi-simple).

<u>Note</u>: G_α not solvable $\Rightarrow$ $\langle \beta, \alpha^\vee \rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$.

True for $\beta = \alpha$;

$\langle \beta, \alpha^\vee \rangle = 0 \Rightarrow (\beta, \alpha) = (s_\alpha.\beta, s_\alpha.\alpha) = (\beta, -\alpha)$.

$P' = \{\alpha \in P \mid G_\alpha \text{ not solvable}\}$

<u>Prop</u> $W = \langle s_\alpha \mid \alpha \in P' \rangle$

<u>Proof</u>:

Let $1 \neq w \in W$.

Assume $w - 1 : X^*(T)_{\mathbb{R}} \xrightarrow{\cong} X^*(T)_{\mathbb{R}}$ is bijective.

Choose $\alpha \in P'$. (Possible since G is not solvable.)

Choose $\beta \in X^*(T)_{\mathbb{R}}$ s.t. $\alpha = (w-1).\beta = w.\beta - \beta$.

$$(\beta, \beta) = (w.\beta, w.\beta) = (\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta)$$

$$\Rightarrow \langle \beta, \alpha^\vee \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = -1$$

$$\Rightarrow s_\alpha.\beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \alpha + \beta = w.\beta$$

$$\Rightarrow s_\alpha w.\beta = \beta.$$

Replace $w \longmapsto s_\alpha w$.

<u>WLOG</u>: $w - 1$ not bijective.

$\psi : T \longrightarrow T$, $\psi(t) = (w^{-1}.t)t^{-1}$ group hom.

<u>Note</u>: $((w-1).\beta)(t) = (w.\beta)(t)\,\beta(t)^{-1} = \beta(\psi(t))$.

$w-1$ not injective $\Rightarrow$ $\psi(T) \subsetneq T$.

$e \neq S = \mathrm{Ker}(\psi)^\circ \subseteq T$ subtorus.

$Z = Z_G(S) \subseteq G$ closed connected.

Induction on $\dim(G)$:

$Z \subsetneq G$: $\dot{w} \in N_Z(T) \Rightarrow w \in W(Z,T)$.

Induction $\Rightarrow$ $W(Z,T) = \langle s_\alpha \mid Z_\alpha$ not solvable$\rangle$.

<u>Note</u>: $Z_\alpha \subseteq G_\alpha$, $W(Z_\alpha, T) \xrightarrow{\cong} W(G_\alpha, T)$.

$Z = G$: $S \subseteq G$ central torus.

$\pi : G \longrightarrow \bar{G} = G/S$. $\bar{T} = T/S$.

$\pi : W(G,T) \xrightarrow{\cong} W(\bar{G}, \bar{T})$.

Induction $\Rightarrow$ $W(\bar{G}, \bar{T}) = \langle s_{\bar{\alpha}} \mid \bar{G}_{\bar{\alpha}}$ not solvable$\rangle$.

Assume $\bar{G}_{\bar{\alpha}}$ not solvable.

<u>Check</u>: $\pi^{-1}(s_{\bar{\alpha}}) \in W(G,T)$ is a reflection.

Choose $n \in N_G(T)$ s.t. $\pi(n) = \dot{s}_{\bar{\alpha}} \in N_{\bar{G}_{\bar{\alpha}}}(\bar{T})$.

$\alpha = \bar{\alpha}\pi : T \longrightarrow \bar{T} \longrightarrow \mathbb{G}_m$. ($G_\alpha$ solvable ?)

$t \in T_\alpha \Rightarrow ntn^{-1} = ts$, $s \in S$.

$s_{\bar{\alpha}}^2 = 1 \Rightarrow n^2 \in Z_G(T) \Rightarrow t = n^2 t n^{-2} = ts^2 \Rightarrow s^2 = 1$.

$T_\alpha \longrightarrow \{s \in S \mid s^2 = 1\}$, $t \mapsto ntn^{-1}t^{-1}$

T_α connected, target finite $\Rightarrow$ $n \in Z_G(T_\alpha) = G_\alpha$.

$\therefore$ n defines reflection $s_\alpha \in W(G,T)$.

$\square$

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Lemma U unipotent connected LAG.

$H \subsetneq U$ proper closed conn. subgp. $\Rightarrow$ $H \subsetneq N_U(H)^\circ$.

Proof

$e \neq U$ nilpotent connected $\Rightarrow$ $Z(U)^\circ \neq e$.

$Z(U)^\circ \nsubseteq H$: $H \subsetneq Z(U)^\circ H \leq N_U(H)^\circ$.

$Z(U)^\circ \subseteq H$: $\bar{H} = H/Z(U)^\circ \subsetneq \bar{U} = U/Z(U)^\circ$.

Induction on $\dim(U)$ $\Rightarrow$ $\bar{H} \subsetneq N_{\bar{U}}(\bar{H})^\circ$.

$N_U(H)/Z(U)^\circ = N_{\bar{U}}(\bar{H})$.

$\therefore$ $H \subsetneq N_U(H)^\circ$.

□

Semi-simple groups of rank 1

Assume $\mathrm{rank}(G) = \mathrm{ssrank}(G) = 1$.

$T \subseteq B \subseteq G$, T max. torus, B Borel.

G not solvable, $W = \mathrm{Aut}(T) = \{1, s\}$, $\dim(G/B) = 1$.

$n = \dot{s} \in N_G(T) - T$. $(G/B)^T = \{1.B, n.B\}$.

Note:
- $ntn^{-1} = s.t = t^{-1}$ $\forall t \in T$.
- $n^2 \in Z_G(T)$.

$U^- = nUn^{-1} = n^{-1}Un$.

__Lemma__ Assume $\mathrm{rank}(G) = \mathrm{ssrank}(G) = 1$.

(1) $G - B = UnB$.

(2) $R(G) = (U \cap U^-)^\circ$

(3) $\dim(U/U\cap U^-) = 1$.

__Proof__

$(G/B)^B = \{1.B\}$.

$Un.B = Bn.B = G/B - \{1.B\}$.

$\therefore\ UnB = G - B$.

Isotropy group: $U_{n.B} = U \cap nBn^{-1} = U \cap U^-$.

$$U/U\cap U^- \longrightarrow Un.B \text{ bijective.}$$

$\therefore\ \dim(U/U\cap U^-) = 1$.

$(U\cap U^-)^\circ \subsetneq N_U((U\cap U^-)^\circ)^\circ \Rightarrow (U\cap U^-)^\circ \lhd U$ normal.

$U, T, \{n\} \subseteq N_G((U\cap U^-)^\circ) \Rightarrow (U\cap U^-)^\circ \lhd G$ normal.

$\therefore\ (U\cap U^-)^\circ \subseteq R(G)$.

$\mathrm{rank}(G) = \mathrm{ssrank}(G) \Rightarrow \mathrm{rank}\, R(G) = 0 \Rightarrow R(G)$ unipotent.

$\therefore\ R(G) \subseteq (U\cap U^-)^\circ$.

$\square$

__Lemma__ Assume G semi-simple, $\mathrm{rank}(G) = 1$.

(1) $\dim(U) = 1$, $Z_G(T) = T$, $U \cap U^- = e$.

(2) $L(G) = L(T) \oplus L(U) \oplus L(U^-)$.

