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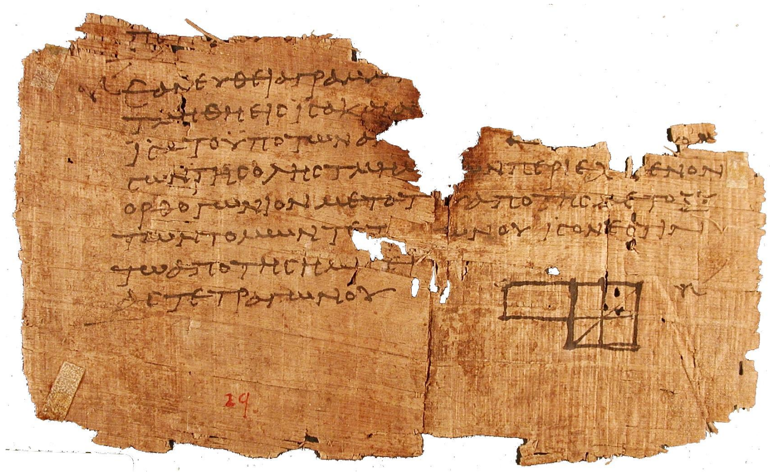
Pi Day? How about Archimedes Day!

by Alex Kontorovich

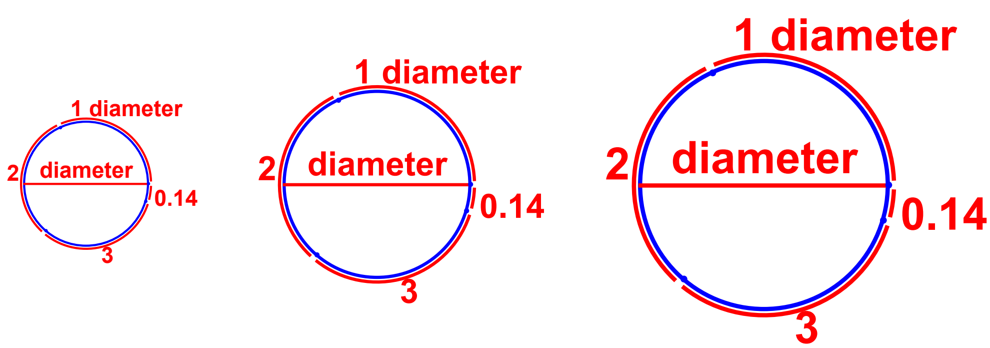
The number $\pi \approx 3.14$, that is, the ratio of circumference to diameter of any circle, has continually captivated mankind since at least the ancient Babylonians four millennia ago. It even appears in the Hebrew Bible, where King Solomon's Temple is described as containing a perfectly round ritual basin ("Molten Sea") measuring exactly 10 cubits from brim to brim and exactly 30 cubits around. This may suggest that π is identically $30/10 = 3$, a mistake in the scripture. But an ancient *midrash*, or commentary, by the 2^{nd} century Rabbi Nehemiah says that this passage in fact conceals in cypher the thickness of the basin! For the last 30 years or so, people around the world have been celebrating "Pi Day" annually on March 14th (3/14) with festivities including the baking of pies and contests of who can recite the most digits of π. But the "real" story of π is much more fascinating!

It starts shortly after 300 BCE, when the great Greek geometer Euclid of Alexandria wrote his *magnum opus*, "The Elements," until the 20th century the second most printed work of all time (behind the Bible). Here is a fragment of The Elements found in the Oxyrhynchus papyri, dated to around 100 CE:

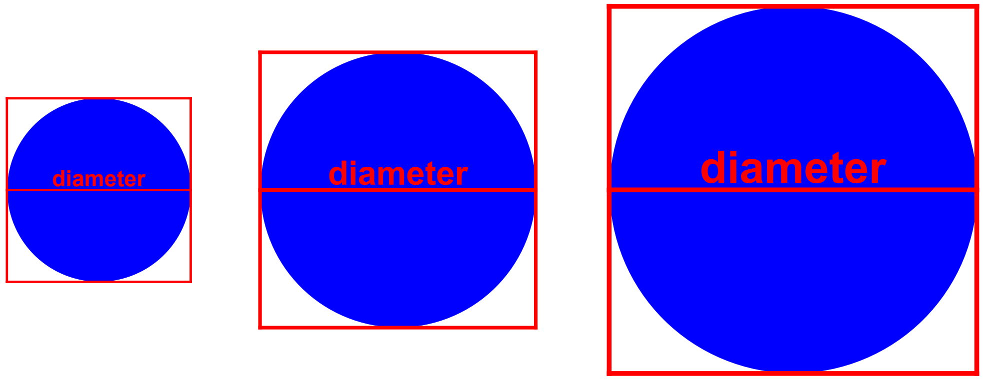


(image source: wikipedia)

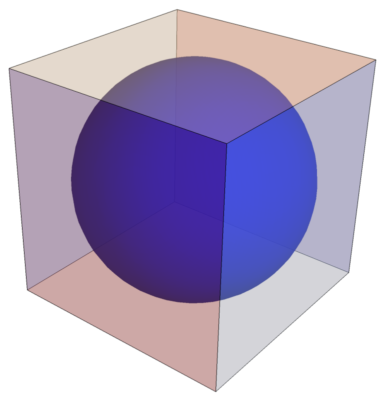
The Elements is a compendium of all known mathematics at the time, and includes, among many, many other "Propositions," the aforementioned but still striking fact that, regardless the size of a circle, its circumference is always some absolute constant (around 3.14) times its diameter:



Actually Euclid doesn't really prove this, but it can be derived from other statements in The Elements. But he certainly does prove (in Book XII, Proposition 2, stated here in more modern language) that: the area inside any circle fills up some constant proportion (around 79%) of the square on its diameter:



Euclid doesn't stop there; turning his attention to 3D space, he proves in Book XII Proposition 18 that a sphere fills up some constant (about 52%) proportion of the volume of the cube on its diameter:



There is no evidence whatsoever in The Elements that Euclid gave any thought to these constants, 3.14, 0.79, 0.52 – to him, they're as boring as $\frac{1}{4}$ or $\sqrt{2}$ – and there's certainly no indication at that time of the constants being in any way related.

The truly amazing thing about π, uncovered almost a century later by the brilliant Archimedes of Syracuse, is its **universality**: Once you name one of the constants, all the others can be expressed elementarily in terms of it:

$$\begin{aligned} 3.14 &\approx \pi, \\ 0.79 &\approx \pi/4, \\ 0.52 &\approx \pi/6. \end{aligned}$$

It is an accident of quite recent history – Leonhard Euler's popular calculus textbooks in the 18^{th} century – that the first one won the title "π." These miraculous discoveries of Archimedes are now basic facts drilled into school children in the more familiar reformulations:

$$\begin{aligned} \text{Circumference of a circle of radius } r &= 2\pi r, \\ \text{Area inside a circle of radius } r &= \pi r^2, \\ \text{Volume inside a sphere of radius } r &= \frac{4}{3}\pi r^3. \end{aligned}$$

It would have made Archimedes's head spin to think about it, but today lots of real-world applications require mathematicians to think about spaces with more dimensions than the three we (seem to) inhabit. With calculus, one can compute that the 4-D hypersphere takes up only

$$\frac{\pi^2}{32} \approx 31\%$$

of a hypercube on its diameter, and in 10 dimensions, that proportion goes down to

