THE KLOOSTERMAN CIRCLE METHOD AND WEIGHTED REPRESENTATION NUMBERS OF POSITIVE DEFINITE QUADRATIC FORMS

By

EDNA LUO JONES

A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Alex Kontorovich

And approved by

New Brunswick, New Jersey

October 2022

ABSTRACT OF THE DISSERTATION

The Kloosterman circle method and weighted representation numbers of positive definite quadratic forms

By EDNA LUO JONES

Dissertation Director:

Alex Kontorovich

We develop a version of the Kloosterman circle method with a bump function that is used to provide asymptotics for weighted representation numbers of positive definite integral quadratic forms. Unlike many applications of the Kloosterman circle method, we explicitly state some constants in the error terms that depend on the quadratic form. This version of the Kloosterman circle method uses Gauss sums, Kloosterman sums, Salié sums, and a principle of nonstationary phase. We briefly discuss a potential application of this version of the Kloosterman circle method to a proof of a strong asymptotic local-global principle for certain Kleinian sphere packings.

ACKNOWLEDGMENTS

First of all, I thank all of my friends, family, and mentors who have supported me over the years. Without them, I would not have been able to complete this dissertation. I am sure that if I started naming the people who have supported me during my time in graduate school that I would accidentally forget to include on my list some of my most important supporters.

Therefore, I am only going to name some of the people who contributed directly to the mathematics and the writing in this dissertation. I thank Alex Kontorovich, Henryk Iwaniec, Stephen D. Miller, and Robert Vaughan for agreeing to be on my thesis defense committee. I thank Alex Kontorovich, Gene Kopp, Louis Gaudet, and Henryk Iwaniec for their key mathematical insights. I thank Alex for giving me the problem of trying to prove a strong asymptotic local-global principle for certain Kleinian sphere packings. I highly appreciate my many enlightening conversations with Gene and Louis as we checked the soundness of my arguments. I also thank Gene for being a sounding board as I questioned my wording of various things. I thank Trevor Wooley for several helpful conversations with him about the circle method and mathematical writing. I also thank Trevor for sharing his arithmetic harmonic analysis notes [Woo20].

Again, the named people directly contributed to the mathematics and the writing in this dissertation, but there have been many people (both named and unnamed) who have supported my emotional, mental, and physical well-being during my years in graduate school. I greatly appreciate each and every one of them.

Table of contents

Chapter 1

Introduction

Kloosterman [Klo26] developed what is now called the Kloosterman circle method to prove an asymptotic formula for the number of representations of an integer by a positive definite diagonal integral quaternary quadratic form. He used ideas from the Hardy-Littlewood circle method and adapted them to handle quaternary quadratic forms. For an overview of the Hardy-Littlewood circle method, see [Vau97]. The Kloosterman circle method is described Section 11.4 in [Iwa97] and in Sections 20.3 and 20.4 in [IK04].

Before developing the Kloosterman circle method, Kloosterman [Klo24] already had used the Hardy-Littlewood circle method to determine an asymptotic for the number of representations of an integer by a positive definite diagonal integral quadratic form in $s \geq 5$ variables. However, as explained by Kloosterman in Section 1 of [Klo26], the error term obtained from the Hardy-Littlewood circle method is too large to provide asymptotic formulas for positive definite quaternary quadratic forms.

We will use the Kloosterman circle method to obtain asymptotics for a weighted number of representations of an integer by a positive definite integral quadratic form in $s \geq 4$ variables. In our version of the Kloosterman circle method, we will use a bump function to ensure the convergence of our generating function for our weighted representation numbers. (In [Iwa97] and [IK04], the generating function has an argument in the upper half plane. This ensures that their generating function converges.) Our bump function provides greater flexibility for our version of the Kloosterman circle method than the version found in [Iwa97] and [IK04].

Because we are discussing asymptotics and error terms, it will be useful to define big O notation and related terminology. If f and g are both functions of x, then the notation $f(x) = O(g(x))$ means that there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in D$, where D is an appropriate domain that can be deduced from the context. The constant C is called the *implied constant*. We take $f \ll g$ to have the same meaning as $f = O(g)$. If the implied constant depends on a parameter α, then we write $f = O_\alpha(g)$ or $f \ll_\alpha g$.

Heath-Brown [HB96] uses the delta method to obtain an asymptotic for a weighted number of representations of an integer by a quadratic form. However, in [HB96], it can be difficult to determine how the implied constants depend on the quadratic form.

In the error terms of our main result, we explicitly state some constants dependent on a quadratic form in s variables. This is unlike what is done in [Iwa97], [IK04], and [HB96]. The implied constants in our main result only depend on the number s, some positive number ε, and the bump function involved.

In many applications of the circle method to representing integers by forms, one does not need to know any of the implied constants that depend on the form. However, there are instances in which it is useful to know some of these implied constants. For instance, Dietmann [Die03] needed to know some constants dependent on a quadratic form in order to provide a search bound for the smallest solution to a quadratic polynomial in s variables with integer coefficients. In order to do this, Dietmann provided an asymptotic for a certain weighted representation number in which the implied constants only depended on s and some $\varepsilon > 0$. (See Theorem 2 of [Die03].) However, the asymptotic Dietmann gives in Theorem 2 of [Die03] does not say anything of use for positive definite quaternary quadratic forms.

A potential application of our version of the Kloosterman circle method is proving a strong asymptotic local-global principle for certain Kleinian sphere packings. (See Chapter 7 for more information on a strong asymptotic local-global principle for certain Kleinian sphere packings.) Because multiple positive definite integral quaternary quadratic forms may be involved at the same time in this potential application, we would like to know some of the constants dependent on the quadratic forms involved.

Before we state our main result, some notation in the result should be mentioned. Let $M_s(R)$ denote the set of $s \times s$ matrices over a ring R. The determinant of a square matrix A is denoted by $\det(A)$.

The ring of integers modulo m is denoted by $\mathbb{Z}/m\mathbb{Z}$. The multiplicative group of integers modulo m is denoted by $(\mathbb{Z}/m\mathbb{Z})^\times$. If $d \in (\mathbb{Z}/m\mathbb{Z})^\times$, then we denote the multiplicative inverse of d modulo m by d^*.

The following notation is standard in analytic number theory. Let $\mathrm{e}(x) = e^{2\pi i x}$. In a product over p, the index of multiplication p is taken to be prime. In general, p is taken to be prime. The gamma function is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

For a statement Y, let $\mathbf{1}_{\{Y\}}$ be the indicator function

$$\mathbf{1}_{\{Y\}} = \begin{cases} 1 & \text{if } Y \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer n, the divisor function $\tau(n)$ is the number of positive divisors of n. It is well-known that for all $\varepsilon > 0$, we have $\tau(n) \ll_\varepsilon n^\varepsilon$. (See, for example, (2.20) in [MV06].)

The space of real-valued and infinitely differentiable functions on $\mathbb{R}^s$ is denoted by $C^\infty(\mathbb{R}^s)$. We call a function f in $C^\infty(\mathbb{R}^s)$ a *smooth* function. For a continuous function f, define the *support* of f (denoted by $\text{supp}(f)$) to be the closure of the set $\{\mathbf{x} \in \mathbb{R}^s : f(\mathbf{x}) \neq 0\}$. A function f is said to be *compactly supported* if $\text{supp}(f)$ is a compact set. The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}^s$ is denoted by $C_c^\infty(\mathbb{R}^s)$. We call a function f in $C_c^\infty(\mathbb{R}^s)$ a *bump* function (or a *test* function).

Let $\psi \in C_c^\infty(\mathbb{R}^s)$ be a bump function. Since ψ is compactly supported, there exists a nonnegative real number ρ such that

$$\text{supp}(\psi) \subseteq \{\mathbf{x} \in \mathbb{R}^s : \|\mathbf{x}\| \leq \rho\}. \tag{1.1}$$

Define ρ_ψ to be the smallest nonnegative ρ that satisfies (1.1). (The number ρ_ψ exists since $\text{supp}(\psi)$ is compact.)

For $\psi \in C_c^\infty(\mathbb{R}^s)$, $X > 0$, and $\mathbf{m} \in \mathbb{R}^s$, let ψ_X be defined by

$$\psi_X(\mathbf{m}) = \psi\left(\frac{1}{X}\mathbf{m}\right). \tag{1.2}$$

Notice that $\text{supp}(\psi_X) \subseteq \{\mathbf{m} \in \mathbb{R}^s : \|\mathbf{m}\| \leq \rho_\psi X\}$ and $\rho_{\psi_X} = \rho_\psi X$. Also, $\psi_1 = \psi$.

For a positive real number X, an integer n, and an integral quadratic form F, let $R_{F,\psi,X}(n)$ be the weighted representation number defined by

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}} \psi_X(\mathbf{m}). \tag{1.3}$$

The main results of this dissertation concern asymptotics for this weighted representation number.

We now define some quantities that appear in our main results. For a positive integer n and an integral quadratic form F in s variables, define $\mathfrak{S}_F(n)$ to be the *singular series*

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right). \tag{1.4}$$

The singular series will appear in the main term for our asymptotic for the weighted representation number $R_{F,\psi,X}(n)$. The singular series is known to contain information modulo q for every positive integer q. See Section 11.5 in [Iwa97] for more information about the singular series.

For a nonsingular quadratic form F, a real number n, and a positive real number X, we define the *real factor* $\sigma_{F,\psi,\infty}(n, X)$ to be

$$\sigma_{F,\psi,\infty}(n, X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m}) - \frac{n}{X^2}\right| < \varepsilon} \psi(\mathbf{m})\, d\mathbf{m}. \tag{1.5}$$

This real factor can be viewed as a weighted density of real solutions to $F(\mathbf{m}) = n/X^2$.

The main result of this dissertation is the following theorem about the weighted representation number $R_{F,\psi,X}(n)$.

Theorem 1.1. *Suppose that n is a positive integer. Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let λ_s be largest eigenvalue of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$.*

Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function. Then for $X \geq 1/\lambda_s$ and $\varepsilon > 0$, the *weighted representation number $R_{F,\psi,X}(n)$ is*

$$
\begin{aligned}
&R_{F,\psi,X}(n)\\
&= \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2}\\
&\qquad + O_{\psi,s,\varepsilon}\left(n^{s/2-1}X^{(3-s)/2+\varepsilon}\lambda_s^{(3-s)/2+\varepsilon}(\det(A))^{-1/2}L^{s/2}\tau(n)\prod_{p\mid 2\det(A)}(1-p^{-1/2})^{-1}\right)\\
&\qquad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon}\lambda_s^{(s+1)/2+\varepsilon}L^{s/2}\tau(n)\prod_{p\mid 2\det(A)}(1-p^{-1/2})^{-1}\right).
\end{aligned}
$$

Remark 1.2. The integer L discussed in Theorem 1.1 exists since $\det(A)A^{-1} \in M_s(\mathbb{Z})$.

Remark 1.3. If $X = n^{1/2}$, then Theorem 1.1 gives a result that is somewhat similar to Theorem 4 in [HB96]. However, the implied constants in Theorem 1.1 only depend on ψ, s, and ε, while the implied constants in Theorem 4 in [HB96] also depend on the quadratic form F.

When we optimize the value of X in Theorem 1.1, we obtain the following asymptotic for the weighted representation number $R_{F,\psi,X}(n)$.

Corollary 1.4. *Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let λ_s be largest eigenvalue of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. Suppose that n is a positive integer that satisfies*

$$n \geq \lambda_s^{2/(s-2)}(\det(A))^{1/(s-2)}.$$

Set X to be

$$X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}.$$

Then the weighted representation number $R_{F,\psi,X}(n)$ is

$$R_{F,\psi,X}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} \\ + O_{\psi,s,\varepsilon}\Bigg(n^{(s-1)/4+\varepsilon}\lambda_s^{(s-3)/(2s-4)-\varepsilon/(s-2)}(\det(A))^{(1-s)/(4s-8)-\varepsilon/(2s-4)} \\ \times L^{s/2}\prod_{p\mid 2\det(A)}(1-p^{-1/2})^{-1}\Bigg)$$

for any $\varepsilon > 0$.