(3) $\exists!\, \alpha \in X^*(T)$: $L(U) = \mathfrak{g}_\alpha$, $L(U^-) = \mathfrak{g}_{-\alpha}$.

(4) $U \times B \xrightarrow{\cong} UnB$, $(u,b) \mapsto unb$ iso. of varieties.

Proof

$(U\cap U^-)^\circ = R(G) = e \Rightarrow U\cap U^-$ finite

$\Rightarrow \dim(U) = 1 \Rightarrow \dim(B) = 2 \Rightarrow \dim(G) = 3.$

$T \subseteq Z_G(T) \subsetneq B$ (since $Z_G(T)$ nilpotent).

$\therefore Z_G(T) = T.$

T normalizes $U\cap U^-$ (finite set)

$\Rightarrow T$ centralizes $U\cap U^-.$

$\therefore U\cap U^- \subseteq G_u \cap Z_G(T) = e.$

$u : \mathbb{G}_a \longrightarrow U$ iso. $T \circlearrowright U$ by conjugation.

$\exists\, \alpha \in X^*(T) : t\, u(x)\, t^{-1} = u(\alpha(t)x)\ \forall\, t\in T,\ x\in \mathbb{G}_a.$

$\therefore L(U) \subseteq \mathfrak{g}_\alpha.$

$U \cap Z_G(T) = e \Rightarrow \alpha \neq 0.$

$T \circlearrowright U^- :\ t\, n u(x) n^{-1} t^{-1} = n\, t^{-1} u(x)\, t\, n^{-1} = n\, u(\alpha(t)^{-1}x)\, n^{-1}$

$\therefore L(U^-) \subseteq \mathfrak{g}_{-\alpha}.$

$\dim(G) = 3 \Rightarrow L(G) = L(T) \oplus L(U) \oplus L(U^-).$

$U^- \times B \circlearrowright G,\ (u,b).g = ugb^{-1}.$

$\phi : U^-\times B \longrightarrow G,\ \phi(u,b) = ub^{-1}$ equivariant map.

$\phi^{-1}(e) = U^-\cap B = U^-\cap U = e \Rightarrow \phi$ injective.

$d\phi : L(U^-)\oplus L(B) \xrightarrow{\cong} L(G),\ (X,Y) \longmapsto X - Y.$

$\therefore \phi : U^-\times B \xrightarrow{\cong} U^-B \subseteq G$ iso. of vars.

$\Rightarrow U\times B \xrightarrow{\cong} UnB$ (since $U^- = n^{-1}Un$).

□

Thm Let G be semi-simple of rank 1.

$\exists\, u : \mathbb{G}_a \xrightarrow{\cong} B_u$, $\alpha^\vee : \mathbb{G}_m \longrightarrow T$, $\alpha : T \longrightarrow \mathbb{G}_m$, $n \in N_G(T) - T$ such that

(1) $\langle \alpha, \alpha^\vee \rangle = 2$

(2) $n^2 = \alpha^\vee(-1)$

(3) $t\,u(x)\,t^{-1} = u(\alpha(t)\,x) \quad \forall\, t \in T,\ x \in \mathbb{G}_a$

(4) $n\,u(y)\,n^{-1} = u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1}) \quad \forall\, y \in \mathbb{G}_m = \mathbb{G}_a - \{0\} = k^\times.$

Example: $G = SL_2$ and $G = PGL_2$:

$u(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $\alpha^\vee(s) = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}$, $\alpha\left(\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}\right) = st^{-1}$, $n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$G = SL_2 \Leftrightarrow \alpha^\vee$ isomorphism. $G = PGL_2 \Leftrightarrow \alpha$ isomorphism.

Note: Given $T \subset B \subset G$:

α and $\alpha^\vee$ are unique: $L(B_u) = \mathfrak{g}_\alpha$.

$u : \mathbb{G}_a \xrightarrow{\cong} B_u$ can be chosen arbitrarily.

n depends on u.

Note: Given $t \in T$, can also use

$u'(y) = t\,u(y)\,t^{-1} = u(\alpha(t)u)$ and $n' = t^2 n$.

Application

Idea: $B = \{s\,u(x)\} \cong \mathbb{G}_m \times \mathbb{G}_a$ (as variety)

$$G - B = B_u n B = \{u(y)\,n\,t\,u(z)\} \cong \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a .$$

Def Given $m \in \{1,2\}$, define

$H^{(m)} = (\mathbb{G}_m \times \mathbb{G}_a) \amalg (\mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a)$

with binary operation:

$$(s,x)\cdot(t,z) = (st,\ t^{-m}x + z)$$

$$(s,x)\cdot(y,t,z) = (s^m(x+y),\ s^{-1}t,\ z)$$

$$(0,1,0)\cdot(t,z) = (0,t,z)$$

$$(0,1,0)\cdot(y,t,z) = \begin{cases} ((-1)^{2/m}t,\ z) & \text{if } y = 0 \\ (-y^{-1},\ (-y)^{2/m}t,\ -y^{-1}t^{-m}+z) & \text{if } y \neq 0 \end{cases}$$

$$(v,s,x)\cdot h = (1,v)\cdot(0,1,0)\cdot(s,x)\cdot h, \quad h \in H^{(m)}.$$

Let G, u, $\alpha^\vee$, α, n be as in Thm. Define maps of sets:

$$\phi : H^{(2)} \longrightarrow G; \quad (s,x) \longmapsto \alpha^\vee(s)\,u(x)$$
$$(v,s,x) \longmapsto u(v)\,n\,\alpha^\vee(s)\,u(x)$$

$$\psi : G \longrightarrow H^{(1)}; \quad t\,u(z) \longmapsto (\alpha(t), z)$$
$$u(y)\,n\,t\,u(z) \longmapsto (y, \alpha(t), z)$$

Exer: (1) - (4) $\Rightarrow$

$$\phi(h_1 \cdot h_2) = \phi(h_1)\phi(h_2)$$
$$\psi(g_1 g_2) = \psi(g_1)\cdot\psi(g_2)$$

Note: $\mathbb{G}_m \xrightarrow{\alpha^\vee} T \xrightarrow{\alpha} \mathbb{G}_m$

$\langle \alpha, \alpha^\vee \rangle = 2 \Rightarrow \alpha^\vee$ iso. or α iso.

Assume $\alpha^\vee$ isomorphism:

$\phi : \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a \xrightarrow{\cong} G - B$ iso. of varieties.

$\phi : H^{(2)} \longrightarrow G$ bijective.

$\therefore \; SL_2 \xrightarrow{\phi^{-1}} H^{(2)} \xrightarrow{\phi} G$ iso. of LAGs.

Assume α isomorphism:

$\psi : G - B \xrightarrow{\cong} \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a$ iso. of varieties.

$\psi : G \longrightarrow H^{(1)}$ bijective.

$\therefore \; G \xrightarrow{\psi} H^{(1)} \xrightarrow{\psi^{-1}} PGL_2$ iso. of LAGs.

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Proof of Thm

Identify $T = \mathbb{G}_m = \mathbb{G}_a - \{0\} = k^\times$.

Choose $u : \mathbb{G}_a \xrightarrow{\cong} B_u$.

For some $m \in \mathbb{Z}$: $t\,u(x)\,t^{-1} = u(t^m x)$ $\forall t \in T$, $x \in \mathbb{G}_a$.

