MATH 535 PROBLEM SET 2
DUE WEDNESDAY 9/27 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

Problem 2. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \dots, x_n]$ for $n \geq 2$.
(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.
(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.
(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Problem 3. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 4. Let $X \subset \mathbb{A}^n$ be a closed subvariety. Identify $\mathbb{A}^n$ with $D_+(x_0) \subset \mathbb{P}^n$ and let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Show that $I(\overline{X}) = I(X)^* \subset k[x_0, \dots, x_n]$. ($I(X)^*$ was defined in class on 9/20.)

Problem 5. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 6. Show that if R is a finitely generated reduced k-algebra then the space with functions Spec-m(R) is an affine variety.

Problem 7. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \to X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to X$ is a morphism.

Problem 8. Let X and Y be spaces with functions and let (P, π_X, π_Y) and (P', π'_X, π'_Y) be two products of X and Y. Show that there is a unique isomorphism $\varphi : P \xrightarrow{\sim} P'$ such that $\pi_X = \pi'_X \circ \varphi$ and $\pi_Y = \pi'_Y \circ \varphi$.