MATH 535 PROBLEM SET 7
DUE WEDNESDAY 11/1 IN CLASS

Try to solve all of the following problems. Write up at least 3 of them.

Problem 1. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

has rank one. Show that X is a non-singular rational closed subvariety of $\mathbb{P}^5$, and find its dimension and degree.

Hint: Intersect X with a hyperplane and find the multiplicities and degrees of the components.

Problem 2. [Mostly Hartshorne I.7.1] In this problem, just find the numbers and give an argument why they are correct that *could* be expanded into a proof.

(a) Find the degree of $v_d(\mathbb{P}^n)$ in $\mathbb{P}^N$ where v_d is the Veronese embedding.

(b) Find the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$.

Problem 3. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X,P} \subset \mathcal{O}_{\mathbb{P}^2,P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2,P}$ such that $f|_{U \cap X} = 0$ for some open set U containing P. The multiplicity $\mu_P(X)$ of X at P is the largest number r such that $I_{X,P} \subset \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{\mathbb{P}^2,P}$ is the maximal ideal.

(a) $P \in X \Leftrightarrow \mu_P(X) \geq 1$.

(b) P is a non-singular point of X iff $\mu_P(X) = 1$.

(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(I_{X,P} + I_{Y,P})$.

(d) $I(X \cdot Y; P) = 1$ iff P is a non-singular point of both X and Y, and the tangent directions at P are different.

(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.

(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through P we have $\mu_P(X) = I(X \cdot L; P)$.

Problem 4. Let R be a Noetherian ring and M a finitely generated R-module. Show that every submodule of M is also finitely generated.

Problem 5. [Hartshorne I.7.2]

Let $Y \subset \mathbb{P}^n$ be an irreducible closed subvariety of dimension r. The *arithmetic genus* of Y is defined as $p_a(Y) = (-1)^r (P_Y(0) - 1)$, where P_Y is the Hilbert polynomial. It can be shown that $p_a(Y)$ does not depend on the embedding $Y \subset \mathbb{P}^n$.

(a) Show that $p_a(\mathbb{P}^n) = 0$.

(b) If $Y \subset \mathbb{P}^n$ is a hypersurface of degree d, then $p_a(Y) = \binom{d-1}{n}$.