WLOG: $m > 0$. (Replace $t \mapsto t^{-1}$)

Choose $n \in N_G(T) - T$.

$\bar{n} \in W(G,T)$ has order 2 $\Rightarrow$

$$n^2 \in T \quad \text{and} \quad n t n^{-1} = t^{-1} \quad \forall t \in T.$$

$\therefore$ $n^2 = n\, n^2 n^{-1} = n^{-2} \Rightarrow n^4 = 1.$

$$\mathbb{G}_a \times T \times \mathbb{G}_a \xrightarrow{\cong} B_u n B = G - B$$
$$(x, t, z) \longmapsto u(x)\, n\, t\, u(z)$$

$B_u \cap n B_u n^{-1} = e \Rightarrow n\, u(y)\, n^{-1} \in G - B \quad \forall y \in T.$

$\exists\, (f, g, h) : T \longrightarrow \mathbb{G}_a \times T \times \mathbb{G}_a$

$$n\, u(y)\, n^{-1} = u(f(y))\, n\, g(y)\, u(h(y)) \quad \forall y \in T.$$

Note: $f(1) \neq 0$, since otherwise

$$u(1)\, n^{-1} = g(1)\, u(h(1)) \Rightarrow n \in B \ \lightning$$

$$\begin{aligned} t\, n\, u(y)\, n^{-1} t^{-1} &= t\, u(f(y))\, n\, g(y)\, u(h(y))\, t^{-1} \\ &= u(t^m f(y))\; t n\; t^{-1} g(y)\; u(t^m h(y)) \end{aligned}$$

Choose $t \in \mathbb{G}_m$ such that $t^m f(1) = -1$.

Replace:
$$\begin{aligned} n &\longmapsto tn \\ f &\longmapsto [y \mapsto t^m f(y)] \\ g &\longmapsto [y \mapsto t^{-1} g(y)] \\ h &\longmapsto [y \mapsto t^m h(y)] \end{aligned}$$

WLOG: $f(1) = -1$ and
$$n\,u(y)\,n^{-1} = u(f(y))\,n\,g(y)\,u(h(y)).$$

$$\begin{aligned} &u(f(-y))\,n\,g(-y)\,u(h(-y)) \\ &\quad = n\,u(-y)\,n^{-1} = (n\,u(y)\,n^{-1})^{-1} \\ &\quad = (u(f(y))\,n\,g(y)\,u(h(y)))^{-1} \\ &\quad = u(-h(y))\,g(y)^{-1}\,n^{-1}\,u(-f(y)) \\ &\quad = u(-h(y))\,n\,n^2 g(y)\,u(-f(y)) \end{aligned}$$

$\therefore\ f(-y) = -h(y)$ and $g(-y) = n^2 g(y)$.

$$\begin{aligned} &u(f(t^m y))\,n\,g(t^m y)\,u(h(t^m y)) \\ &\quad = n\,u(t^m y)\,n^{-1} = t^{-1}\,n\,u(y)\,n^{-1}\,t \\ &\quad = t^{-1}\,u(f(y))\,n\,g(y)\,u(h(y))\,t \\ &\quad = u(t^{-m} f(y))\,n\,t^2 g(y)\,u(t^{-m} h(y)) \end{aligned}$$

For $s = t^m$, $y = 1$:
$$\begin{aligned} f(s) &= s^{-1} f(1) = -s^{-1}. \\ h(s) &= -f(-s) = -s^{-1}. \end{aligned}$$

$$\therefore\ n\,u(y)\,n^{-1} = u(-y^{-1})\,n\,g(y)\,u(-y^{-1})$$

$$\begin{aligned} u(-(y+1)^{-1})\, n B &= n\, u(y+1)\, n^{-1} B \\ &= n\, u(y)\, n^{-1}\, n\, u(1)\, n^{-1} B \\ &= u(-y^{-1})\, n\, g(y)\, u(-y^{-1})\, u(-1)\, n B \\ &= u(-y^{-1})\, g(y)^{-1}\, n\, u(-y^{-1}-1)\, n^{-1} B \\ &= u(-y^{-1})\, g(y)^{-1}\, u((y^{-1}+1)^{-1})\, n B \\ &= u(-y^{-1})\, u(g(y)^{-m}(y^{-1}+1)^{-1})\, n B \end{aligned}$$

$$\therefore\; -(y+1)^{-1} = -y^{-1} + g(y)^{-m}(y^{-1}+1)^{-1}$$

$$\Rightarrow\; g(y)^{-m} = \left(y^{-1} - (y+1)^{-1}\right)\left(y^{-1}+1\right) = y^{-2}$$

$m = 1$: $g(y) = y^2$

$m = 2$: $g(y)^2 = y^2$, $g(y) = \varepsilon y$, $\varepsilon = \pm 1$.

$n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \varepsilon y\, u(-y^{-1})$

Note: $\varepsilon\, u(y)\, \varepsilon^{-1} = u(\varepsilon^2 y) = u(y)$.

Replace: $n \mapsto n\varepsilon$. WLOG: $g(y) = y$.

Define: $\alpha : T \longrightarrow \mathbb{G}_m$, $\alpha(t) = t^m$

$\alpha^\vee : \mathbb{G}_m \longrightarrow T$, $\alpha^\vee(t) = g(t) = t^{2/m}$

Now:
- $\langle \alpha, \alpha^\vee \rangle = 2$
- $t\, u(x)\, t^{-1} = u(\alpha(t)x)$
- $n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1})$

$$(-y)^{2/m} = g(-y) = n^2 g(y) = n^2 y^{2/m}.$$

$m = 1$: $n^2 = 1 = \alpha^\vee(-1)$. $m = 2$: $n^2 = -1 = \alpha^\vee(-1)$.

□

Prop G reductive LAG.

(1) $R(G) = Z(G)^\circ$ is a central torus.

(2) $R(G) \cap (G,G)$ is finite.

Proof

$R(G)$ connected solvable, $R(G)_u = e \Rightarrow R(G)$ torus.

$Z_G(R(G))^\circ = N_G(R(G))^\circ = G \Rightarrow R(G) \subseteq Z(G)$.

$Z(G)^\circ \subseteq G$ closed conn. solvable normal $\Rightarrow Z(G)^\circ \subseteq R(G)$.

$G \subseteq GL(V)$ closed.

$$V = \bigoplus_{\chi} V_\chi \quad , \qquad \chi \in X^*(R(G)).$$

$v \in V_\chi$, $g \in G$, $s \in R(G) \Rightarrow s.(g.v) = g.(s.v) = \chi(s)\, g.v$

$\therefore$ $G.V_\chi = V_\chi$.

Let $g \in R(G) \cap (G,G)$.

$g : V_\chi \longrightarrow V_\chi$ mult. by $\chi(g)$.

$g \in (G,G) \Rightarrow 1 = \det(g : V_\chi \to V_\chi) = \chi(g)^{\dim(V_\chi)}$.

$\square$

Note: G not solvable $\Rightarrow \dim(G) \geq 3$.

Must have $e \neq T \subsetneq B \subsetneq G$.

Reductive of semi-simple rank 1

Assume G is reductive, $\mathrm{ssrank}(G) = 1$.

Fix $T \subseteq B \subseteq G$, T max. torus, B Borel.

