MATH 535 PROBLEM SET 6
DUE WEDNESDAY 10/25 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. [Mostly Hartshorne I.6.3]
Give examples of varieties X and Y, a point $P \in X$, and a morphism $\varphi : X \smallsetminus \{P\} \to Y$ such that φ can't be extended to a morphism on all of X in each of the cases:
(a) X is a non-singular curve and Y is not projective.
(b) X is a curve, P is a singular point on X, Y is projective.
(c) X is non-singular of dimension at least two, Y is projective.

Problem 2. Let X and Y be curves and $\varphi : X \to Y$ a birational morphism.
(a) X_{sing} is a proper closed subset of X.
(b) $\varphi(X_{\text{sing}}) \subset Y_{\text{sing}}$.
(c) If $y \in Y$ is a non-singular point, then $\varphi^{-1}(y)$ contains at most one point.

Problem 3. Two non-singular projective curves are isomorphic if and only if they have the same function field.

Problem 4. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \smallsetminus \{P\}$ is affine.

Problem 5. [Hartshorne I.6.2]
Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, $\text{char}(k) \neq 2$.
(a) E is a non-singular curve.
(b) The units in $k[E]$ are the non-zero elements of k. [Hints: Define an automorphism $\sigma : k[E] \to k[E]$ fixing x and sending y to $-y$. Then define a norm $N : k[E] \to k[x]$ by $N(a) = a\,\sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$.]
(c) $k[E]$ is not a unique factorization domain.
(d) Show that E is not rational.

Problem 6. A morphism $f : X \to Y$ of varieties is called *affine* if for every open affine set $V \subset Y$ the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for all open affine $V \subset Y$.
Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\ i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

Problem 7. Resolution of singularities for curves.
Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a "proper" morhism φ.)

Problem 8. (a) Any unique factorization domain is integrally closed. In particular, any DVR is integrally closed.

(b) If R is an integrally closed domain and $S \subset R$ is any multiplicative subset, then the localization $S^{-1}R$ is integrally closed.

Problem 9. Let $K = k(t)$. Find all DVRs of K/k, i.e. all discrete valuation rings R such that $k \subset R \subset K$ and K is the field of fractions of R.