By fixing a particular bump function ψ in Theorem 1.1 or in Corollary 1.4, we obtain the following corollary about the (unweighted) representation number of the positive definite integral quadratic form F. (The representation number of a positive definite integral form F is the number of integral solutions to $F(\mathbf{m}) = n$.) This corollary is Theorem 11.2 in [Iwa97] and Theorem 20.9 in [IK04].

Corollary 1.5. *Suppose that n is a positive integer. Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Then the number of integral solutions to $F(\mathbf{m}) = n$ is*

$$|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}| = \mathfrak{S}_F(n)\frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}}n^{s/2-1} + O_{F,\varepsilon}\left(n^{(s-1)/4+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Remark 1.6. The implied constant in Corollary 1.5 depends on the quadratic form F, because the choice of ψ depends on F.

If a bump function ψ is chosen first and then a positive definite quadratic form F is chosen based on ψ (and possibly on $n, X > 0$), we can have a result in which all of the implied constants only depend on ψ, s, and ε. We hope that such a result can be used towards a proof of a strong asymptotic local-global principle for certain Kleinian sphere packings.

The remainder of this dissertation is organized as follows. In Chapter 2, we define some notation that will be used throughout the dissertation. In Chapter 3, we set up the Kloosterman circle method and apply it to our particular problem. Once we apply the Kloosterman circle method to our problem, we obtain an arithmetic part and an archimedean part. We analyze the arithmetic part in Chapter 4 and the archimedean part in Chapter 5. In Chapter 6, we put together estimates from previous chapters and complete our proofs of Theorem 1.1 and Corollaries 1.4 and 1.5. In Chapter 7, we briefly discuss a potential application of our version of the Kloosterman circle method: a proof of a strong asymptotic local-global principle for certain Kleinian sphere packings.

Chapter 2

Some additional notation

In this chapter, we state some notation used throughout this dissertation. This is not a comprehensive list of notation used in this dissertation, but most of the notation used in this dissertation is listed here, in the previous chapter, or in the next chapter.

The greatest common divisor of integers $a_1, \ldots, a_m$ is denoted by $\gcd(a_1, \ldots, a_m)$.

A vector $\mathbf{m} \in \mathbb{Z}^s$ is viewed as an $s \times 1$ column vector. Let jth entry of a vector $\mathbf{m}$ be denoted by m_j. The entry in the jth row and the kth column of a matrix A is denoted by a_{jk}. For a vector $\mathbf{m}$ (or matrix A), let $\mathbf{m}^\top$ (or $A^\top$) denote the transpose of $\mathbf{m}$ (or A). A diagonal $s \times s$ matrix with diagonal entries $d_1, d_2, \ldots, d_s$ is denoted by $\operatorname{diag}(d_1, d_2, \ldots, d_s)$.

The *dot product* of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^s$ is $\mathbf{x}^\top \mathbf{y} = \sum_{j=1}^s x_j y_j$ and is denoted by $\mathbf{x} \cdot \mathbf{y}$. The *Euclidean norm* of a vector $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. For $\mathbf{x} \in \mathbb{R}^s$ and a nonempty subset U of $\mathbb{R}^s$, define the *distance* between $\mathbf{x}$ and U to be

$$\operatorname{dist}(\mathbf{x}, U) = \inf_{\mathbf{m} \in U} \|\mathbf{x} - \mathbf{m}\|.$$

We use $\mathbf{0}$ to denote the zero vector in $\mathbb{R}^s$.

For a nonsingular symmetric matrix $A \in M_s(\mathbb{R})$, we define the *signature* of A to be the number of positive eigenvalues of A minus the number of negative eigenvalues of A. The signature of A is denoted by $\operatorname{sgnt}(A)$.

As is standard in analytic number theory, the natural logarithm of z is denoted by $\log(z)$, not by $\ln(z)$.

For $z \in \mathbb{C}$, the notation $\overline{z}$ means the complex conjugate of z. The real part of a complex number z is denoted by $\operatorname{Re}(z)$.

For a nonnegative integer k and a function $f\colon \mathbb{R} \to \mathbb{R}$ with a kth derivative, we let $f^{(k)}$ denote the kth derivative of f. Note that $f^{(0)} = f$, $f^{(1)} = f'$, and $f^{(2)} = f''$.

We use the notation $\lfloor x \rfloor$ for the greatest integer less than or equal to x. Similarly, $\lceil x \rceil$ denotes the least integer greater than or equal to x.

For a sufficiently nice function $f\colon \mathbb{R}^d \to \mathbb{R}$ (where d is a positive integer), define the *Fourier transform* $\widehat{f}$ of f to be

$$\widehat{f}(\mathbf{y}) = \int_{\mathbb{R}^d} f(\mathbf{x})\mathrm{e}(-\mathbf{x}\cdot\mathbf{y}) \; d\mathbf{x} \tag{2.1}$$

for $\mathbf{y} \in \mathbb{R}^d$. Assuming that f is sufficiently nice, the inverse Fourier transform of $\widehat{f}$ is f. That is,

$$f(\mathbf{y}) = \int_{\mathbb{R}^d} \widehat{f}(\mathbf{x})\mathrm{e}(\mathbf{x}\cdot\mathbf{y}) \; d\mathbf{x}. \tag{2.2}$$

Whenever we use the Fourier transform or the inverse Fourier transform of a function, we assume that the function is sufficiently nice. There are various ways that "sufficiently nice" can be made precise. For example, a function is sufficiently nice if it is a Schwartz function. (See Corollary 8.23 of [Fol99].)

Chapter 3

Setting up the Kloosterman circle method

In this chapter, we set up the Kloosterman circle method for our weighted representation number $R_{F,\psi,X}(n)$. Unless otherwise specified, any notation mentioned here will be used for the remainder of this dissertation.

As in the statement of Theorem 1.1, we let F be a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Observe that A is symmetric with each of its diagonal entries in $2\mathbb{Z}$. Also, $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^s$. Let $\{\lambda_j\}_{j=1}^s$ be the set of eigenvalues of A, where $0 < \lambda_1 \leq \cdots \leq \lambda_s$. (Notice that all the eigenvalues of A are positive since A is positive definite.)

Let F^* be the adjoint quadratic form

$$F^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top A^{-1}\mathbf{x}$$

for any $\mathbf{x} \in \mathbb{R}^s$. We note that A^{-1} might not be an integral matrix. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. (Such an L exists since $\det(A)A^{-1} \in M_s(\mathbb{Z})$.)

Let $\Theta_{F,\psi,X}(x)$ be the real analytic function with $R_{F,\psi,X}(n)$ as the Fourier coefficients; i.e., let

$$\Theta_{F,\psi,X}(x) = \sum_{n=0}^{\infty} R_{F,\psi,X}(n)\mathrm{e}(nx) \tag{3.1}$$

for $x \in \mathbb{R}$. Notice that $\Theta_{F,\psi,X}(x+1) = \Theta_{F,\psi,X}(x)$. The function $\Theta_{F,\psi,X}$ can be viewed as a generating function for $\{R_{F,\psi,X}(n)\}_{n=0}^{\infty}$. We call $\Theta_{F,\psi,X}$ a *weighted theta series* of F.

Using the Fourier transform, we see that

$$R_{F,\psi,X}(n) = \int_0^1 \Theta_{F,\psi,X}(x)\mathrm{e}(-nx)\ dx. \tag{3.2}$$

The fact that the nth Fourier coefficient of $\Theta_{F,\psi,X}(x)$ is $R_{F,\psi,X}(n)$ suggests that the function $\Theta_{F,\psi,X}$ is related to F. The next lemma states how $\Theta_{F,\psi,X}$ is directly related to F.

Lemma 3.1. *For $x \in \mathbb{R}$ and $X > 0$, the function $\Theta_{F,\psi,X}(x)$ is*

$$\Theta_{F,\psi,X}(x) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m}).$$

Proof. It suffices to show that the nth Fourier coefficient of $\sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m})$ is $R_{F,\psi,X}(n)$. By the Fubini-Tonelli theorem, the nth Fourier coefficient of $\sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m})$ is

$$\begin{aligned}\int_0^1 \mathrm{e}(-nx) \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m})\ dx &= \sum_{\mathbf{m}\in\mathbb{Z}^s} \psi_X(\mathbf{m}) \int_0^1 \mathrm{e}(-nx)\,\mathrm{e}(xF(\mathbf{m}))\ dx \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^s} \psi_X(\mathbf{m}) \int_0^1 \mathrm{e}(x(F(\mathbf{m})-n))\ dx.\end{aligned}$$

Now

$$\int_0^1 \mathrm{e}(x(F(\mathbf{m})-n))\ dx = \mathbf{1}_{\{F(\mathbf{m})-n=0\}} = \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

so

$$\begin{aligned}\int_0^1 \mathrm{e}(-nx)\sum_{\mathbf{m}\in\mathbb{Z}^s}\mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m})\ dx &= \sum_{\mathbf{m}\in\mathbb{Z}^s}\psi_X(\mathbf{m})\mathbf{1}_{\{F(\mathbf{m})=n\}}\\ &= R_{F,\psi,X}(n).\end{aligned}$$

□

3.1 Using a Farey dissection

To use the Kloosterman circle method, we want to break up a unit interval (say $[z, z+1)$ for $z \in \mathbb{C}$) into smaller intervals (or "arcs") using Farey sequences. This unit interval is where the "circle" in the "circle method" comes from. A unit interval is sometimes considered a circle since $\mathrm{e}([z, z+1))$ is a circle in the complex plane.

For $Q \geq 1$, the Farey sequence $\mathfrak{F}_Q$ of order Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$. (Traditionally, Q is required to be an integer, but we will find that allowing Q to be not an integer will remove some technicalities later. See Section 6.6.) An element of $\mathfrak{F}_Q$ is called a *Farey point*. We will state some well-known properties of $\mathfrak{F}_Q$. For more background on Farey sequences of order Q, please see Chapter III of [HW08] or pp. 451–452 of [IK04].

Let $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ be adjacent Farey points of order Q (so that $q', q, q'' \leq Q$ and $\frac{a}{q}$ is the only fraction in $\mathfrak{F}_Q$ that is greater than $\frac{a'}{q'}$ and less than $\frac{a''}{q''}$). The denominators q', q'' are determined by the conditions:

$$Q - q < q' \leq Q, \qquad aq' \equiv 1 \pmod{q}, \tag{3.3}$$

$$Q - q < q'' \leq Q, \qquad aq'' \equiv -1 \pmod{q}. \tag{3.4}$$

For a positive integer q and an integer a coprime to q, let $j_{a,q}$ be the interval

$$j_{a,q} = \left[\frac{a}{q} - \frac{q}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right), \tag{3.5}$$

where $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ are adjacent Farey points of order Q. The interval $j_{a,q}$ is called a *Farey arc*. It is known that

$$\left[-\frac{1}{1+\lfloor Q\rfloor}, 1 - \frac{1}{1+\lfloor Q\rfloor}\right) = \bigsqcup_{q=1}^{\lfloor Q\rfloor} \bigsqcup_{\substack{a=0\\ \gcd(a,q)=1}}^{q-1} j_{a,q}. \tag{3.6}$$

We say that the set

$$\{j_{a,q}\}_{1\le q\le Q,\ 0\le a\le q-1,\ \gcd(a,q)=1} \tag{3.7}$$

of Farey arcs is a *Farey dissection of order Q* of the circle.

We now have all the tools to express an idea that is crucial in the the Kloosterman circle method.

Lemma 3.2. *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then*

$$\int_0^1 f(x)\ dx = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q<d\le q+Q\\ qdx<1\\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\ dx\right) \tag{3.8}$$

$$= 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \int_0^1 \frac{1}{qd}\ f\left(\frac{x}{qd} - \frac{d^*}{q}\right)\ dx\right), \tag{3.9}$$

where d^* is the multiplicative inverse of d modulo q.

