MATH 535 PROBLEM SET 4
DUE WEDNESDAY 10/11 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. Let E be the elliptic curve $V_+(y^2z-x^3+xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x:y:z) = (x:z)$ and $g(x:y:z) = (y:z)$. (These are just projections to the x and y axis on the open subset $D_+(z)$.)

(a) Find the maximal open sets in E where f and g are defined as morphisms.

(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g.

(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what "typical" means.)

Problem 2. Let X be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that φ is defined as a morphism on all of $\mathbb{P}^1$.

Problem 3. (a) If X has components $X_1, \dots, X_m$ then $\dim(X) = \max \dim(X_i)$.

(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 4. The commutative algebra result *lying over* states that if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\varphi : X \to Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y.

(b) If $\varphi : X \to Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y.

Problem 5. If X is a variety and $x \in X$, we define the Zariski cotangent space to X at x to be $\mathfrak{m}_x/\mathfrak{m}_x^2$. The Zariski tangent space is the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Show that if $f : X \to Y$ is a morphism of varieties with $f(x) = y$, then f induces linear maps $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ and $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$.

Problem 6. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \to G$ and $i : G \to G$, and an identity element $e \in G$, such that G is a group in the usual sense when m is used to define multiplication and i maps any element to its inverse element.

(a) Show that $\mathrm{GL}_n(k)$ is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that $\mathbb{P}^1$ is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $i : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the group axioms.

(d) Challenge: How about $\mathbb{P}^n$ for $n \geq 2$?