MATH 535 PROBLEM SET 8
DUE WEDNESDAY 11/8 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them. The first two are important, give them another try!

Problem 1. A morphism $f : X \to Y$ of varieties is called *affine* if for every open affine set $V \subset Y$ the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for all open affine $V \subset Y$.

Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\, i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

Hint: Note that the affine cover of Y can be refined by replacing V_i with smaller sets $(V_i)_h$ for $h \in k[V_i]$. If $V \subset Y$ is any open affine, then find $h_1, \dots, h_r \in k[V]$ such that $f^{-1}(V_{h_i})$ is affine for each i. Use problem 6 from PS 3.

Problem 2. Resolution of singularities for curves.

Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a "proper" morhism φ.)

Hint: Let $\tilde{X} \subset C_K$ be the maximal open subset where the birational map $\varphi : C_K \dashrightarrow X$ is defined as a morphism, $K = k(X)$. If $V \subset X$ is any open affine subvariety, then $\varphi^{-1}(V) = \text{Spec-m}(\overline{k[V]})$.

Problem 3. Let $\mathcal{F}$ be a sheaf on X and $p \in X$ a point. Prove the following from the definition of the stalk $\mathcal{F}_p$:

(a) Each element of $\mathcal{F}_p$ has the form s_p for some section $s \in \mathcal{F}(U)$, $p \in U$.

(b) Let $s \in \mathcal{F}(U)$, $p \in U$. Then $s_p = 0 \Leftrightarrow s|_V = 0$ for some $p \in V \subset U$.

(c) Let $s \in \mathcal{F}(U)$. Prove that $s = 0$ if and only if $s_p = 0\ \forall\, p \in U$.

Problem 4. [Hartshorne II.1.2]

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $U = \bigcup V_i$ of U and sections $t_i \in \mathcal{F}(V_i)$ such that $\varphi_{V_i}(t_i) = s|_{V_i}$ for all i.

Problem 5. [Hartshorne II.1.14]

Let $\mathcal{F}$ be a sheaf on X and $s \in \mathcal{F}(X)$ a global section. Show that the set $\{p \in X \mid s_p \neq 0\}$ is a closed subset of X.

Problem 6. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\operatorname{Im}(\varphi)_p = \operatorname{Im}(\varphi_p)$ for all $p \in X$.

Problem 7. Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ a sheaf on Y. Show that $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$.