Remark 3.3. When analytic number theorists talk about using the Kloosterman circle method, they often mean that they are using something similar to Lemma 3.2. One of the most notable applications of Lemma 3.2 is writing the function $\mathbf{1}_{\{n=0\}}$ as a sum of exponential functions by having $f(x) = \mathrm{e}(nx)$ in (3.9). (See Proposition 11.1 in [Iwa97] and Proposition 20.7 in [IK04].)

Remark 3.4. We note that Lemma 3.2 is general enough to work with a variety of representation problems by having $f(x) = \mathrm{e}(-nx)\sum_{m\in\mathbb{Z}} R(m)\mathrm{e}(mx)$ for a given representation number $R(n)$. Notice that Lemma 3.2 is applicable to the function $f(x) = \mathrm{e}(-nx)\,\Theta_{F,\psi,X}(x)$ and so can give us an expression of $R_{F,\psi,X}(m)$ containing an integral with a Farey dissection.

Proof of Lemma 3.2. Because f has a period of 1, we have the following equality of integrals:

$$\int_0^1 f(x)\ dx = \int_{-\frac{1}{1+\lfloor Q\rfloor}}^{1-\frac{1}{1+\lfloor Q\rfloor}} f(x)\ dx.$$

Using our Farey dissection of $\left[-\frac{1}{1+\lfloor Q\rfloor}, 1-\frac{1}{1+\lfloor Q\rfloor}\right)$, notice that

$$\begin{aligned}\int_0^1 f(x)\ dx &= \sum_{1\le q\le Q}\ \sum_{\substack{a=0\\ \gcd(a,q)=1}}^{q-1} \int_{j_{a,q}} f(x)\ dx\\ &= \sum_{1\le q\le Q}\ \sum_{\substack{a=0\\ \gcd(a,q)=1}}^{q-1} \int_{\frac{a}{q}-\frac{1}{q(q+q')}}^{\frac{a}{q}+\frac{1}{q(q+q'')}} f(x)\ dx\\ &= \sum_{1\le q\le Q}\ \sum_{\substack{a=0\\ \gcd(a,q)=1}}^{q-1} \int_{-\frac{1}{q(q+q')}}^{\frac{1}{q(q+q'')}} f\left(x+\frac{a}{q}\right)\ dx\end{aligned}$$

by letting $x \mapsto x + a/q$ in the last integral.

By breaking up and rearranging the integrals, we obtain

$$\int_0^1 f(x)\ dx = J_1 + J_2, \tag{3.10}$$

where

$$J_1 = \sum_{1 \le q \le Q} \sum_{\substack{a=0 \\ \gcd(a,q)=1}}^{q-1} \int_{-\frac{1}{q(q+q')}}^{0} f\left(x + \frac{a}{q}\right)\, dx$$

and

$$J_2 = \sum_{1 \le q \le Q} \sum_{\substack{a=0 \\ \gcd(a,q)=1}}^{q-1} \int_{0}^{\frac{1}{q(q+q'')}} f\left(x + \frac{a}{q}\right)\, dx.$$

In J_1, we let $d = q + q'$. By (3.3), the expression $d = q + q'$ is equivalent to the combination of conditions $Q < d \le q + Q$ and $a \equiv d^* \pmod{q}$. Thus, when we use the periodicity of f, we have

$$J_1 = \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_{-\frac{1}{qd}}^{0} f\left(x + \frac{d^*}{q}\right)\, dx.$$

We now map $x \mapsto -x$, so

$$\begin{aligned} J_1 &= \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_{\frac{1}{qd}}^{0} -f\left(-x + \frac{d^*}{q}\right)\, dx \\ &= \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_{0}^{\frac{1}{qd}} f\left(-x + \frac{d^*}{q}\right)\, dx \\ &= \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_{0}^{\frac{1}{qd}} f\left(-\left(x - \frac{d^*}{q}\right)\right)\, dx. \end{aligned}$$

Since $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$, we conclude that

$$J_1 = \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} \overline{f\left(x - \frac{d^*}{q}\right)}\, dx$$

$$= \overline{\sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} f\left(x - \frac{d^*}{q}\right)\, dx}. \tag{3.11}$$

In J_2, we let $d = q + q''$. By (3.4), the expression $d = q + q''$ is equivalent to the combination of conditions $Q < d \le q + Q$ and $a \equiv -d^* \pmod q$. Thus, when we use the periodicity of f, we have

$$J_2 = \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} f\left(x - \frac{d^*}{q}\right)\, dx. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we obtain

$$\int_0^1 f(x)\, dx = \overline{\sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} f\left(x - \frac{d^*}{q}\right)\, dx}$$

$$+ \sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} f\left(x - \frac{d^*}{q}\right)\, dx$$

$$= 2\,\mathrm{Re}\left(\sum_{1 \le q \le Q} \sum_{\substack{Q < d \le q+Q \\ \gcd(d,q)=1}} \int_0^{\frac{1}{qd}} f\left(x - \frac{d^*}{q}\right)\, dx\right). \tag{3.13}$$

By switching the order of summation and integration in (3.13), we deduce (3.8). By mapping $x \mapsto \frac{x}{qd}$ in (3.13), we obtain (3.9). $\square$

We now apply Lemma 3.2 to (3.2) to decompose the "circle." Taking $f(x) =$

$\Theta_{F,\psi,X}(x)\mathrm{e}(-nx)$ in (3.8) of Lemma 3.2, we see that

$$\begin{aligned}
R_{F,\psi,X}(n) &= 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q<d\le q+Q \\ qdx<1 \\ \gcd(d,q)=1}} \Theta_{F,\psi,X}\left(x-\frac{d^*}{q}\right) \mathrm{e}\biggl(-n\left(x-\frac{d^*}{q}\right)\biggr)\, dx \right) \\
&= 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \int_0^{\frac{1}{qQ}} \mathrm{e}(-nx) \sum_{\substack{Q<d\le q+Q \\ qdx<1 \\ \gcd(d,q)=1}} \Theta_{F,\psi,X}\left(x-\frac{d^*}{q}\right) \mathrm{e}\biggl(n\frac{d^*}{q}\biggr)\, dx \right).
\end{aligned} \tag{3.14}$$

The appearance of $\Theta_{F,\psi,X}\left(x-d^*/q\right)$ leads us to try to evaluate $\Theta_{F,\psi,X}\left(x-d^*/q\right)$ in the next section.

3.2 Examining a weighted theta series

Due to the appearance of $\Theta_{F,\psi,X}\left(x-d^*/q\right)$ in our expression of $R_{F,\psi,X}(n)$ in (3.14), we would like to evaluate the weighted theta series $\Theta_{F,\psi,X}\left(x-d^*/q\right)$. By Lemma 3.1, we see that

$$\begin{aligned}
\Theta_{F,\psi,X}\left(x-\frac{d^*}{q}\right) &= \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}\biggl(\left(x-\frac{d^*}{q}\right)F(\mathbf{m})\biggr)\,\psi_X(\mathbf{m}) \\
&= \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}\biggl(-\frac{d^*}{q}F(\mathbf{m})\biggr)\,\mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m}).
\end{aligned}$$

Because the value $\mathrm{e}\Bigl(-\frac{d^*}{q}F(\mathbf{m})\Bigr)$ only depends on $\mathbf{m}$ modulo q, we would like to split the sum over $\mathbf{m}\in\mathbb{Z}^s$ into sums over congruence classes modulo q. Doing this results in the following:

$$\Theta_{F,\psi,X}\left(x-\frac{d^*}{q}\right)=\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\sum_{\mathbf{m}\equiv\mathbf{h}\ (\mathrm{mod}\, q)}\mathrm{e}\left(-\frac{d^*}{q}F(\mathbf{m})\right)\mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m})$$
$$=\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\mathrm{e}\left(-\frac{d^*}{q}F(\mathbf{h})\right)\sum_{\mathbf{m}\equiv\mathbf{h}\ (\mathrm{mod}\, q)}\mathrm{e}(xF(\mathbf{m}))\,\psi_X(\mathbf{m}).$$

By replacing $\mathbf{m}$ by $\mathbf{h}+q\mathbf{m}$ (where $\mathbf{m}\in\mathbb{Z}^s$), we obtain

$$\Theta_{F,\psi,X}\left(x-\frac{d^*}{q}\right)=\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\mathrm{e}\left(-\frac{d^*}{q}F(\mathbf{h})\right)\sum_{\mathbf{m}\in\mathbb{Z}^s}\mathrm{e}(xF(\mathbf{h}+q\mathbf{m}))\,\psi_X(\mathbf{h}+q\mathbf{m}). \tag{3.15}$$

3.3 Using Poisson summation

To obtain asymptotics on $\Theta_{F,\psi,X}\left(x-d^*/q\right)$, it looks like we need to estimate

$$\sum_{\mathbf{m}\in\mathbb{Z}^s}\mathrm{e}(xF(\mathbf{h}+q\mathbf{m}))\,\psi_X(\mathbf{h}+q\mathbf{m}). \tag{3.16}$$

However, if we could estimate (3.16) directly, we probably would have estimated $\Theta_{F,\psi,X}(x-d^*/q)$ directly in the first place. Since we did not, we need to use another tool. To understand (3.16), we use Poisson summation in the next lemma. Poisson summation allows us to concentrate the bulk of the sum into one term.

Lemma 3.5. *For $\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s$, $x\in\mathbb{R}$, and positive $q\in\mathbb{Z}$, we have*

$$\sum_{\mathbf{m}\in\mathbb{Z}^s}\mathrm{e}(xF(\mathbf{h}+q\mathbf{m}))\,\psi_X(\mathbf{h}+q\mathbf{m})$$
$$=\sum_{\mathbf{r}\in\mathbb{Z}^s}\frac{1}{q^s}\mathrm{e}\left(\frac{1}{q}\mathbf{h}\cdot\mathbf{r}\right)\int_{\mathbb{R}^s}\mathrm{e}\left(xF(\mathbf{m})-\frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\ d\mathbf{m}.$$

Proof. We use the Poisson summation formula

$$\sum_{\mathbf{m}\in\mathbb{Z}^s} f(\mathbf{m}) = \sum_{\mathbf{r}\in\mathbb{Z}^s} \widehat{f}(\mathbf{r}), \tag{3.17}$$

where $\widehat{f}$ is the Fourier transform of f.

The Fourier transform of $f(\mathbf{m}) = \mathrm{e}(xF(\mathbf{h}+q\mathbf{m}))\,\psi_X(\mathbf{h}+q\mathbf{m})$ is

$$\widehat{f}(\ell) = \int_{\mathbb{R}^s} \mathrm{e}(xF(\mathbf{h}+q\mathbf{m}))\,\psi_X(\mathbf{h}+q\mathbf{m})\mathrm{e}(-\mathbf{m}\cdot\mathbf{r})\ \ d\mathbf{m}.$$

Map $\mathbf{m}$ to $\frac{1}{q}(\mathbf{m}-\mathbf{h})$ to obtain

$$\widehat{f}(\mathbf{r}) = \frac{1}{q^s}\mathrm{e}\left(\frac{1}{q}\mathbf{h}\cdot\mathbf{r}\right)\int_{\mathbb{R}^s} \mathrm{e}\left(xF(\mathbf{m}) - \frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\ d\mathbf{m}.$$

Putting this into the Poisson summation formula (3.17), we obtain the result of this lemma. □

By applying Lemma 3.5 to (3.15), we obtain

$$\begin{aligned}
\Theta_{F,\psi,X}\left(x - \frac{d^*}{q}\right) &= \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(-\frac{d^*}{q}F(\mathbf{h})\right)\sum_{\mathbf{r}\in\mathbb{Z}^s}\frac{1}{q^s}\mathrm{e}\left(\frac{1}{q}\mathbf{h}\cdot\mathbf{r}\right) \\
&\qquad\times\int_{\mathbb{R}^s}\mathrm{e}\left(xF(\mathbf{m}) - \frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\ d\mathbf{m} \\
&= \sum_{\mathbf{r}\in\mathbb{Z}^s}\frac{1}{q^s}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\mathrm{e}\left(\frac{1}{q}\left(-d^*F(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r}\right)\right) \\
&\qquad\times\int_{\mathbb{R}^s}\mathrm{e}\left(xF(\mathbf{m}) - \frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\ d\mathbf{m} \\
&= \sum_{\mathbf{r}\in\mathbb{Z}^s}\frac{1}{q^s}G_{\mathbf{r}}(-d^*,q)\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q),
\end{aligned} \tag{3.18}$$

where $G_{\mathbf{r}}(d,q)$ is the Gauss sum

$$G_{\mathbf{r}}(d,q) = \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r})\right) \tag{3.19}$$

and

$$\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) = \int_{\mathbb{R}^s} \mathrm{e}\left(xF(\mathbf{m}) - \frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\, d\mathbf{m}. \tag{3.20}$$