$$\frac{\pi^5}{122880} \approx 0.25\%.$$

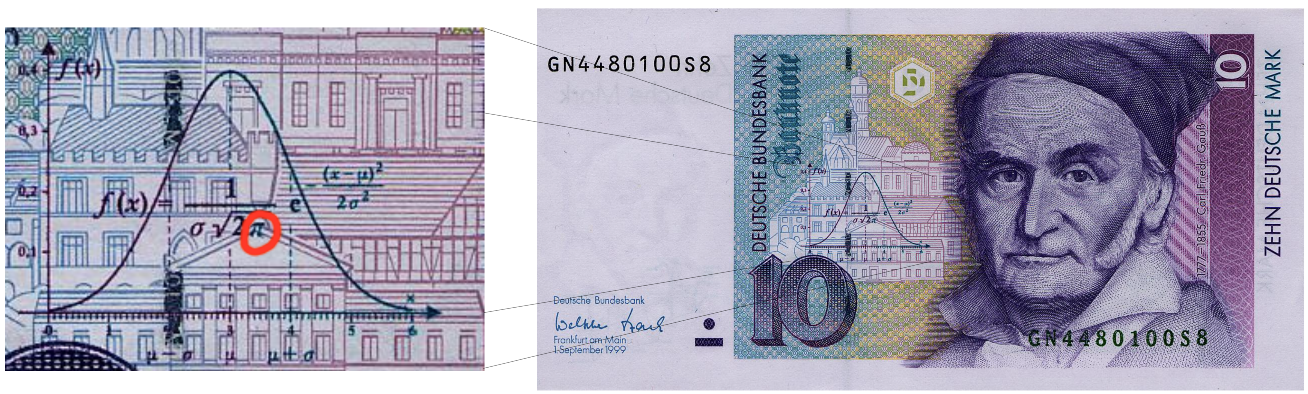
Until quite recently, the densest way to pack equal spheres in space was only known in up to 3 dimensions. In a stunning breakthrough[1] in 2016, Ukrainian mathematician Maryna Viazovska (now a Professor at the École Polytechnique Fédérale de Lausanne, Switzerland) and her collaborators solved the densest sphere packing problem in two more dimensions: 8 and 24. The proportion of space one can fill with spheres in 8-D is

$$\frac{\pi^4}{384} \approx 25\%,$$

and in 24-D it is

$$\frac{\pi^{12}}{479001600} \approx 0.2\%.$$

Perhaps you're thinking, "ok, π appears in all these places, because it has to do with 'round' things." But the plot thickens. Here is the now-discontinued German 10-Deutsche Mark banknote celebrating Carl Friedrich Gauss and the "standard normal distribution," being used for a financial transaction right now as you read these words.



An even more striking and unexpected appearance of π is in the resolution of the so-called "Basel Problem." Posed around 1650 by Italian mathematician Pietro Mengoli, the problem is to understand the exact value of the sum of the inverse squares:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots = \boxed{???} \approx 1.644934.$$

Sure, we can compute lots of its digits, but what exactly is this mystery number? This problem fascinated the Bernoulli family of Basel, Switzerland, especially rival brothers Johann and Jacob, who were the first "AP Calc" students, having learned the art from its co-creator (along with Sir Isaac Newton), Gottfried Wilhelm Leibniz. The Bernoullis were fascinated and utterly stumped until 1734, when the aforementioned Euler, their student, burst to international fame with the problem's resolution:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots = \boxed{\frac{\pi^2}{6}} \approx 1.644934.$$

What on earth do sums of inverse squares have to do with circles[2] or spheres?!

[1] The interested reader is encouraged to peruse the Quanta Magazine article https://www.quantamagazine.org/sphere-packing-solved-in-higher-dimensions-20160330

[2] See the answer on the popular youtube channel 3blue1brown at https://youtu.be/d-o3eB9sfls

By the same ingenious (and at first, not entirely rigorous) method, Euler simultaneously computed exactly the sum of inverse fourth-powers:

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots \;=\; 1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\cdots \;=\; \boxed{\frac{\pi^4}{90}} \;\approx\; 1.0823.$$

What about the sum of inverse cubes?

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\cdots \;=\; 1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots \;=\; \boxed{???} \;\approx\; 1.202.$$

Even armed with all our modern technological advances over the centuries and millennia, this stubborn problem remains as impenetrable today as in Euler's time!

In conclusion, although *a priori* all the above could have been unrelated mysterious constants of the universe (like, presumably, the last), these and many other formulae are understood today in terms of one fantastically fascinating number, π. And for that, we have the prescient Archimedes to thank. So on Pi Day, let's not recount the (totally arbitrary and meaningless) digits of π, but instead celebrate the life and work of Archimedes, one of the greatest minds ever to grace our Earth.

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