(1) $\dim(G/T) = 2$, $\dim(G/B) = 1$.

$G/R(G)$ semi-simple of rank 1.

$\dim(G/R(G)) = 3$. $\dim(T/R(G)) = 1$.

(2) $Z_G(T) = T$.

$T \subseteq Z_G(T) \subsetneq B$.

(3) (G,G) semi-simple of rank 1.

(G,G) not solvable $\Rightarrow \dim(G,G) \geq 3$.

$(G,G) \longrightarrow G/R(G)$ finite and surjective.

$R((G,G)) \longrightarrow \{e\} \Rightarrow R((G,G)) = e$.

Must have $\mathrm{rank}(G,G) = 1$.

(4) $T_1 = (G,G) \cap T$, $B_1 = (G,G) \cap B = T_1 \ltimes B_u$.

$T_1 \subseteq B_1 \subseteq (G,G)$ max. torus, Borel.

$R(G) \subseteq T$. $(G,G) \twoheadrightarrow G/T \twoheadrightarrow G/B$.

$\dim((G,G) \cap T) = 1$, $\dim((G,G) \cap B) = 2$.

$T_1^\circ \subseteq B_1^\circ \subseteq (G,G)$ max. torus, Borel. $(B_1 := (G,G) \cap B.)$

$T_1 \subseteq Z_{(G,G)}(T_1^\circ) = T_1^\circ$.

$\dim B_u = \dim(B_1^\circ)_u = 1 \Rightarrow B_u = (B_1^\circ)_u \subseteq (G,G)$.

$B_1 = (G,G) \cap (T \ltimes B_u) = T_1 \ltimes B_u$.

<u>Note</u>: $T = T_1 R(G)$, $B = B_1 R(G)$.

Choose $u : \mathbb{G}_a \xrightarrow{\cong} B_u$.

$\exists!\, \alpha \in X^*(T) : \ t u(x) t^{-1} = u(\alpha(t) x) \quad \forall\, t \in T,\ x \in \mathbb{G}_a$.

(5) $L(G) = L(T) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$; $\quad \mathfrak{g}_\alpha = L(B_u)$.

$B = T \ltimes B_u$. $\quad L(B) = L(T) \oplus L(B_u)$.

Def: $R(G,T) = \{\alpha, -\alpha\}$ roots of (G,T).

Note: $\alpha \in X^*(T)$ root of (G,T) $\Leftrightarrow$ $\alpha \neq 0$ and $L(G)_\alpha \neq 0$.

(6) $L(G,G) = L(T_1) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

(7) $W((G,G), T_1) \xrightarrow{\cong} W(G,T)$

$s_\alpha \in W$ represented by $\dot{s}_\alpha \in N_{(G,G)}(T_1)$.

(8) $\alpha^\vee \in X_*(T_1) \subseteq X_*(T)$:

Def. $\alpha^\triangledown : \mathbb{G}_m \longrightarrow T_1$ by $\langle \alpha, \alpha^\triangledown \rangle = 2$.

Show: $s_\alpha . \alpha^\triangledown = -\alpha^\triangledown$.

$(s_\alpha . \alpha^\triangledown)(t) = s_\alpha . (\alpha^\triangledown(t)) = \dot{s}_\alpha \alpha^\triangledown(t) \dot{s}_\alpha^{-1} = \alpha^\triangledown(t)^{-1}$.

(9) $\exists\, n \in N_{(G,G)}(T_1) - T_1$ such that:

$n^2 = \alpha^\vee(-1)$ and

$n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1}) \quad \forall\, y \in \mathbb{G}_m$.

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Note: Assume $T \subseteq B \subseteq G$, B Borel.

Any character $\chi \in X^*(T)$ extends uniquely to $\chi : B \longrightarrow \mathbb{G}_m$.

$B = T \ltimes B_u$. $\chi(tu) = \chi(t)$.

Prop G semi-simple of rank 1, $T \subseteq B \subseteq G$, B Borel,

$L(B) = L(T) \oplus L(G)_\alpha$.

Assume $\chi \in X^*(T)$, $0 \neq f \in k[G]$ satisfy:

$$\forall g \in G,\ b \in B :\ f(gb) = \chi(b) f(g). \quad \text{Then:}$$

(1) $\langle \chi, \alpha^\vee \rangle \geq 0$.

(2) $\langle \chi, \alpha^\vee \rangle = 0 \Leftrightarrow f$ constant.

Proof

$n B_u n^{-1} B \subseteq G$ dense subset $\Rightarrow$ f not zero on $n B_u n^{-1}$.

$$\begin{aligned}\underbrace{f(n\,u(y)\,n^{-1})}_{\text{poly}(y)\,\neq\,0} &= f(u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1})) \\ &= y^{\langle \chi, \alpha^\vee \rangle}\, \underbrace{f(u(-y^{-1})\, n)}_{\text{poly}(y^{-1}).}\end{aligned}$$

$\therefore$ $\langle \chi, \alpha^\vee \rangle \geq 0$.

$\langle \chi, \alpha^\vee \rangle = 0 \Rightarrow f(n\,u(y)\,n^{-1})$ constant

$\Rightarrow f$ constant $\Rightarrow \langle \chi, \alpha^\vee \rangle = 0$.

□

Assume $\operatorname{ssrank}(G) = 1$:

$T \subset B \subset G$ max. torus, Borel.

$$\begin{tikzcd}
G \arrow[r, "\pi"] & \bar{G} = G/R_u(G) & \\
\cup & \cup & \\
B \arrow[r] & \bar{B} = B/R_u(G) & \\
\cup & \cup & \\
T \arrow[r, "\cong", "\pi"'] & \bar{T} \arrow[r, "\bar{\alpha}"] & \mathbb{G}_m \\
& \mathbb{G}_m \arrow[u, "\bar{\alpha}^\vee"'] &
\end{tikzcd}$$

$R(\bar{G}, \bar{T}) = \{\pm \bar{\alpha}\}$.

Def: $R(G,T) = \{\pm \alpha\}$, $\alpha = \bar{\alpha}\pi$.

Note: Given $\beta \in X^*(T)$:

$\beta \in R(G,T) \Leftrightarrow \beta \neq 0$ and $L(G/R_u(G))_\beta \neq 0$.

Note: $\alpha^\vee = \pi^{-1}\bar{\alpha}^\vee \in X_*(T)$.

Roots

G connected LAG. $T \subseteq G$ max. torus.

Given $\alpha \in X^*(T)$:

$T_\alpha = \operatorname{Ker}(\alpha)^\circ \subseteq T$, $G_\alpha = Z_G(T_\alpha)$, $R_\alpha = R_u(G_\alpha)$.

Recall: $\operatorname{ssrank}(G_\alpha) \leq 1$.

Def: $R(G,T) = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } L(G_\alpha/R_\alpha)_\alpha \neq 0\}$.

Lemma $\alpha \neq \beta \in R(G,T) \Rightarrow \alpha^\vee \neq \beta^\vee$.

Proof

Assume $\alpha^\vee = \beta^\vee$.

Let $(-,-)$ be W-invariant, sym., pos. definite.