Substituting (3.18) into (3.14), we see that

$$\begin{aligned} &R_{F,\psi,X}(n) \\ &= 2\,\mathrm{Re}\left(\sum_{1\le q\le Q}\int_0^{\frac{1}{qQ}} \mathrm{e}(-nx) \sum_{\substack{Q<d\le q+Q\\ qdx<1\\ \gcd(d,q)=1}} \sum_{\mathbf{r}\in\mathbb{Z}^s} \frac{1}{q^s} G_{\mathbf{r}}(-d^*,q)\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)\mathrm{e}\left(n\frac{d^*}{q}\right)\, dx\right). \end{aligned}$$

Switching the order of the $\mathbf{r}$ and d sums, we obtain

$$R_{F,\psi,X}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q}\frac{1}{q^s}\int_0^{\frac{1}{qQ}} \mathrm{e}(-nx)\sum_{\mathbf{r}\in\mathbb{Z}^s}\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)T_{\mathbf{r}}(q,n;x)\, dx\right), \tag{3.21}$$

where

$$T_{\mathbf{r}}(q,n;x) = \sum_{\substack{Q<d\le q+Q\\ qdx<1\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q). \tag{3.22}$$

We call the expression $T_{\mathbf{r}}(q,n;x)$ the *arithmetic part*, because $T_{\mathbf{r}}(q,n;x)$ contains information modulo q. We call the integral $\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)$ the *archimedean part*. We will use Gauss sums, Kloosterman sums, and Salié sums to obtain estimates on the arithmetic part $T_{\mathbf{r}}(q,n;x)$. We will use a principle of nonstationary phase to obtain estimates on the archimedean part $\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)$.

Chapter 4

Analyzing the arithmetic part

In this chapter, we analyze the arithmetic part $T_{\mathbf{r}}(q, n; x)$. We do this by completing the sum $T_{\mathbf{r}}(q, n; x)$, using Gauss sums, and obtaining estimates of our complete sums. An upper bound for the absolute value of the arithmetic part is given in the last section of this chapter.

4.1 Completing the sum

In general, finding the value of a sum of a periodic function over an arbitrary range of integer is more difficult than finding the value of a sum of a periodic function over its period. In our calculation of $R_{F,\psi,X}(n)$, we currently have the sum $T_{\mathbf{r}}(q, n; x)$. We notice that $T_{\mathbf{r}}(q, n; x)$ is an incomplete sum since it is a sum over d over the range $(Q, q+Q]$ such that $qdx < 1$ and d is coprime to q. We use the following lemma to complete this sum so that a sum over the period of the summand function appears.

Lemma 4.1. *For $\mathbf{r} \in \mathbb{Z}^s$, positive $q \in \mathbb{Z}$, $n \in \mathbb{Z}$, and $x \in \mathbb{R}$, the sum $T_{\mathbf{r}}(q, n; x)$ is*

$$T_{\mathbf{r}}(q, n; x) = \sum_{\ell \,(\mathrm{mod}\, q)} \gamma(\ell) K(\ell, n, \mathbf{r}; q),$$

where

$$\gamma(\ell) = \frac{1}{q} \sum_{Q < b \leq \min\{q+Q, \lceil 1/(qx) - 1 \rceil\}} \mathrm{e}\left(-\frac{b\ell}{q}\right)$$

and

$$K(\ell, n, \mathbf{r}; q) = \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \mathrm{e}\left(\frac{\ell d + n d^*}{q}\right) G_{\mathbf{r}}(-d^*, q).$$

Remark 4.2. Note that $K(\ell, n, \mathbf{r}; q)$ is a complete sum, that is, it is a sum that is over the entire period of its summand function.

Before we prove Lemma 4.1, we state a formula related to an indicator function that comes up frequently in number theory and will appear in the proof of Lemma 4.1.

Lemma 4.3. *Let $a, q \in \mathbb{Z}$ and $q > 0$. Then*

$$\begin{aligned} \sum_{j \in \mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{aj}{q}\right) &= q \mathbf{1}_{\{a \equiv 0 \pmod{q}\}} \\ &= \begin{cases} q & \text{if } a \equiv 0 \pmod{q}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. This lemma follows from the fact that the exponential sum appearing in the formula can be viewed as a geometric sum with j ranging from 0 to $q-1$. Alternatively, since $\mathrm{e}\left(\frac{aj}{q}\right)$ is a character of the additive group of $\mathbb{Z}/q\mathbb{Z}$, the formula also follows from the orthogonality relations for characters [Apo76, Theorem 6.10]. $\square$

Using Lemma 4.3, we now give a proof of Lemma 4.1.

Proof of Lemma 4.1. We want to split up the sum $T_{\mathbf{r}}(q, n; x)$ into sums over residue classes. To do this, we note the following fact: For any integer d,

$$\mathbf{1}_{\{\gcd(d,q)=1\}} = \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \mathbf{1}_{\{b\equiv d \pmod q\}} = \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \frac{1}{q} \sum_{\ell \pmod q} \mathrm{e}\left(\frac{\ell b - \ell d}{q}\right). \tag{4.1}$$

The last equality follows from Lemma 4.3 with $a = b - d$ and $j = \ell$.

We rewrite (3.22) as

$$\begin{aligned} T_{\mathbf{r}}(q,n;x) &= \sum_{\substack{Q<d\le q+Q\\ d<1/(qx)\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q) \\ &= \sum_{\substack{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q) \\ &= \sum_{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}} \mathbf{1}_{\{\gcd(d,q)=1\}}\mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q). \end{aligned}$$

By applying (4.1) to the last expression, we obtain

$$T_{\mathbf{r}}(q,n;x) = \sum_{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}} \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \mathbf{1}_{\{b\equiv d \pmod q\}}\mathrm{e}\left(\frac{nd^*}{q}\right) G_{\mathbf{r}}(-d^*,q).$$

Because $\mathbf{1}_{\{b\equiv d \pmod q\}}\mathrm{e}\left(\frac{nd^*}{q}\right) G_{\mathbf{r}}(-d^*,q)$ is periodic modulo q and the inner sum only contributes if $b\equiv d \pmod q$, we can change some of the d's to b's to obtain

$$T_{\mathbf{r}}(q,n;x) = \sum_{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}} \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \mathbf{1}_{\{b\equiv d \pmod q\}}\mathrm{e}\left(\frac{nb^*}{q}\right) G_{\mathbf{r}}(-b^*,q).$$

Applying (4.1) again, we see that

$$\begin{aligned}
T_{\mathbf{r}}(q,n;x) &= \sum_{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}} \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \frac{1}{q}\sum_{\ell \,(\mathrm{mod}\, q)} \mathrm{e}\left(\frac{\ell b-\ell d}{q}\right)\mathrm{e}\left(\frac{nb^*}{q}\right) G_{\mathbf{r}}(-b^*,q) \\
&= \sum_{\ell \,(\mathrm{mod}\, q)} \frac{1}{q} \sum_{Q<d\le\min\{q+Q,\lceil 1/(qx)-1\rceil\}} \mathrm{e}\left(\frac{-\ell d}{q}\right) \\
&\qquad\qquad \times \sum_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \mathrm{e}\left(\frac{\ell b+nb^*}{q}\right) G_{\mathbf{r}}(-b^*,q) \\
&= \sum_{\ell \,(\mathrm{mod}\, q)} \gamma(\ell) K(\ell,n,\mathbf{r};q). \qquad \square
\end{aligned}$$

We have now separated $T_{\mathbf{r}}(q,n;x)$ into a sum $\gamma(\ell)$ and a complete sum $K(\ell,n,\mathbf{r};q)$. We will examine the sum $K(\ell,n,\mathbf{r};q)$ more in later sections. For now, we will focus on $\gamma(\ell)$.

Lemma 4.4. *If* $|\ell|\le \frac{q}{2}$, *then*

$$|\gamma(\ell)| \le (1+|\ell|)^{-1}.$$

Proof. Let $Y=\min\{q+Q,\lceil 1/(qx)-1\rceil\}$.

If $\ell=0$, then

$$\gamma(\ell)=\frac{1}{q}\sum_{Q<b\le Y}\mathrm{e}\left(-\frac{b\ell}{q}\right)=\frac{1}{q}\sum_{Q<b\le Y}1=\frac{\lfloor Y-Q\rfloor}{q}\le\frac{\lfloor (q+Q)-Q\rfloor}{q}=\frac{q}{q}=1$$

since $Y\le q+Q$ by definition. Thus, $\gamma(\ell)\le(1+|\ell|)^{-1}$ when $\ell=0$.

Now suppose that $0 < |\ell| \leq \frac{q}{2}$. Then

$$\begin{aligned}\gamma(\ell) &= \frac{1}{q}\sum_{Q<b\leq Y} \mathrm{e}\left(-\frac{b\ell}{q}\right) = \frac{1}{q}\sum_{Q<b\leq Y}\left(\mathrm{e}\left(-\frac{\ell}{q}\right)\right)^b \\ &= \frac{1}{q}\mathrm{e}\left(-\frac{\ell(Q+1)}{q}\right)\sum_{0\leq b\leq\lfloor Y-Q-1\rfloor}\left(\mathrm{e}\left(-\frac{\ell}{q}\right)\right)^b \\ &= \frac{1}{q}\mathrm{e}\left(-\frac{\ell(Q+1)}{q}\right)\frac{1-\mathrm{e}\left(-\frac{\ell}{q}\right)^{\lfloor Y-Q\rfloor}}{1-\mathrm{e}\left(-\frac{\ell}{q}\right)}.\end{aligned}$$

By multiplying the numerator and the denominator by $\mathrm{e}\left(\frac{\ell}{2q}\right)$, we obtain

$$\begin{aligned}\gamma(\ell) &= \frac{1}{q}\mathrm{e}\left(-\frac{\ell(Q+1)}{q}\right)\mathrm{e}\left(\frac{\ell}{2q}\right)\frac{1-\mathrm{e}\left(-\frac{\ell}{q}\right)^{\lfloor Y-Q\rfloor}}{\mathrm{e}\left(\frac{\ell}{2q}\right)-\mathrm{e}\left(-\frac{\ell}{2q}\right)} \\ &= \frac{1}{q}\mathrm{e}\left(-\frac{\ell(Q+1)}{q}\right)\mathrm{e}\left(\frac{\ell}{2q}\right)\frac{1-\mathrm{e}\left(-\frac{\ell}{q}\right)^{\lfloor Y-Q\rfloor}}{2i\sin\left(\frac{\pi\ell}{q}\right)} \qquad (4.2)\end{aligned}$$

since $\sin(2\pi x) = \dfrac{\mathrm{e}(x)-\mathrm{e}(-x)}{2i}$. By taking absolute values, we obtain

$$|\gamma(\ell)| = \frac{1}{q}\frac{\left|1-\mathrm{e}\left(-\frac{\ell}{q}\right)^{\lfloor Y-Q\rfloor}\right|}{\left|2\sin\left(\frac{\pi\ell}{q}\right)\right|}.$$

Since $\left|1-\mathrm{e}\left(-\frac{\ell}{q}\right)^{\lfloor Y-Q\rfloor}\right| \leq 2$, we have

$$|\gamma(\ell)| \leq \frac{1}{q}\frac{1}{\left|\sin\left(\frac{\pi\ell}{q}\right)\right|}. \qquad (4.3)$$

For $|x| \leq \frac{\pi}{2}$, we have $|\sin(x)| \geq \frac{2|x|}{\pi}$. Because $|\ell| \leq \frac{q}{2}$, the statement $\left|\frac{\pi\ell}{q}\right| \leq \frac{\pi}{2}$ is true. Therefore,

$$\left|\sin\left(\frac{\pi\ell}{q}\right)\right| \geq \frac{2|\ell|}{q}.$$

Substituting this into (4.3), we see that

$$|\gamma(\ell)| \leq \frac{1}{2|\ell|}.$$

Observe that $(2|\ell|)^{-1} \leq (1+|\ell|)^{-1}$ since $|\ell| \geq 1$. Therefore, $\gamma(\ell) \leq (1+|\ell|)^{-1}$ when $0 < |\ell| \leq \frac{q}{2}$.

