MATH 535 PROBLEM SET 10
DUE WEDNESDAY 11/29 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. (a) Let X be a complete variety and $f : X \to Y = \text{Spec-m}(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety X and a birational morphism $f : X \to Y$ such that $f_*\mathcal{O}_X$ is not locally free on Y.

Problem 2. (a) $Y \subset \mathbb{P}^n$ is a hypersurface of degree d with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Show that $\mathcal{I}_Y \cong \mathcal{O}(-d)$.

(b) Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding, $N = \binom{n+d}{n} - 1$. Show that $(v_d)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.

Problem 3. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be any non-constant morphism. Then $\dim \varphi(\mathbb{P}^n) = n$. Furthermore, φ is the composition of a Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$, a projection $\mathbb{P}(k^N) - \mathbb{P}(L) \to \mathbb{P}(k^N/L)$ for some linear subspace $L \subset k^N$, and an inclusion of a linear subspace $\mathbb{P}(k^N/L) \subset \mathbb{P}^m$.

Problem 4. (a) Let $\varphi : X \to Y$ be an affine morphism of pre-varieties. Show that if Y is separated then so is X.

(b) X is an irreducible affine variety, $U \subset X$ an open affine subset, $\bar{U} \subset \bar{X}$ their normalizations, and $\pi : \bar{X} \to X$ the normalization map. Show that $\pi^{-1}(U) = \bar{U}$.

(c) If X is any irreducible variety then $\pi : \bar{X} \to X$ is a finite morphism. Conclude that $\bar{X}$ is separated.

Problem 5. (a) If Y is a normal variety and $f : Y \to X$ a dominant morphism, then there exists a unique morphism $\bar{f} : Y \to \bar{X}$ such that $f = \pi \circ \bar{f}$.

(b) Give a counter example to (a) when f is not dominant.

Problem 6. $X = V(xy - z^2) \subset \mathbb{A}^3$ is normal. [Hint: $k[X] = k[x, xt, xt^2] \subset k(x, t)$ where $t = z/x$. Write this ring as the intersection of two normal rings.]

Problem 7. If X is any normal rational variety then $\mathrm{C}\ell(X)$ is a finitely generated Abelian group.