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \langle \alpha, \beta^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2 = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$$

$$\Rightarrow \quad (\alpha,\alpha) = (\alpha,\beta) = (\beta,\beta).$$

$$(\alpha-\beta, \alpha-\beta) = (\alpha,\alpha) - 2(\alpha,\beta) + (\beta,\beta) = 0.$$

$\square$

Root datum: $\Psi = \Psi(G,T) = (X, R, X^\vee, R^\vee)$

$X = X^*(T) \cong \mathbb{Z}^n$ lattice.

$R = R(G,T) \subseteq X$.

$X^\vee = X_*(T)$ dual lattice.

$R^\vee = \{\alpha^\vee \mid \alpha \in R\}$

Implicit: Bijection $R \longleftrightarrow R^\vee$, $\alpha \leftrightarrow \alpha^\vee$.

Properties/Axioms

(1) $\langle \alpha, \alpha^\vee \rangle = 2 \quad \forall \alpha \in R$.

(2) $s_\alpha . R = R$ and $s_\alpha . R^\vee = R^\vee$.

True since $G_{s_\alpha . \beta} = \dot{s}_\alpha G_\beta \dot{s}_\alpha^{-1}$.

$\Psi(R,T)$ is **reduced**: $\alpha \in R \Rightarrow \mathbb{R}\alpha \cap R = \{\pm\alpha\}$.

True since $\beta \in \mathbb{R}\alpha \Rightarrow T_\beta = T_\alpha \Rightarrow G_\beta = G_\alpha$.

Notes

- R is a root system in $\mathrm{Span}_{\mathbb{R}}(R) \subseteq X^*(T)_{\mathbb{R}}$.
- Dual root datum $(X^\vee, R^\vee, X, R)$ satisfies same axioms.
- Weyl group of $(X, R, X^\vee, R^\vee)$: $W = \langle s_\alpha : \alpha \in R \rangle \subseteq \mathrm{Aut}(X)$.

Positive roots

Let $\alpha \in R$. $L(G_\alpha/R_\alpha) = L(T) \oplus k_\alpha \oplus k_{-\alpha}$

$\mathcal{B}(G_\alpha)^T = \{B_\alpha, B_{-\alpha}\}$.

$B_\alpha \subseteq G_\alpha$ unique Borel s.t. $T \subseteq B_\alpha$ and $L(B_\alpha/R_\alpha)_\alpha \neq 0$.

Let $B \subseteq G$ Borel, $T \subseteq B \subseteq G$.

$R^+(B) = \{\alpha \in R \mid B_\alpha \subseteq B\}$, $R^-(B) = \{\alpha \in R \mid B_{-\alpha} \subseteq B\}$.

<u>Note</u>:
- $R(G,T) = R^+(B) \amalg R^-(B)$
- $\alpha \in R^+(B) \Leftrightarrow B_\alpha = G_\alpha \cap B \Leftrightarrow L(G_\alpha \cap B/R_\alpha)_\alpha \neq 0$.

<u>Prop</u> $R^+(B) \subseteq R(G,T)$ is a system of positive roots:

$$\exists\, \chi \in X^*(T) : \ R^+(B) = \{\alpha \in R \mid \langle \chi, \alpha^\vee \rangle > 0\}.$$

<u>Proof</u>

$G/B \subseteq \mathbb{P}(V)$ equiv. proj. embedding.

$G \longrightarrow GL(V)$ rat. rep., $1.B = [v] \in \mathbb{P}(V)^B$.

$\exists\, \chi : B \longrightarrow \mathbb{G}_m : \ b.v = \chi(b)v \ \ \forall b \in B$.

Choose $\ell : V \longrightarrow k$ linear such that

$\ell(n_\alpha . v) \neq \ell(v) \ \ \forall \alpha \in R$ where $n_\alpha \in (G_\alpha, G_\alpha) \setminus B$.

Def. $F : G \longrightarrow k$, $F(g) = \ell(g.v)$.

$F \in k[G]$ and $F(gb) = \chi(b)F(g) \ \ \forall g \in G,\ b \in B$.

Let $\alpha \in R^+(B)$. $\overline{G}_\alpha = G_\alpha/R_\alpha$ reductive, $\operatorname{ssrank}(\overline{G}_\alpha) = 1$.

$\chi(R_\alpha) = 1 \Rightarrow F|_{G_\alpha} \in k[G_\alpha]^{R_\alpha} = k[\overline{G}_\alpha]$.

$B_\alpha \subseteq B \Rightarrow F(gb) = \chi(b)F(g) \ \ \forall g \in \overline{G}_\alpha,\ b \in \overline{B}_\alpha$.

$F(n_\alpha) \neq F(e) \Rightarrow F|_{G_\alpha}$ not constant on $(\overline{G}_\alpha, \overline{G}_\alpha)$.

$\therefore \langle \chi, \alpha^\vee \rangle > 0$.

$\square$

Two roots

$(X, R, X^\vee, R^\vee)$ root datum.

Assume $\alpha, \beta \in R$ not parallel.

$\sigma = s_\alpha s_\beta \in GL(V)$, $V = \mathrm{Span}_{\mathbb{R}}\{\alpha, \beta\}$.

$\sigma \neq 1$ and $\det(\sigma) = 1 \Rightarrow \chi_\sigma(1) \neq 0$.

$a = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$.

Prop $a \in \{0, 1, 2, 3\}$, $\mathrm{ord}(\sigma) \in \{2, 3, 4, 6\}$,
and $\langle \alpha, \beta^\vee \rangle = 0 \Leftrightarrow \langle \beta, \alpha^\vee \rangle = 0$.

Proof

$$\begin{aligned} \sigma.\alpha &= s_\alpha.(\alpha - \langle \alpha, \beta^\vee \rangle \beta) &&= (a-1)\alpha - \langle \alpha, \beta^\vee \rangle \beta \\ \sigma.\beta &= -s_\alpha.\beta &&= \langle \beta, \alpha^\vee \rangle \alpha - \beta \end{aligned}$$

$\chi_\sigma(\lambda) = (\lambda + 1 - a)(\lambda + 1) + a = (\lambda+1)^2 - a(\lambda+1) + a.$

$\chi_\sigma(\lambda) = 0 \Leftrightarrow \lambda = \frac{1}{2}a - 1 \pm \frac{1}{2}\sqrt{a^2 - 4a}.$

$\sigma \in W$ has finite order $\Rightarrow$ σ diagonalizable $/\mathbb{C}$.

Eigenvalues are roots of unity $\neq 1$.

$a < 0$ or $a \geq 4$: Real eigenvalue $\neq -1$. ↯

$a = 0$: $\lambda = -1$, $\sigma = -1$. $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$.

$a = 1$: $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. $\mathrm{ord}(\sigma) = 3$.

$a = 2$: $\lambda = \pm\sqrt{-1}$ $\mathrm{ord}(\sigma) = 4$.

$a = 3$: $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. $\mathrm{ord}(\sigma) = 6$.

$\square$

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Assume $\alpha, \beta \in R$ not parallel.

Cor $\langle \alpha, \beta^\vee \rangle < 0 \Rightarrow \alpha + \beta \in R$

$\langle \alpha, \beta^\vee \rangle > 0 \Rightarrow \alpha - \beta \in R$

Proof

$\langle \alpha, \beta^\vee \rangle = -1 \Rightarrow \alpha + \beta = s_\beta.\alpha \in R.$

$\langle \beta, \alpha^\vee \rangle = -1 \Rightarrow \alpha + \beta = s_\alpha.\beta \in R.$

$\langle \alpha, \beta^\vee \rangle > 0 \Rightarrow \langle -\alpha, \beta^\vee \rangle < 0 \Rightarrow -\alpha + \beta \in R.$

□

Let $R^+ \subseteq R$ be a choice of positive roots.