Combining this result with the result when $\ell = 0$, we have $\gamma(\ell) \leq (1+|\ell|)^{-1}$. $\square$

4.2 Using Gauss sums

To examine $K(\ell, n, \mathbf{r}; q)$, we first analyze the Gauss sum $G_{\mathbf{r}}(d,q)$. We first develop a bound for $G_{\mathbf{r}}(d,q)$ when d and q are coprime.

Lemma 4.5. *If* $\gcd(d,q) = 1$ *and* $\mathbf{r} \in \mathbb{Z}^s$, *then*

$$|G_{\mathbf{r}}(d,q)| \leq (\gcd(L,q))^{s/2} q^{s/2}.$$

Proof. We have

$$\begin{aligned}
|G_{\mathbf{r}}(d,q)|^2 &= \left(\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r})\right)\right) \overline{\left(\sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{j}) + \mathbf{j}\cdot\mathbf{r})\right)\right)} \\
&= \left(\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r})\right)\right) \left(\sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(-\frac{1}{q}(dF(\mathbf{j}) + \mathbf{j}\cdot\mathbf{r})\right)\right) \\
&= \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(d(F(\mathbf{h}) - F(\mathbf{j})) + (\mathbf{h}-\mathbf{j})\cdot\mathbf{r})\right). \qquad (4.4)
\end{aligned}$$

Let $\mathbf{k} = \mathbf{h} - \mathbf{j}$. Then

$$\begin{aligned}
F(\mathbf{h}) &= F(\mathbf{j} + \mathbf{k}) \\
&= \frac{1}{2}(\mathbf{j} + \mathbf{k})^\top A(\mathbf{j} + \mathbf{k}) \\
&= \frac{1}{2}\mathbf{j}^\top A\mathbf{j} + \frac{1}{2}\mathbf{j}^\top A\mathbf{k} + \frac{1}{2}\mathbf{k}^\top A\mathbf{j} + \frac{1}{2}\mathbf{k}^\top A\mathbf{k}.
\end{aligned} \tag{4.5}$$

Because $\mathbf{k}^\top A\mathbf{j} \in \mathbb{R}$ and A is symmetric, we have

$$\begin{aligned}
\mathbf{k}^\top A\mathbf{j} &= (\mathbf{k}^\top A\mathbf{j})^\top \\
&= \mathbf{j}^\top A^\top \mathbf{k} \\
&= \mathbf{j}^\top A\mathbf{k}.
\end{aligned}$$

Substituting this into (4.5), we obtain

$$\begin{aligned}
F(\mathbf{h}) &= F(\mathbf{j}) + \mathbf{j}^\top A\mathbf{k} + F(\mathbf{k}) \\
&= F(\mathbf{j}) + \mathbf{j} \cdot (A\mathbf{k}) + F(\mathbf{k}).
\end{aligned}$$

Using this in (4.4), we see that

$$\begin{aligned}
|G_{\mathbf{r}}(d,q)|^2 &= \sum_{\mathbf{k} \in (\mathbb{Z}/q\mathbb{Z})^s} \sum_{\mathbf{j} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(d(\mathbf{j} \cdot (A\mathbf{k}) + F(\mathbf{k})) + \mathbf{k} \cdot \mathbf{r})\right) \\
&= \sum_{\mathbf{k} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{k}) + \mathbf{k} \cdot \mathbf{r})\right) \sum_{\mathbf{j} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\mathbf{j} \cdot (A\mathbf{k})\right).
\end{aligned} \tag{4.6}$$

Now let $\mathbf{w} = A\mathbf{k}$. Then

$$\begin{aligned}
\sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\mathbf{j}\cdot(A\mathbf{k})\right) &= \sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\mathbf{j}\cdot\mathbf{w}\right) \\
&= \sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\sum_{\ell=1}^{s} j_\ell w_\ell\right) \\
&= \sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \prod_{\ell=1}^{s} \mathrm{e}\left(\frac{dw_\ell j_\ell}{q}\right). \qquad (4.7)
\end{aligned}$$

Because $\mathbf{j}$ runs over all vectors in $(\mathbb{Z}/q\mathbb{Z})^s$, we have

$$\sum_{\mathbf{j}\in(\mathbb{Z}/q\mathbb{Z})^s} \prod_{\ell=1}^{s} \mathrm{e}\left(\frac{dw_\ell j_\ell}{q}\right) = \prod_{\ell=1}^{s}\left(\sum_{j_\ell\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{dw_\ell j_\ell}{q}\right)\right). \qquad (4.8)$$

By Lemma 4.3, this expression is equal to

$$\begin{aligned}
\prod_{\ell=1}^{s}\left(\sum_{j_\ell\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{dw_\ell j_\ell}{q}\right)\right) &= \prod_{\ell=1}^{s}\left(q\mathbf{1}_{\{dw_\ell\equiv 0 \pmod q\}}\right) \\
&= q^s\prod_{\ell=1}^{s}\mathbf{1}_{\{dw_\ell\equiv 0 \pmod q\}}. \qquad (4.9)
\end{aligned}$$

Because d is coprime to q, the last expression is equal to

$$\begin{aligned}
q^s\prod_{\ell=1}^{s}\mathbf{1}_{\{dw_\ell\equiv 0 \pmod q\}} &= q^s\prod_{\ell=1}^{s}\mathbf{1}_{\{w_\ell\equiv 0 \pmod q\}} \\
&= q^s\mathbf{1}_{\{\mathbf{w}\equiv\mathbf{0} \pmod q\}} \\
&= q^s\mathbf{1}_{\{A\mathbf{k}\equiv\mathbf{0} \pmod q\}}. \qquad (4.10)
\end{aligned}$$

Combining (4.6), (4.7), (4.8), (4.9), and (4.10), we obtain

$$|G_{\mathbf{r}}(d,q)|^2 = q^s\sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{k})+\mathbf{k}\cdot\mathbf{r})\right)\mathbf{1}_{\{A\mathbf{k}\equiv\mathbf{0} \pmod q\}}.$$

Therefore,

$$
\begin{aligned}
|G_{\mathbf{r}}(d,q)|^2 &= \left| q^s \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{k})+\mathbf{k}\cdot\mathbf{r})\right) \mathbf{1}_{\{A\mathbf{k}\equiv\mathbf{0}\ (\mathrm{mod}\, q)\}} \right| \\
&\le q^s \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \left| \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{k})+\mathbf{k}\cdot\mathbf{r})\right) \mathbf{1}_{\{A\mathbf{k}\equiv\mathbf{0}\ (\mathrm{mod}\, q)\}} \right| \\
&= q^s \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathbf{1}_{\{A\mathbf{k}\equiv\mathbf{0}\ (\mathrm{mod}\, q)\}} \\
&= q^s \left| \{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s : A\mathbf{k}\equiv\mathbf{0} \pmod q\}\right| .
\end{aligned} \tag{4.11}
$$

Recall that L is the smallest positive integer such that $LA^{-1}\in M_s(\mathbb{Z})$. Thus,

$$
\begin{aligned}
|\{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s : A\mathbf{k}\equiv\mathbf{0} \pmod q\}| &\le \left|\left\{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s : LA^{-1}A\mathbf{k}\equiv LA^{-1}\mathbf{0} \pmod q\right\}\right| \\
&= |\{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s : L\mathbf{k}\equiv\mathbf{0} \pmod q\}| \\
&= (\gcd(L,q))^s.
\end{aligned}
$$

Substituting this into (4.11), we conclude that

$$
|G_{\mathbf{r}}(d,q)|^2 \le q^s(\gcd(L,q))^s.
$$

By taking square roots of the previous inequality, we obtain the conclusion of this lemma. $\square$

Lemma 4.5 gives an upper bound for the absolute value of the Gauss sum $G_{\mathbf{r}}(d,q)$. If possible, we would like some exact calculations for the Gauss sum $G_{\mathbf{r}}(d,q)$. In order to do that, we need to look some other Gauss sums called quadratic Gauss sums.

Let $G\left(\dfrac{d}{q}\right)$ be the quadratic Gauss sum

$$
G\left(\frac{d}{q}\right) = \sum_{h\ (\mathrm{mod}\, q)} \mathrm{e}\left(\frac{dh^2}{q}\right).
$$

For an odd integer q, let

$$\varepsilon_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4, \\ i & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

The following lemma is a rephrasing of Theorem 1.5.2 in [BEW98] and gives a value for $G\left(\frac{d}{q}\right)$ when q is odd and d is coprime to q.

Lemma 4.6 (Theorem 1.5.2 in [BEW98]). *Let d be an integer and q be a positive odd integer. If $\gcd(d, q) = 1$, then*

$$G\left(\frac{d}{q}\right) = \left(\frac{d}{q}\right) \varepsilon_q \sqrt{q},$$

where $\left(\frac{\cdot}{q}\right)$ is the Jacobi symbol.

To use Lemma 4.6, we need to be able to diagonalize the Hessian matrix A modulo q when q is coprime $2\det(A)$. We will actually be able to diagonalize A modulo q whenever q is odd. To begin with, our next lemma says that you can diagonalize A when q is an odd prime power. It is a rephrasing of Theorem 31 in [Wat60].

Lemma 4.7 (Theorem 31 in [Wat60]). *Let F be an integral quadratic form with a Hessian matrix A. Suppose that p is an odd prime and k is a positive integer. Then there exist $D, P \in M_s(\mathbb{Z})$ such that D is diagonal, $p \nmid \det(P)$, and*

$$D \equiv P^{\top} A P \pmod{p^k}. \tag{4.12}$$

With Lemma 4.8, we have what we need to diagonalize the Hessian matrix A modulo q when q is odd.

Lemma 4.8. *Let F be an integral quadratic form with a Hessian matrix A. Suppose that q is odd. Then there exist $D, P \in M_s(\mathbb{Z})$ such that D is diagonal,* $\det(P)$ *and q*

are coprime, and

$$D \equiv P^{\top} A P \pmod{q}. \tag{4.13}$$

Proof. If $q = 1$, then there is nothing to prove. (Notice that every integer is congruent to any integer modulo 1, and the greatest common divisor of 1 and any integer is 1.)

If q is an odd prime power (say $q = p^k$), then we can apply Lemma 4.7 directly and obtain the result of this lemma.

Now suppose that $q = \prod_{j=1}^{\ell} p_j^{e_j}$, where the p_j are distinct primes. As already mentioned, for each j, we can find $D_j, P_j \in M_s(\mathbb{Z})$ such that D_j is diagonal, $\det(P_j)$ and q are coprime, and (4.13) is satisfied with $D = D_j$, $P = P_j$, and $q = p_j^{e_j}$. By applying the Chinese remainder theorem to create D from the D_j and P from the P_j, we obtain the result of this lemma. □

Since we have now developed the necessary tools, we explicitly compute $G_{\mathbf{r}}(d, q)$ when q is coprime to $2\det(A)d$.

