ERRATUM TO: ON REPRESENTATIONS OF INTEGERS IN THIN SUBGROUPS OF $\mathrm{SL}_2(\mathbb{Z})$ GAFA 20:5 (2010)

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Let $\Gamma < \mathrm{SL}(2,\mathbb{Z})$ be finitely generated, free, with no parabolic elements, and having critical exponent $1/2 < \delta < 1$. As such, there exists a finite symmetric set $\mathcal{A} := \{A_1, A_1^{-1}, A_2, A_2^{-1}, \dots, A_k, A_k^{-1}\}$ of generators with no relations, so that every $\gamma \in \Gamma$ is expressed uniquely as some reduced word $\gamma = B_1 B_2 \cdots B_m$ with $B_j \in \mathcal{A}$. (Reduced means no annihilations, $B_j B_{j+1} \neq I$). Mimicing [BK, §3, (3.2)], we construct the following exponential sum.

Fix $N \gg 1$ and $0 < \sigma < 1/4$. Let Ξ be a subset of Γ containing elements of Frobenius norm at most $N^{1/2}$, such that all elements $\xi \in \Xi$, when written as a reduced word in the generators of Γ, start with the same letter.

Similarly, let Π be a subset of Γ containing elements of Frobenius norm at most $N^{1/2-\sigma}$, and all elements $\varpi \in \Pi$, when written as a reduced word in the generators of Γ, ending in the same letter (ensuring that it is not the inverse of the letter which starts all elements of Ξ).

Then for $\theta \in [0,1]$, and primitive $v_0, w_0 \in \mathbb{Z}^2$, let

$$S_N(\theta) := \sum_{\xi\in\Xi} \sum_{\varpi\in\Pi} \sum_{\substack{\gamma\in\Gamma \\ \|\gamma\|<N^\sigma}} e\big(\langle v_0 \cdot \gamma\varpi\xi, w_0\rangle\theta\big) \,. \tag{3.2$'$}$$

(We have already changed the definitions of Ξ, Π and S_N to fix a minor notational inconsistency: in [BK, §3-4], every appearance of "$\xi\varpi$" should be replaced by "$\varpi\xi$", and similarly "${}^t\varpi\,{}^t\xi$" by "${}^t\xi\,{}^t\varpi$". The latter appearances are correct.)

The purpose of this construction was twofold. First, by the pigeonhole principle, Ξ and Π can be chosen so that they are "as large as they should be," that is, $|\Xi| \gg N^\delta$ and $|\Pi| \gg N^{\delta(1-2\sigma)}$. Second, and most importantly, the concatenation $\Pi \cdot \Xi$ should be unique, that is,

$$\text{if } \varpi\xi = \varpi'\xi', \text{ with } \varpi, \varpi' \in \Pi \text{ and } \xi, \xi' \in \Xi, \text{ then } \varpi = \varpi' \text{ and } \xi = \xi'. \tag{$*$}$$

By forcing the ending letter of ϖ to differ from the inverse of the starting letter of ξ, we have ensured that there is no annihilation in the concatenation $\varpi \cdot \xi$.

Unfortunately, this does not guarantee $(*)$. Indeed if $A, B \in \Gamma$ are free, then the concatenation of $\Pi = \{A, ABA\}$ with $\Xi = \{B, BAB\}$ contains both $A \cdot BAB$ and $ABA \cdot B$.

On the other hand, $(*)$ *would* be guaranteed if we restricted Ξ, say, to contain elements which, in addition to having the same starting letter, were also all of the same length.

So we amend the construction as follows. For a geometrically finite group with no parabolics, the wordlength metric $\ell(\cdot)$ is related to Frobenius norm $\|\cdot\|$ by

[F, Lem., p. 213]:

$$\log \|\gamma\| \ll \ell(\gamma) \ll \log \|\gamma\|\,,$$

with implied constants depending only on Γ. Hence in the ball in Γ of elements having Frobenius norm at most $N^{1/2}$, the wordlengths $\ell(\gamma)$ are dominated by a constant multiple of $\log N$. Then by the pigeonhole principle, there is a subset $\tilde{\Xi}$ consisting of words of the same wordlength, having size

$$|\tilde{\Xi}| \gg N^{\delta}/\log N.$$

We now proceed as before, selecting from within this set a subset Ξ consisting of elements with the same starting letter. The rest of the argument works in the same way, save a log loss in the major arcs estimate, where Theorem 4.1 is replaced by the estimate

$$\mathcal{M}_N(n) \gg \frac{1}{\log\log(|n|+10)} \frac{N^{2\delta-1}}{\log N}\,,$$

if n is admissible and not in the exceptional set.

Fortunately for us, such a loss is inconsequential, due to the power savings on the minor arcs (Theorem 8.18). The proof of Theorem 8.19 now bounds the size of the exceptional set by

$$|\mathfrak{E}(N)| \ll N^{1-\eta}(\log N)^2(\log\log N)^2,$$

which is still plenty for our purposes.

References

[BK] J. Bourgain, A. Kontorovich, On representations of integers in thin subgroups of $\mathrm{SL}_2(\mathbb{Z})$, Geom. Funct. Anal. 20:5 (2010), 1144–1174.

[F] W.J. Floyd, Group completions and limit sets of Kleinian groups, Invent. Math. 57:3 (1980), 205–218.

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