Cor $\exists\, w \in \langle s_\alpha, s_\beta \rangle : w.\alpha > 0$ and $w.\beta > 0$.

Proof

$\sigma = s_\alpha s_\beta.$

$\sigma^{12} = 1$ and $\chi_\sigma(1) \neq 0 \Rightarrow 1 + \sigma + \cdots + \sigma^{10} + \sigma^{11} = 0.$

WLOG: $\beta > 0$ (Else replace $\alpha \mapsto s_\beta.\alpha$, $\beta \mapsto s_\beta.\beta$.)

Choose $d \geq 0$ minimal with $\sigma^d.\alpha > 0$ or $\sigma^d s_\alpha.\beta > 0$.

$\sigma^d.\alpha < 0$: $w = \sigma^d s_\alpha$ works.

$\sigma^d.\alpha > 0$: $w = \sigma^d$ works

since $\sigma^d.\beta = \sigma^{d-1} s_\alpha.(-\beta) > 0.$

□

Unipotent Radical

G connected LAG, $T \subseteq G$ max. torus.

$$C = \Big(\bigcap_{\substack{B \supset T \\ \text{Borel}}} B_u \Big)^{\circ}$$

Given $\alpha \in R(G,T)$, set $C_\alpha = \langle C, (B_\alpha)_u \rangle$.

Lemma C_α is unipotent and $\dim(C_\alpha / C) \leq 1$.

Proof

T normalizes C and C_α.

$L(C) = \bigoplus L(C)_\gamma \subseteq L(C_\alpha) = \bigoplus L(C_\alpha)_\gamma, \quad \gamma \in X^*(T)$.

Show: $L(C)_\gamma = L(C_\alpha)_\gamma$ for $\gamma \neq \alpha$ and
$\dim L(C_\alpha)_\alpha \leq \dim L(C)_\alpha + 1$.

$\gamma = 0$: $L(G)_0 = L(G_0), \quad G_0 = Z_G(T) = T \times Z_G(T)_u$
$Z_G(T)_u \subseteq C$. $L(C)_0 = L(Z_G(T)_u) = L(C_\alpha)_0$.

Assume $\gamma \neq 0$.

Note: $B \supset T \Rightarrow R_\gamma \subseteq (B \cap G_\gamma)_u \subseteq B_u$.
$\therefore R_\gamma \subseteq C$.

$L(R_\gamma)_\gamma \subseteq L(C)_\gamma \subseteq L(G)_\gamma = L(G_\gamma)_\gamma$

$\dim\left(L(G)_\gamma / L(C)_\gamma\right) \leq \dim L(G_\gamma / R_\gamma)_\gamma \leq 1$.

Enough: $L(C)_\gamma \subsetneq L(C_\alpha)_\gamma \Rightarrow \gamma = \alpha$.

Assume $L(C)_\gamma \subsetneq L(C_\alpha)_\gamma$ and $\gamma \neq \alpha$.

Then $\gamma \in R(G,T)$ and $L(C_\alpha)_\gamma = L(G)_\gamma$.

Choose $B \supset T$.

Choose $w \in W$ s.t. $w.\alpha \in R^+(B)$, $w.\gamma \in R^-(B)$.

Replace $B \mapsto \dot{w}^{-1} B \dot{w}$.

$\alpha \in R^+(B)$, $\gamma \in R^-(B)$. $B_\alpha \subseteq B$, $B_{-\gamma} \subseteq B$.

$C_\alpha = \langle C, (B_\alpha)_u \rangle \subseteq B_u \Rightarrow C_\alpha$ unipotent.

$L(G_\gamma / R_\gamma) = L(T) \oplus k_\gamma \oplus k_{-\gamma}$

$L(B_{-\gamma}/R_\gamma) = L(T) \oplus k_{-\gamma}$

$\therefore\ L(C_\alpha)_\gamma \subseteq L(B)_\gamma = L(B_{-\gamma})_\gamma \subsetneq L(G_\gamma)_\gamma$. $\lightning$

$\square$

<u>Thm</u> $R_u(G) = \left(\bigcap_{B \supset T} B_u \right)^\circ$

<u>Proof</u>

$B \supset T \Rightarrow R_u(G) \subseteq B_u$

$\therefore\ R_u(G) \subseteq C = \left(\bigcap_{B \supset T} B_u \right)^\circ$.

C closed connected unipotent.

<u>Enough</u>: $C \lhd G$ normal.

G generated by G_γ for all $\gamma \in X^*(T)$.

G generated by T, C, C_α for all $\alpha \in R(G,T)$.

C_α conn. unipotent, $C \subsetneq C_\alpha$ proper closed conn.

$\Rightarrow C \subsetneq N_{C_\alpha}(C)^\circ$. (LAG 23)

$\therefore$ C normalized by C_α.

$\therefore$ $C \lhd G$ normal.

$\square$

Reductive groups

Assume G is reductive (and connected).

$T \subseteq G$ max. torus. $R = R(G,T)$.

(1) $S \subseteq G$ subtorus $\Rightarrow Z_G(S)$ is reductive.

WLOG $S \subseteq T$. $Z = Z_G(S)$. $T \subseteq Z$ max. torus.

$$R_u(Z) = \Big(\bigcap_{T \subseteq B' \subseteq Z} B'_u\Big)^\circ$$
$$= \Big(\bigcap_{T \subseteq B \subseteq G} (Z \cap B)_u\Big)^\circ \subseteq \Big(\bigcap_{B \supset T} B_u\Big)^\circ = e.$$

(2) $Z_G(T) = T$.

$Z_G(T)$ is reductive and nilpotent.

(3) $Z(G) \subseteq T$.

(4) $R = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } L(G)_\alpha \neq 0\}$

$\alpha \in$ RHS $\Rightarrow G_\alpha = Z_G(T_\alpha)$ is reductive.

$L(G_\alpha)_\alpha = L(G)_\alpha \neq 0$.

(5) $\alpha \in R \Rightarrow \dim L(G)_\alpha = 1$.

(6) $L(G) = L(T) \oplus \bigoplus_{\alpha \in R} L(G)_\alpha$

$L(B) = L(T) \oplus \bigoplus_{\alpha \in R^+} L(G)_\alpha$, $B \supset T$ Borel.

(7) $\dim(G) = \dim(T) + |R|$.

$\dim(B) = \dim(T) + \frac{1}{2}|R|$.

Connected LAG

G any connected LAG. $T \subseteq G$ max. torus.

$\pi : G \longrightarrow \bar{G} = G/R_u(G)$ reductive quotient.

Identify $T = \pi(T)$, max. torus in $\bar{G}$.

(1) $U \lhd G$ closed unipotent normal $\Rightarrow$ $U \subseteq R_u(G)$

$U^\circ \subseteq R_u(G)$.

$\pi(U) \subseteq \bar{G}$ finite normal unipotent.

$\pi(U) \subseteq Z(\bar{G}) \subseteq T$.