Lemma 4.9. *Suppose that* $\gcd(q, 2\det(A)d) = 1$ *and* $\mathbf{r} \in \mathbb{Z}^s$. *Choose* $\alpha \in \mathbb{Z}$ *so that* $\alpha A^{-1} \in M_s(\mathbb{Z})$ *and* $\gcd(\alpha, q) = 1$. *(Such an* α *exists since* $\det(A)A^{-1} \in M_s(\mathbb{Z})$ *and* $\gcd(\det(A), q) = 1$.*) Then*

$$G_{\mathbf{r}}(d, q) = \left(\frac{\det(A)}{q}\right) \left(\varepsilon_q \left(\frac{2d}{q}\right) \sqrt{q}\right)^s \mathrm{e}\left(\frac{-d^*}{q} 2^* \alpha^* 2\alpha F^*(\mathbf{r})\right),$$

where $d^*, \alpha^*, 2^* \in \mathbb{Z}/q\mathbb{Z}$ *are such that* $dd^* \equiv \alpha\alpha^* \equiv 2(2^*) \equiv 1 \pmod{q}$.

Remark 4.10. Lemma 4.9 is similar to Lemma 20.13 in [IK04]. However, the lemmas concern slightly different quantities. Also, the statement of Lemma 20.13 in [IK04] has a few errors that we correct in our statement of Lemma 4.9.

Proof of Lemma 4.9. We apply Lemma 4.8 to obtain $D, P \in M_s(\mathbb{Z})$ such that D is diagonal, $\gcd(\det(P), q) = 1$, and

$$D \equiv P^\top AP \pmod{q}. \tag{4.14}$$

We have

$$\begin{aligned}
G_{\mathbf{r}}(d,q) &= \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r})\right) \\
&= \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}\left(\frac{1}{2}d\mathbf{h}^\top A\mathbf{h} + \mathbf{h}^\top\mathbf{r}\right)\right) \\
&= \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}\left(2^*d\mathbf{h}^\top A\mathbf{h} + \mathbf{h}^\top\mathbf{r}\right)\right).
\end{aligned}$$

(Note that we are able to use 2^* instead of $\frac{1}{2}$, because 2 divides $\mathbf{x}^\top A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{Z}^s$.) We do a change of variables $\mathbf{h} = P\mathbf{k}$ to obtain

$$\begin{aligned}
G_{\mathbf{r}}(d,q) &= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}\left(2^*d(P\mathbf{k})^\top AP\mathbf{k} + (P\mathbf{k})^\top\mathbf{r}\right)\right) \\
&= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}\left(2^*d\mathbf{k}^\top P^\top AP\mathbf{k} + \mathbf{k}^\top P^\top\mathbf{r}\right)\right) \\
&= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{2^*}{q}\left(d\mathbf{k}^\top D\mathbf{k} + 2\mathbf{k}\cdot(P^\top\mathbf{r})\right)\right).
\end{aligned} \tag{4.15}$$

Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_s)$ and $\mathbf{b} = P^\top \mathbf{r}$. Then (4.15) becomes

$$\begin{aligned}
G_{\mathbf{r}}(d,q) &= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{2^*}{q}\left(d\mathbf{k}^\top D\mathbf{k} + 2\mathbf{k}\cdot\mathbf{b}\right)\right) \\
&= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{2^*}{q}\sum_{j=1}^{s}\left(dd_jk_j^2 + 2b_jk_j\right)\right) \\
&= \sum_{\mathbf{k}\in(\mathbb{Z}/q\mathbb{Z})^s} \prod_{j=1}^{s} \mathrm{e}\left(\frac{2^*}{q}\left(dd_jk_j^2 + 2b_jk_j\right)\right) \\
&= \prod_{j=1}^{s}\left(\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*}{q}\left(dd_jk_j^2 + 2b_jk_j\right)\right)\right).
\end{aligned}$$

Note that each d_j is coprime to q since q is coprime to $\det(D) = \prod_{j=1}^s d_j$. (This is true, because q is coprime to $\det(A)$ and $\det(P)$ and we have (4.14).) Let d_j^* be the multiplicative inverse of d_j modulo q. Then by completing the square, we obtain

$$\begin{aligned}
G_{\mathbf{r}}(d,q) &= \prod_{j=1}^{s}\left(\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*dd_j}{q}\left(k_j^2 + 2d^*d_j^*b_jk_j\right)\right)\right) \\
&= \prod_{j=1}^{s}\left(\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*dd_j}{q}\left((k_j + d^*d_j^*b_j)^2 - (d^*d_j^*b_j)^2\right)\right)\right) \\
&= \prod_{j=1}^{s}\left(\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*}{q}\left(dd_j(k_j + d^*d_j^*b_j)^2 - d^*d_j^*b_j^2\right)\right)\right) \\
&= \prod_{j=1}^{s} \mathrm{e}\left(\frac{-2^*d^*d_j^*b_j^2}{q}\right)\left(\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*dd_j}{q}(k_j + d^*d_j^*b_j)^2\right)\right). \qquad (4.16)
\end{aligned}$$

Now we make a change of variables $\ell_j = k_j + d^*d_j^*b_j$ to see that

$$\sum_{k_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*dd_j}{q}(k_j + d^*d_j^*b_j)^2\right) = \sum_{\ell_j\in\mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^*dd_j}{q}\ell_j^2\right).$$

We apply Lemma 4.6 to obtain

$$\sum_{k_j \in \mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^* d d_j}{q}(k_j + d^* d_j^* b_j)^2\right) = \left(\frac{2^* d d_j}{q}\right) \varepsilon_q \sqrt{q}.$$

Because $\left(\frac{2^*}{q}\right) = \left(\frac{2}{q}\right)$, we have

$$\sum_{k_j \in \mathbb{Z}/q\mathbb{Z}} \mathrm{e}\left(\frac{2^* d d_j}{q}(k_j + d^* d_j^* b_j)^2\right) = \left(\frac{2 d d_j}{q}\right) \varepsilon_q \sqrt{q}.$$

Substituting this into (4.16), we see that

$$\begin{aligned} G_{\mathbf{r}}(d,q) &= \prod_{j=1}^{s} \mathrm{e}\left(\frac{-2^* d^* d_j^* b_j^2}{q}\right) \left(\frac{2 d d_j}{q}\right) \varepsilon_q \sqrt{q} \\ &= \left(\frac{\prod_{j=1}^{s} d_j}{q}\right) \left(\left(\frac{2d}{q}\right) \varepsilon_q \sqrt{q}\right)^s \prod_{j=1}^{s} \mathrm{e}\left(\frac{-2^* d^* d_j^* b_j^2}{q}\right) \\ &= \left(\frac{\det(D)}{q}\right) \left(\left(\frac{2d}{q}\right) \varepsilon_q \sqrt{q}\right)^s \mathrm{e}\left(\frac{-d^*}{q} 2^* \sum_{j=1}^{s} d_j^* b_j^2\right). \end{aligned} \tag{4.17}$$

Because $D \equiv P^\top A P \pmod{q}$, we know that $\det(D) \equiv \det(A)\det(P)^2 \pmod{q}$. Thus,

$$\left(\frac{\det(D)}{q}\right) = \left(\frac{\det(A)\det(P)^2}{q}\right) = \left(\frac{\det(A)}{q}\right), \tag{4.18}$$

and

$$G_{\mathbf{r}}(d,q) = \left(\frac{\det(A)}{q}\right) \left(\left(\frac{2d}{q}\right) \varepsilon_q \sqrt{q}\right)^s \mathrm{e}\left(\frac{-d^*}{q} 2^* \sum_{j=1}^{s} d_j^* b_j^2\right). \tag{4.19}$$

For a matrix B that is invertible over $\mathbb{Z}/q\mathbb{Z}$, let B^* be the multiplicative inverse of B modulo q. Note that $D^* = \operatorname{diag}(d_1^*, \ldots, d_s^*)$ and

$$D^* = (P^\top AP)^* = P^* A^* (P^\top)^*.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{s} d_j^* b_j^2 &\equiv \mathbf{b}^\top D^* \mathbf{b} \pmod q\\ &\equiv \mathbf{b}^\top P^* A^* (P^\top)^* \mathbf{b} \pmod q\,.\end{aligned}$$

Since $\mathbf{b} = P^\top \mathbf{r}$, we have

$$\begin{aligned}\sum_{j=1}^{s} d_j{}^* b_j^2 &\equiv (P^\top \mathbf{r})^\top P^* A^* (P^\top)^* P^\top \mathbf{r} \pmod q\\ &\equiv \mathbf{r}^\top P P^* A^* (P^\top)^* P^\top \mathbf{r} \pmod q\\ &\equiv \mathbf{r}^\top A^* \mathbf{r} \pmod q\,. \end{aligned} \tag{4.20}$$

A short calculation shows that

$$A^* \equiv \alpha^* \alpha A^{-1} \pmod q\,. \tag{4.21}$$

Substituting this into (4.20), we obtain

$$\begin{aligned}\sum_{j=1}^{s} d_j^* b_j^2 &\equiv \alpha^* \alpha \mathbf{r}^\top A^{-1} \mathbf{r} \pmod q\\ &\equiv 2\alpha^* \alpha F^*(\mathbf{r}) \pmod q\,.\end{aligned}$$

We substitute this into (4.19) and conclude that

$$G_{\mathbf{r}}(d,q) = \left(\frac{\det(A)}{q}\right) \left(\left(\frac{2d}{q}\right) \varepsilon_q \sqrt{q} \right)^s \mathrm{e}\!\left(\frac{-2^* d^*}{q} 2\alpha^* \alpha F^*(\mathbf{r}) \right). \qquad \square$$

4.3 Decomposing some complete sums

In the previous section, we derived some estimates for the Gauss sum $G_{\mathbf{r}}(d,q)$. We would like to apply these estimates to the complete sum $K(\ell,n,\mathbf{r};q)$. However, we have additional estimates for the Gauss sum when q is coprime to $2\det(A)$. We would like to exploit as much as we can from these estimates, so we split q into $q = q_0q_1$ such that q_0 is the largest factor of q having all of its prime divisors dividing $2\det(A)$ so that $\gcd(q_1, 2\det(A)) = 1$. To do this, we use the following lemma.

Lemma 4.11. *Let* $q = q_0q_1$ *such that* $\gcd(q_0,q_1) = 1$. *Let*

$$K^{(q_1)}(\ell,n,\mathbf{r};q_0) = \sum_{d_0\in(\mathbb{Z}/q_0\mathbb{Z})^\times} \mathrm{e}\left(\frac{q_1^*(\ell d_0 + nd_0^*)}{q_0}\right) G_{q_1^*\mathbf{r}}(-q_1^*d_0^*, q_0)$$

and

$$K^{(q_0)}(\ell,n,\mathbf{r};q_1) = \sum_{d_1\in(\mathbb{Z}/q_1\mathbb{Z})^\times} \mathrm{e}\left(\frac{q_0^*(\ell d_1 + nd_1^*)}{q_1}\right) G_{q_0^*\mathbf{r}}(-q_0^*d_1^*, q_1),$$

where $d_0^*, q_1^* \in \mathbb{Z}/q_0\mathbb{Z}$ *are such that* $d_0d_0^* \equiv q_1q_1^* \equiv 1 \pmod{q_0}$ *and* $d_1^*, q_0^* \in \mathbb{Z}/q_1\mathbb{Z}$ *are such that* $d_1d_1^* \equiv q_0q_0^* \equiv 1 \pmod{q_1}$. *Then*

$$K(\ell,n,\mathbf{r};q) = K^{(q_1)}(\ell,n,\mathbf{r};q_0)K^{(q_0)}(\ell,n,\mathbf{r};q_1). \tag{4.22}$$

