MATH 535 PROBLEM SET 5
DUE WEDNESDAY 10/18 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. [Hartshorne I.5.2]
Assume $\mathrm{char}(k) \neq 2$. Locate the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$, and $Z = V(xy + x^3 + y^3)$ in $\mathbb{A}^3$. (Take a look at the nice pictures in Hartshorne!)

Problem 2. Assume $\mathrm{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \dots, x_n]$.
(a) Show that $X_{\mathrm{sing}} = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$.
(b) Show that $X_{\mathrm{sing}} \neq X$.

Problem 3. [Shafarevich II.1.13]
(a) Show that an intersection of r hypersurfaces in $\mathbb{P}^r$ is never empty.
(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that X is singular. [Hint: Choose the coordinates on $\mathbb{P}^n$ such that $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n) \subset \mathbb{P}^n$.]

Problem 4. [Shafarevich II.1.10].
Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Problem 5. Let $m_0, m_1, \dots, m_N \in k[x_0, \dots, x_n]$ be all the monomials of degree d. The *Veronese embedding* is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$v_d(x_0 : \cdots : x_n) = (m_0(x_0, \dots, x_n) : \cdots : m_N(x_0, \dots, x_n)) .$$

(a) Show that v_d is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.
(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $S = V_+(f)$ where $f \in k[x_0, \dots, x_n]$ is a form of degree d. Show that $S = {v_d}^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 6. Let L_1, L_2, and L_3 be lines in $\mathbb{P}^3$ such that none of them meet.
(a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2, and L_3. [Hint: Start by applying an automorphism of $\mathbb{P}^3$ to make the lines nice.]
(b) S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2, and L_3.
(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2, or L_3. Then the number of lines meeting L_1, L_2, L_3, and L_4 is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.

Problem 7. Let G be an irreducible algebraic group acting on a variety X, i.e. we have a morphism $G \times X \to X$ such that the axioms for a group action are satisfied.
(a) Show that each orbit in X is locally closed.
(b) Each orbit is a non-singular variety.

Problem 8. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $F = R/\mathfrak{m}$.

(a) Any set of generators for the ideal $\mathfrak{m}$ contains at least $\dim(R)$ elements.
(b) $\dim_F(\mathfrak{m}/\mathfrak{m}^2)$ is the smallest possible number of generators for $\mathfrak{m}$.

[Hints: for (a), use the Principal Ideal Theorem: If S is a Noetherian ring with elements $f_1, \ldots, f_c \in S$, and if P is minimal among primes containing $f_1, \ldots, f_c$, then $\dim(S_P) \leq c$.

For (b), use Nakayama's Lemma: If $(R, \mathfrak{m})$ is a local ring and M a finitely generated R-module such that $\mathfrak{m}M = M$, then $M = 0$. Thanks to Charlie for fixing this hint!]