$\therefore$ $\pi(U) = e$.

Let $S \subseteq G$ be a subtorus. Set $Z = Z_G(S)$.

(2) $R_u(Z) = Z \cap R_u(G)$.

WLOG $S \subseteq T$.

$R_u(G) \lhd G$ normal

$\Rightarrow Z \cap R_u(G) \lhd Z$ normal & unipotent.

$\Rightarrow Z \cap R_u(G) \subseteq R_u(Z)$.

$$R_u(Z) = \Big(\bigcap_{T \subseteq B' \subseteq Z} B'_u\Big)^\circ$$

$$= \Big(\bigcap_{T \subseteq B \subseteq G} (Z \cap B)_u\Big)^\circ \subseteq R_u(G).$$

(3) $Z/R_u(Z) \xrightarrow{\subseteq} Z_{\bar{G}}(S)$ is injective.

(4) $\alpha \in X^*(T) \Rightarrow G_\alpha/R_\alpha \xrightarrow{\cong} \bar{G}_\alpha$.

$G_\alpha/R_\alpha \xrightarrow{\subseteq} \bar{G}_\alpha$.

Both groups are reductive of ssrank ≤ 1.

Same max. torus T.

$\operatorname{ssrank}(G_\alpha/R_\alpha) = 1 \Rightarrow \operatorname{ssrank}(\bar{G}_\alpha) = 1$.

Assume $\operatorname{ssrank}(\bar{G}_\alpha) = 1$.

$s_\alpha \in W(\bar{G}_\alpha, T) \subseteq W(\bar{G}, T) = W(G, T)$.

$\dot{s}_\alpha \in N_G(T)$ representative.

$T_\alpha \subseteq Z(\bar{G}_\alpha) \Rightarrow s_\alpha$ identity on T_α

$\Rightarrow \dot{s}_\alpha \in Z_G(T_\alpha) = G_\alpha$.

$\therefore\ s_\alpha \in W(G_\alpha, T) = W(G_\alpha/R_\alpha, T)$

$\Rightarrow \operatorname{ssrank}(G_\alpha/R_\alpha) = 1$

$L(T) \xrightarrow{\cong} L(\pi(T))$, $L(G_\alpha/R_\alpha)_\alpha \xrightarrow{\cong} L(\bar{G}_\alpha)_\alpha$.

(5) $R(G,T) = R(\bar{G},T)$.

$L(G_\alpha/R_\alpha)_\alpha \neq 0 \Leftrightarrow L(\bar{G}_\alpha)_\alpha \neq 0$.

(3^+) $Z/R_u(Z) \xrightarrow{\cong} Z_{\bar{G}}(s)$ isomorphism.

$R(Z,T) = \{\alpha \in R(G,T) \mid \alpha(s) = 1\} = R(Z_{\bar{G}}(s), T)$

$\dim(Z/R_u(Z)) = \dim(T) + |R(Z,T)| = \dim(Z_{\bar{G}}(s))$.

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From now: G reductive (connected) LAG. $T \subseteq G$ max. torus.

$R = R(G,T)$ root system.

$W = W(G,T) = N_G(T)/T$ Weil group.

Given $\alpha \in R$: $T_\alpha = \mathrm{Ker}(\alpha)^\circ$, $G_\alpha = Z_G(T_\alpha)$.

$B_\alpha \subseteq G_\alpha$ unique Borel s.t. $T \subseteq B_\alpha$ and $L(B_\alpha)_\alpha \neq 0$.

$U_\alpha = (B_\alpha)_u \cong \mathbb{G}_a$.

<u>Note</u>: $G = \langle T, U_\alpha : \alpha \in R \rangle$

$$L(G) = L(T) \oplus \Big(\bigoplus_{\alpha \in R} L(U_\alpha) \Big)$$

<u>Note</u>: $(G_\alpha, G_\alpha) = \langle U_\alpha, U_{-\alpha} \rangle$.

$H = \langle U_{\pm\alpha} \rangle \subseteq (G_\alpha, G_\alpha)$ by LAG 24.

$H \subsetneq (G_\alpha, G_\alpha) \Rightarrow \dim H = 2$

$\Rightarrow H \subseteq (G_\alpha, G_\alpha)$ Borel $\Rightarrow H_u \cong \mathbb{G}_a$ ↯

<u>Thm</u> G semi-simple $\Rightarrow G = (G,G) = \langle U_\alpha : \alpha \in R \rangle$

<u>Proof</u>

$$H = \Big(\bigcap_{\alpha \in R} \mathrm{Ker}(\alpha) \Big)^\circ \subseteq T.$$

$H \subseteq T_\alpha \subseteq Z(G_\alpha)$ for all $\alpha \in R$ $\Rightarrow H \subseteq Z(G)^\circ = e$.

$\Rightarrow \mathrm{Span}_{\mathbb{R}}(R) = X^*(T)_{\mathbb{R}} \Rightarrow T = \langle \alpha^\vee(\mathbb{G}_m) : \alpha \in R \rangle$.

$\alpha^\vee(\mathbb{G}_m) \subseteq (G_\alpha, G_\alpha) = \langle U_{\pm\alpha} \rangle$.

$\therefore$ $G = \langle T, U_\alpha : \alpha \in R \rangle = \langle U_\alpha : \alpha \in R \rangle$.

$U_\alpha \subseteq (G_\alpha, G_\alpha) \subseteq (G,G) \Rightarrow G = (G,G)$.

□

Cor G reductive.

(1) $G = R(G)(G,G)$.

(2) (G,G) is semi-simple.

(3) $(G,G) \twoheadrightarrow G/R(G)$ is finite and surjective.

Proof

$\bar{G} = G/R(G)$ semi-simple $\Rightarrow \bar{G} = (\bar{G},\bar{G}) \Rightarrow G = R(G)(G,G)$.

Recall: $R(G) \cap (G,G)$ is finite.

$U_\alpha \subseteq (G_\alpha, G_\alpha) \subseteq (G,G) \quad \forall \alpha \in R$.

$\therefore (G,G) \twoheadrightarrow G/R(G)$ finite & surjective.

$R((G,G)) \mapsto e \Rightarrow R((G,G)) = e$.

□

Prop. 1 H connected solvable LAG, $S \subseteq H$ max. torus, $U_1, \dots, U_n \subseteq H$ closed subgps. Assume:

(1) $U_i \cong \mathbb{G}_a$

(2) $L(H_u) = L(U_1) \oplus \cdots \oplus L(U_n)$.

(3) $S \subseteq N_G(U_i)$.

(4) $L(U_i) = L(H)_{\beta_i}$, $\beta_i \in X^*(S)$.

(5) $i \neq j \Rightarrow \beta_i \notin \mathbb{Q}\beta_j$.

Then $U_1 \times U_2 \times \cdots \times U_n \xrightarrow{\cong} H_u$ iso. of varieties.

$(u_1, u_2, \dots, u_n) \longmapsto u_1 u_2 \cdots u_n$.

Proof

Show: $S \times U_1 \times \cdots \times U_n \xrightarrow{\cong} H$

WLOG $n \geq 1$.

$\exists$ closed normal $N \triangleleft H$ s.t. $N \cong \mathbb{G}_a$ and $N \subseteq Z(H_u)$ (LAG 18).