Proof. For a given d, we have $d_0 \in \mathbb{Z}/q_0\mathbb{Z}$ and $d_1 \in \mathbb{Z}/q_1\mathbb{Z}$ such that $d_0 \equiv d \pmod{q_0}$ and $d_1 \equiv d \pmod{q_1}$. By the Chinese remainder theorem, we see that if d coprime to q, then $d \equiv d_0q_1q_1^* + d_1q_0q_0^* \pmod{q}$ and $d^* \equiv d_0^*q_1q_1^* + d_1^*q_0q_0^* \pmod{q}$. Applying this to $K(\ell, n, \mathbf{r}; q)$, we see that

$$\begin{aligned} K(\ell, n, \mathbf{r}; q) &= \sum_{d_0 \in (\mathbb{Z}/q_0\mathbb{Z})^\times} \sum_{d_1 \in (\mathbb{Z}/q_1\mathbb{Z})^\times} \mathrm{e}\left(\frac{\ell(d_0 q_1 q_1^* + d_1 q_0 q_0^*) + n(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*)}{q_0 q_1} \right) \\ &\qquad \times G_{\mathbf{r}}(-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*), q_0 q_1) \\ &= \sum_{d_0 \in (\mathbb{Z}/q_0\mathbb{Z})^\times} \sum_{d_1 \in (\mathbb{Z}/q_1\mathbb{Z})^\times} \mathrm{e}\left(\frac{\ell d_0 q_1^* + n d_0^* q_1^*}{q_0} \right) \mathrm{e}\left(\frac{\ell d_1 q_0^* + n d_1^* q_0^*}{q_1} \right) \\ &\qquad \times G_{\mathbf{r}}(-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*), q_0 q_1). \end{aligned} \tag{4.23}$$

We now take a look at the Gauss sum

$$G_{\mathbf{r}}(-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*), q_0 q_1) = \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0 q_1} (-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*) F(\mathbf{h}) + \mathbf{h} \cdot \mathbf{r} \right). \tag{4.24}$$

For a given $\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s$, we have $\mathbf{j} \in (\mathbb{Z}/q_0\mathbb{Z})^s$ and $\mathbf{k} \in (\mathbb{Z}/q_1\mathbb{Z})^s$ such that $\mathbf{j} \equiv \mathbf{h} \pmod{q_0}$ and $\mathbf{k} \equiv \mathbf{h} \pmod{q_1}$. By the Chinese remainder theorem, we see that $\mathbf{h} \equiv q_1 q_1^* \mathbf{j} + q_0 q_0^* \mathbf{k} \pmod{q}$. Therefore, (4.24) becomes

$$\begin{aligned} &G_{\mathbf{r}}(-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*), q_0 q_1) \\ &= \sum_{\mathbf{j} \in (\mathbb{Z}/q_0\mathbb{Z})^s} \sum_{\mathbf{k} \in (\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0 q_1} (-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*) F(q_1 q_1^* \mathbf{j} + q_0 q_0^* \mathbf{k}) \right) \\ &\qquad \times \mathrm{e}\left(\frac{1}{q_0 q_1} (q_1 q_1^* \mathbf{j} + q_0 q_0^* \mathbf{k}) \cdot \mathbf{r} \right). \\ &= \sum_{\mathbf{j} \in (\mathbb{Z}/q_0\mathbb{Z})^s} \sum_{\mathbf{k} \in (\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0 q_1} (-(d_0^* q_1 q_1^* + d_1^* q_0 q_0^*) F(q_1 q_1^* \mathbf{j} + q_0 q_0^* \mathbf{k})) \right) \\ &\qquad \times \mathrm{e}\left(\frac{q_1^*}{q_0} \mathbf{j} \cdot \mathbf{r} \right) \mathrm{e}\left(\frac{q_0^*}{q_1} \mathbf{k} \cdot \mathbf{r} \right). \end{aligned} \tag{4.25}$$

We now take a closer look at the quadratic form appearing in the Gauss sum:

$$\begin{aligned} F(q_1q_1^*\mathbf{j}+q_0q_0^*\mathbf{k}) &= \frac{1}{2}(q_1q_1^*\mathbf{j}+q_0q_0^*\mathbf{k})^\top A(q_1q_1^*\mathbf{j}+q_0q_0^*\mathbf{k}) \\ &= \frac{1}{2}(q_1q_1^*)^2\mathbf{j}^\top A\mathbf{j} + q_0q_0^*q_1q_1^*\mathbf{j}^\top A\mathbf{k} + \frac{1}{2}(q_0q_0^*)^2\mathbf{k}^\top A\mathbf{k} \\ &= (q_1q_1^*)^2F(\mathbf{j}) + q_0q_0^*q_1q_1^*\mathbf{j}^\top A\mathbf{k} + (q_0q_0^*)^2F(\mathbf{k}) \\ &= (q_1q_1^*)^2F(\mathbf{j}) + (q_0q_0^*)^2F(\mathbf{k}) \pmod{q_0q_1}. \end{aligned} \tag{4.26}$$

Substituting this into (4.25), we obtain

$$\begin{aligned} &G_{\mathbf{r}}(-(d_0^*q_1q_1^*+d_1^*q_0q_0^*), q_0q_1) \\ &= \sum_{\mathbf{j}\in(\mathbb{Z}/q_0\mathbb{Z})^s}\sum_{\mathbf{k}\in(\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0q_1}(-(d_0^*q_1q_1^*+d_1^*q_0q_0^*)((q_1q_1^*)^2F(\mathbf{j})+(q_0q_0^*)^2F(\mathbf{k})))\right) \\ &\qquad\qquad \times \mathrm{e}\left(\frac{q_1^*}{q_0}\mathbf{j}\cdot\mathbf{r}\right)\mathrm{e}\left(\frac{q_0^*}{q_1}\mathbf{k}\cdot\mathbf{r}\right) \\ &= \sum_{\mathbf{j}\in(\mathbb{Z}/q_0\mathbb{Z})^s}\sum_{\mathbf{k}\in(\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0}(-d_0^*q_1^*((q_1q_1^*)^2F(\mathbf{j})+(q_0q_0^*)^2F(\mathbf{k})))\right) \\ &\qquad\qquad \times \mathrm{e}\left(\frac{1}{q_1}(-d_1^*q_0^*((q_1q_1^*)^2F(\mathbf{j})+(q_0q_0^*)^2F(\mathbf{k})))\right)\mathrm{e}\left(\frac{q_1^*}{q_0}\mathbf{j}\cdot\mathbf{r}\right)\mathrm{e}\left(\frac{q_0^*}{q_1}\mathbf{k}\cdot\mathbf{r}\right) \\ &= \sum_{\mathbf{j}\in(\mathbb{Z}/q_0\mathbb{Z})^s}\sum_{\mathbf{k}\in(\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0}(-d_0^*q_1^*F(\mathbf{j}))\right)\mathrm{e}\left(\frac{1}{q_1}(-d_1^*q_0^*F(\mathbf{k}))\right)\mathrm{e}\left(\frac{q_1^*}{q_0}\mathbf{j}\cdot\mathbf{r}\right)\mathrm{e}\left(\frac{q_0^*}{q_1}\mathbf{k}\cdot\mathbf{r}\right) \\ &= \sum_{\mathbf{j}\in(\mathbb{Z}/q_0\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_0}(-d_0^*q_1^*F(\mathbf{j})+q_1^*\mathbf{j}\cdot\mathbf{r})\right)\sum_{\mathbf{k}\in(\mathbb{Z}/q_1\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q_1}(-d_1^*q_0^*F(\mathbf{k})+q_0^*\mathbf{k}\cdot\mathbf{r})\right) \\ &= G_{q_1^*\mathbf{r}}(-d_0^*q_1^*, q_0)G_{q_0^*\mathbf{r}}(-d_1^*q_0^*, q_1). \end{aligned} \tag{4.27}$$

Substituting this into (4.23), we obtain (4.22). □

From now on, unless otherwise specified, let q_0 be the largest factor of q having all of its prime divisors dividing $2\det(A)$ and $q_1 = q/q_0$ so that $\gcd(q_1, 2\det(A)) = 1$. Note that q_0 and q_1 are coprime.

4.4 Estimating some complete sums

Now that we are able to decompose $K(\ell, n, \mathbf{r}; q)$ into other sums, we can provide better estimates on $K(\ell, n, \mathbf{r}; q)$. We begin by bounding $K^{(q_1)}(\ell, n, \mathbf{r}; q_0)$.

Lemma 4.12. *The sum $K^{(q_1)}(\ell, n, \mathbf{r}; q_0)$ satisfies the following:*

$$|K^{(q_1)}(\ell, n, \mathbf{r}; q_0)| \le (\gcd(L, q_0))^{s/2} q_0^{s/2+1}.$$

Proof. Observe that

$$\begin{aligned} |K^{(q_1)}(\ell, n, \mathbf{r}; q_0)| &= \left| \sum_{d_0 \in (\mathbb{Z}/q_0\mathbb{Z})^\times} \mathrm{e}\left(\frac{q_1^*(\ell d_0 + n d_0^*)}{q_0} \right) G_{q_1^* \mathbf{r}}(-q_1^* d_0^*, q_0) \right| \\ &\le \sum_{d_0 \in (\mathbb{Z}/q_0\mathbb{Z})^\times} \left| G_{q_1^* \mathbf{r}}(-q_1^* d_0^*, q_0) \right|. \end{aligned} \tag{4.28}$$

By applying Lemma 4.5 to $\left| G_{q_1^* \mathbf{r}}(-q_1^* d_0^*, q_0) \right|$ in (4.28), we obtain

$$\begin{aligned} |K^{(q_1)}(\ell, n, \mathbf{r}; q_0)| &\le \sum_{d_0 \in (\mathbb{Z}/q_0\mathbb{Z})^\times} (\gcd(L, q_0))^{s/2} q_0^{s/2} \\ &\le (\gcd(L, q_0))^{s/2} q_0^{s/2+1}. \end{aligned}$$

□

Since q_1 is coprime to $2\det(A)$, we can do a better job in evaluating and possibly bounding $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$. We first write $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$ in terms of a Kloosterman sum or a Salié sum. We obtain the following by applying Lemma 4.9 to the definition of $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$.

Lemma 4.13. *Let $\alpha \in \mathbb{Z}$ be such that $\alpha A^{-1} \in M_s(\mathbb{Z})$ and $\gcd(\alpha, q_1) = 1$. The sum*

$K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$ has the following evaluation:

$$K^{(q_0)}(\ell, n, \mathbf{r}; q_1) = \left(\frac{\det(A)}{q_1}\right)\left(\varepsilon_{q_1}\left(\frac{-2q_0^*}{q_1}\right)\sqrt{q_1}\right)^s \kappa_{s,q_1}(q_0^*(\ell + 2^*\alpha^*2\alpha F^*(\mathbf{r})), q_0^* n), \tag{4.29}$$

where

$$\kappa_{s,q_1}(a,b) = \sum_{d\,(\mathrm{mod}\,q_1)} \left(\frac{d}{q_1}\right)^s \mathrm{e}\left(\frac{ad+bd^*}{q_1}\right) \tag{4.30}$$

is either a Kloosterman sum (if s is even) or a Salié sum (if s is odd).

Proof. We first evaluate $G_{q_0^*\mathbf{r}}(-q_0^* d^*, q_1)$ when d is coprime to q_1. Lemma 4.9 shows us that

$$\begin{aligned} G_{q_0^*\mathbf{r}}(-q_0^* d^*, q_1) &= \left(\frac{\det(A)}{q_1}\right)\left(\varepsilon_{q_1}\left(\frac{-2q_0^* d^*}{q_1}\right)\sqrt{q_1}\right)^s \mathrm{e}\left(-\frac{(-q_0^* d^*)^*}{q_1} 2^*\alpha^*2\alpha F^*(q_0^*\mathbf{r})\right) \\ &= \left(\frac{\det(A)}{q_1}\right)\left(\varepsilon_{q_1}\left(\frac{-2q_0^*}{q_1}\right)\sqrt{q_1}\right)^s \left(\frac{d^*}{q_1}\right)^s \mathrm{e}\left(\frac{q_0 d}{q_1} 2^*\alpha^*2\alpha(q_0^*)^2 F^*(\mathbf{r})\right) \\ &= \left(\frac{\det(A)}{q_1}\right)\left(\varepsilon_{q_1}\left(\frac{-2q_0^*}{q_1}\right)\sqrt{q_1}\right)^s \left(\frac{d}{q_1}\right)^s \mathrm{e}\left(\frac{dq_0^* 2^*\alpha^*2\alpha F^*(\mathbf{r})}{q_1}\right) \end{aligned} \tag{4.31}$$

since $\left(\frac{d^*}{q_1}\right) = \left(\frac{d}{q_1}\right)$.

