MATH 535 PROBLEM SET 3
DUE WEDNESDAY 10/4 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of P^N, where $N = nm + n + m$.

Problem 2. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.

Problem 3. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 4. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.
(a) Show that X is separated.
(b) Show that $\mathbb{P}^n$ has this property.

Problem 5. Let $\varphi : X \to Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 6. [Hartshorne II.2.16 and II.2.17]
Let X be any variety and $f \in k[X]$ a regular function.
(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i.]
(b) $k[D(f)] = k[X]_f$.
(c) Suppose $f_1, \ldots, f_r \in k[X]$ satisfy $(f_1, \ldots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.
[Hint for (c): Use Problem 5.]

Problem 7. Let $f : X \to Y$ be a continuous map of topological spaces, and let $W \subset X$ be a subset.
(a) $\overline{W} = X$ if and only if $W \cap U \neq \emptyset$ for every non-empty open subset $U \subset X$.
(b) If $\overline{W} = X$ and $\overline{f(X)} = Y$, then $\overline{f(W)} = Y$.
(c) If X is irreducible and $U \subset X$ is a non-empty open subset, then $\overline{U} = X$.
(d) W is irreducible if and only if $\overline{W}$ is irreducible. [By definition W is irreducible if, whenever $W \subset F_1 \cup F_2$ with $F_i \subset X$ closed, we have $W \subset F_i$ for some i.]
(e) If W is irreducible then $f(W)$ is irreducible.