$L(N) \subseteq L(H_u)$ S-stable subspace.

$L(N) = L(U_j)$ for some j.

$H' = Z_H(\mathrm{Ker}(\beta_j)^\circ)$.

$$L(H') = \bigoplus_{\gamma \in \mathbb{Q}\beta_j} L(H)_\gamma = L(S) \oplus L(U_j).$$

$S \ltimes N = H' = S \ltimes U_j$.

$\therefore$ $N = H'_u = U_j$.

WLOG: $N = U_n$ (since $N \subseteq Z(H_u)$.)

$\pi : H \longrightarrow H/U_n$. $\mathrm{Ker}(d\pi) = L(U_n)$.

Note: $U_i \cap U_n = e$ for $i \neq n$.

$U_i \xrightarrow{\cong} \pi(U_i)$, $S \xrightarrow{\cong} \pi(S)$.

Induction on $n \Rightarrow S \times U_1 \times \cdots \times U_{n-1} \xrightarrow{\cong} H/U_n$.

$\psi : S \times U_1 \times \cdots \times U_n \longrightarrow H$ bijective.

$d\psi_{(e,e,\ldots,e)}$ bijective $\Rightarrow$ ψ birational.

Zariski $\Rightarrow$ ψ isomorphism.

□

Fix total order on R.

Cor $T \subseteq B \subseteq G$, B Borel $\Rightarrow \prod_{\alpha \in R^+(B)} U_\alpha \xrightarrow{\cong} B_u$.

Choose $u_\alpha : \mathbb{G}_a \xrightarrow{\cong} U_\alpha$ for each $\alpha \in R$.

<u>Prop. 2</u> Let $\alpha, \beta \in R$, $\alpha \neq \pm\beta$. $\exists$ constants $c_{ij} \in k$:

$$(u_\alpha(x), u_\beta(y)) = \prod_{\substack{i\alpha + j\beta \in R \\ i,j > 0}} u_{i\alpha+j\beta}(c_{ij}\, x^i y^j)$$

<u>Proof</u>

WLOG: $\alpha, \beta \in R^+ = R^+(B)$, $T \subseteq B \subseteq G$.

$(u_\alpha(x), u_\beta(y)) \in B_u \Rightarrow (u_\alpha(x), u_\beta(y)) = \prod_{\gamma \in R^+} u_\gamma(P_\gamma(x,y))$.

$P_\gamma(x,y) = \sum_{i,j \geq 0} c^\gamma_{ij}\, x^i y^j \in k[x,y]$.

$$t\,(u_\alpha(x), u_\beta(y))\,t^{-1} = \prod_{\gamma \in R^+} t\, u_\gamma(P_\gamma(x,y))\, t^{-1}$$

$$\Rightarrow (u_\alpha(\alpha(t)x), u_\beta(\beta(t)y)) = \prod_{\gamma \in R^+} u_\gamma(\gamma(t)\, P_\gamma(x,y)).$$

$$\therefore\ \gamma(t)\, P_\gamma(x,y) = P_\gamma(\alpha(t)x, \beta(t)y) = \sum_{i,j \geq 0} c^\gamma_{ij}\, (\alpha(t)x)^i\, (\beta(t)y)^j$$

$$= \sum_{i,j \geq 0} (c^\gamma_{ij}\, x^i y^j)\, (i\alpha + j\beta)(t)$$

Linear independence of characters:

$$c^\gamma_{ij} \neq 0 \Rightarrow \gamma = i\alpha + j\beta,\ P_\gamma(x,y) = c^\gamma_{ij}\, x^i y^j.$$

<u>Note</u>: $e = (u_\alpha(x), u_\beta(0)) = u_\alpha(c^\alpha_{10} x)$.

$\therefore\ c^\alpha_{10} = 0$.

Symmetry: $c^\beta_{01} = 0$.

□

Choose system of positive roots $R^+ \subseteq R$.

Def The height of $\alpha \in R^+$ is

$$ht(\alpha) = \max\{n \in \mathbb{N} \mid \alpha \in \underbrace{R^+ + R^+ + \cdots + R^+}_{n}\}.$$

Prop. 3 Assume $A \subseteq R^+$ satisfies

$$\alpha, \beta \in A \Rightarrow (\mathbb{N}\alpha + \mathbb{N}\beta) \cap R^+ \subseteq A.$$

Then $H = \langle U_\alpha : \alpha \in A \rangle$ is unipotent and

$\prod_{\alpha \in A} U_\alpha \xrightarrow{\cong} H$ iso. of varieties.

Proof

Chose $\beta \in A$ with $ht(\beta)$ minimal.

$A' = A - \{\beta\}$, $H' = \langle U_\alpha : \alpha \in A' \rangle$.

Induction on $|A|$ $\Rightarrow$ H' unipotent and $\prod_{\alpha \in A'} U_\alpha \xrightarrow{\cong} H'$.

Prop. 2 $\Rightarrow$ U_β normalizes H'.

$\therefore$ $H = U_\beta \ltimes H'$.

$(H, H) \subseteq H'$ $\Rightarrow$ H solvable.

$U_\beta \subseteq H_u$ and $H_u \subseteq H$ subgroup $\Rightarrow$ $H = H_u$ unipotent.

T normalizes H $\Rightarrow$ $T \ltimes H$ solvable.

Prop. 1 $\Rightarrow$ $\prod_{\alpha \in A} \xrightarrow{\cong} H$.

□

Cor $R^+ \subseteq R$ system of positive roots

$\Rightarrow$ $B = \langle T, U_\alpha : \alpha \in R^+ \rangle \subseteq G$ Borel subgroup.

Proof

$H = \langle U_\alpha : \alpha \in R^+ \rangle$ is unipotent, $T \subseteq N_G(H)$.

$B = T \ltimes H$ is solvable, $\dim(B) = \dim(T) + |R^+|$.

□

Note: $\mathcal{B}^T = \{B \supseteq T\} \longleftrightarrow \{R^+ \subseteq R \text{ system of pos roots}\}$

$$B \longmapsto R^+(B)$$

$$\langle T, U_\alpha : \alpha \in R^+ \rangle \longleftarrow R^+$$

$W \circlearrowright \mathcal{B}^T$ simply transitive.

$R^+(w.B) = w.R^+(B)$

Def: $R^+, \tilde{R}^+$ are adjacent if $|R^+ \cap \tilde{R}^+| = |R^+| - 1$.

Lemma $R^+, \tilde{R}^+$ adjacent $\Rightarrow \exists!\ \beta \in R^+ : \tilde{R}^+ = s_\beta . R^+$

Proof

$A = R^+ \cap \tilde{R}^+$. $R^+ = A \cup \{\beta\}$, $\tilde{R}^+ = A \cup \{-\beta\}$.

Let $\alpha \in A$. $s_\beta . \alpha = \alpha - \langle \alpha, \beta^\vee \rangle \beta$.

$\langle \alpha, \beta^\vee \rangle \leq 0$: $s_\beta . \alpha \in R^+ \setminus \{\beta\} = A$.

$\langle \alpha, \beta^\vee \rangle \geq 0$: $s_\beta . \alpha \in \tilde{R}^+ \setminus \{-\beta\} = A$.

$\therefore s_\beta . A = A$. $s_\beta . \beta = -\beta$.

□