Substituting (4.31) into the definition of $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$, we obtain

$$\begin{aligned} &K^{(q_0)}(\ell, n, \mathbf{r}; q_1) \\ &= \left(\frac{\det(A)}{q_1}\right)\left(\varepsilon_{q_1}\left(\frac{-2q_0^*}{q_1}\right)\sqrt{q_1}\right)^s \\ &\quad \times \sum_{\substack{d\,(\mathrm{mod}\,q_1) \\ \gcd(d,q_1)=1}} \mathrm{e}\left(\frac{q_0^*((\ell + 2^*\alpha^*2\alpha F^*(\mathbf{r}))d + nd^*)}{q_1}\right)\left(\frac{d}{q_1}\right)^s, \end{aligned}$$

which implies (4.29) since $\left(\frac{d}{q_1}\right) = 0$ if $\gcd(d, q_1) \neq 1$. □

To obtain a bound on $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$, we need bounds on Kloosterman sums and Salié sums. When s is even, we use a corollary (Corollary 11.12 in [IK04]) of Weil's bound for Kloosterman sums.

Lemma 4.14 (Corollary 11.12 in [IK04])**.** *If s is even, a and b are integers, and q is a positive integer, then*

$$|\kappa_{s,q}(a, b)| \leq \tau(q)(\gcd(a, b, q))^{1/2} q^{1/2}.$$

When s is odd, notice that $\kappa_{s,q}(a, b) = \kappa_{1,q}(a, b)$. To obtain a bound for Salié sums, we use Lemma 12.4 in [IK04] to evaluate Salié sums under certain circumstances.

Lemma 4.15 (Lemma 12.4 in [IK04])**.** *Suppose that a and b are integers, q is a positive integer, and* $\gcd(q, 2a) = 1$. *Then*

$$\kappa_{1,q}(a, b) = \varepsilon_q q^{1/2} \left(\frac{a}{q}\right) \sum_{\substack{v \,(\mathrm{mod}\, q) \\ v^2 \equiv ab \,(\mathrm{mod}\, q)}} \mathrm{e}\left(\frac{2v}{q}\right).$$

We would like to partially remove the coprimallity condition for the evaluation of Salié sums. To do this, we notice that $\kappa_{1,q}(a, b) = \kappa_{1,q}(b, a)$ since $\left(\frac{x}{q}\right) = \left(\frac{x^*}{q}\right)$. Therefore, we only need to consider the case when a, b, and q have a common divisor. (For our purposes, q will already be odd.)

To do this, we first decompose Salié sums so that we only have to concern ourselves with the case that q is a prime power, i.e., $q = p^k$ for some odd prime p. This decomposition is made precise with the following lemma about the twisted multiplicatiive property of Salié sums.

Lemma 4.16. *Suppose that a and b are integers and q_1 and q_2 are positive odd integers such that* $\gcd(q_1, q_2) = 1$. *Then*

$$\kappa_{1,q_1 q_2}(a, b) = \kappa_{1,q_1}(a q_2^*, b q_2^*) \kappa_{1,q_2}(a q_1^*, b q_1^*),$$

where $q_2q_2^* \equiv 1 \pmod{q_1}$ *and* $q_1q_1^* \equiv 1 \pmod{q_2}$.

Proof. By definition,

$$\begin{aligned}
\kappa_{1,q_1q_2}(a,b) &= \sum_{d \,(\mathrm{mod}\, q_1q_2)} \left(\frac{d}{q_1q_2}\right) \mathrm{e}\left(\frac{ad+bd^*}{q_1q_2}\right) \\
&= \sum_{d \,(\mathrm{mod}\, q_1q_2)} \left(\frac{d}{q_1}\right)\left(\frac{d}{q_2}\right) \mathrm{e}\left(\frac{ad+bd^*}{q_1q_2}\right).
\end{aligned} \tag{4.32}$$

For a given d coprime to q_1q_2, we have $d_1 \in \mathbb{Z}/q_1\mathbb{Z}$ and $d_2 \in \mathbb{Z}/q_2\mathbb{Z}$ such that $d_1 \equiv d \pmod{q_1}$ and $d_2 \equiv d \pmod{q_2}$. Let $d_1^* \in \mathbb{Z}/q_1\mathbb{Z}$ be such that $d_1d_1^* \equiv 1 \pmod{q_1}$, and let $d_2^* \in \mathbb{Z}/q_2\mathbb{Z}$ be such that $d_2d_2^* \equiv 1 \pmod{q_2}$. By the Chinese remainder theorem, we have $d \equiv d_1q_2q_2^* + d_2q_1q_1^* \pmod{q_1q_2}$ and $d^* \equiv d_1^*q_2q_2^* + d_2^*q_1q_1^* \pmod{q_1q_2}$. Applying these facts to (4.32), we see that

$$\begin{aligned}
&\kappa_{1,q_1q_2}(a,b) \\
&= \sum_{d_1 \,(\mathrm{mod}\, q_1)} \sum_{d_2 \,(\mathrm{mod}\, q_2)} \left(\frac{d_1}{q_1}\right)\left(\frac{d_2}{q_2}\right) \mathrm{e}\left(\frac{a(d_1q_2q_2^*+d_2q_1q_1^*)+b(d_1^*q_2q_2^*+d_2^*q_1q_1^*)}{q_1q_2}\right) \\
&= \sum_{d_1 \,(\mathrm{mod}\, q_1)} \left(\frac{d_1}{q_1}\right) \mathrm{e}\left(\frac{aq_2^*d_1+bq_2^*d_1^*}{q_1}\right) \sum_{d_2 \,(\mathrm{mod}\, q_2)} \left(\frac{d_2}{q_2}\right) \mathrm{e}\left(\frac{aq_1^*d_2+bq_1^*d_2^*}{q_2}\right) \\
&= \kappa_{1,q_1}(aq_2^*, bq_2^*)\kappa_{1,q_2}(aq_1^*, bq_1^*).
\end{aligned}$$

□

The previous lemma can be applied repeatedly to $\kappa_{1,c}(a,b)$ to obtain a product of Salié sums with each of their denominators being prime powers. This allows us to only consider the case of q being a prime power. We would like to bound on Salié sums when q is a prime power. To do this, we need the following bound on an exponential sum.

Lemma 4.17. *Suppose that a is an integer, p is an odd prime, and k is a positive* *integer. Then*

$$\left| \sum_{\substack{v \pmod{p^k} \\ v^2 \equiv a \pmod{p^k}}} \mathrm{e}\left(\frac{2v}{p^k}\right) \right| \leq 2. \tag{4.33}$$

Proof. First suppose that $a \equiv 0 \pmod{p^k}$. Then the sum in (4.33) is a sum over v of the form $v = p^{\lceil k/2 \rceil} u$, where $u \in \mathbb{Z}/p^{\lfloor k/2 \rfloor}\mathbb{Z}$. That is,

$$\begin{aligned} \sum_{\substack{v \pmod{p^k} \\ v^2 \equiv a \pmod{p^k}}} \mathrm{e}\left(\frac{2v}{p^k}\right) &= \sum_{u \in \mathbb{Z}/p^{\lfloor k/2 \rfloor}\mathbb{Z}} \mathrm{e}\left(\frac{2p^{\lceil k/2 \rceil} u}{p^k}\right) \\ &= \sum_{u \in \mathbb{Z}/p^{\lfloor k/2 \rfloor}\mathbb{Z}} \mathrm{e}\left(\frac{2u}{p^{\lfloor k/2 \rfloor}}\right) \\ &= \mathbf{1}_{\{\lfloor k/2 \rfloor = 0\}} \end{aligned}$$

by Lemma 4.3 and the fact that p is odd. Therefore, if $a \equiv 0 \pmod{p^k}$, then (4.33) holds.

Now suppose that $a \not\equiv 0 \pmod{p^k}$. Let $a_0 p^{a_1} \equiv a \pmod{p^k}$, where $0 \leq a_1 \leq k-1$ and $a_0 \in (\mathbb{Z}/p^{k-a_1}\mathbb{Z})^\times$. We first explore the possible solutions to

$$v^2 \equiv a \pmod{p^k}. \tag{4.34}$$

If there exists a solution $v \in \mathbb{Z}/p^k\mathbb{Z}$ to (4.34), then $v \not\equiv 0 \pmod{p^k}$ and there exist integers v_0 and v_1 such that $0 \leq v_1 \leq k-1$, $0 \leq v_0 \leq p^{k-v_1} - 1$, $\gcd(v_0, p^{k-v_1}) = 1$, and $v \equiv v_0 p^{v_1} \pmod{p^k}$. In order for $v^2 \equiv a \pmod{p^k}$, we must have

$$v_1 = \frac{a_1}{2} \tag{4.35}$$

and

$$v_0^2 \equiv a_0 \pmod{p^{k-a_1}}. \tag{4.36}$$

Because $(\mathbb{Z}/p^{k-a_1}\mathbb{Z})^\times$ is cyclic, there are either zero or two solutions $x_0 \in \mathbb{Z}$ such that $0 \leq x_0 \leq p^{k-a_1} - 1$, $\gcd(x_0, p^{k-a_1}) = 1$, and x_0 satisfies

$$x_0^2 \equiv a_0 \pmod{p^{k-a_1}}. \tag{4.37}$$

Each solution x_0 lifts to $p^{a_1-v_1} = p^{\frac{a_1}{2}}$ solutions for $v_0 \in (\mathbb{Z}/p^{k-v_1}\mathbb{Z})^\times = (\mathbb{Z}/p^{k-\frac{a_1}{2}}\mathbb{Z})^\times$. Each lift is of the form

$$v_0 \equiv x_0 + x_1 p^{k-a_1} \pmod{p^{k-\frac{a_1}{2}}}, \tag{4.38}$$

where $x_1 \in \mathbb{Z}/p^{\frac{a_1}{2}}\mathbb{Z}$. One can verify that each v_0 of the form in (4.38) contributes to a unique solution $v = v_0 p^{\frac{a_1}{2}}$ to (4.34). Therefore, we have shown that there are either 0 or $2p^{\frac{a_1}{2}}$ solutions to (4.34) modulo $p^{k-\ell}$.

It is clear that if there are no solutions to (4.34), then

$$\sum_{\substack{v \pmod{p^k} \\ v^2 \equiv a \pmod{p^k}}} \mathrm{e}\left(\frac{2v}{p^k}\right) = 0.$$

Thus, we only need to concern ourselves with the case that there are $2p^{\frac{a_1}{2}}$ solutions to (4.34). Let $x_0 \in \mathbb{Z} \cap [0, p^{k-a_1} - 1]$ be coprime to p^{k-a_1} and satisfy (4.37). The only other integer in $[0, p^{k-a_1} - 1]$ that is coprime to p^{k-a_1} and satisfies (4.37) is $p^{k-a_1} - x_0$. (Note that x_0 and $p^{k-a_1} - x_0$ are distinct modulo p^{k-a_1} since p is odd.) Therefore, using (4.38), we find that

$$\begin{aligned}
&\sum_{\substack{v \pmod{p^k} \\ v^2 \equiv a \pmod{p^k}}} \mathrm{e}\left(\frac{2v}{p^k}\right) \\
&= \sum_{x_1 \in \mathbb{Z}/p^{\frac{a_1}{2}}\mathbb{Z}} \left(\mathrm{e}\left(\frac{2(x_0 + x_1 p^{k-a_1})p^{\frac{a_1}{2}}}{p^k}\right) + \mathrm{e}\left(\frac{2(p^{k-a_1} - x_0 + x_1 p^{k-a_1})p^{\frac{a_1}{2}}}{p^k}\right)\right) \\
&= \left(\mathrm{e}\left(\frac{2x_0}{p^{k-\frac{a_1}{2}}}\right) + \mathrm{e}\left(\frac{2(p^{k-a_1} - x_0)}{p^{k-\frac{a_1}{2}}}\right)\right) \sum_{x_1 \in \mathbb{Z}/p^{\frac{a_1}{2}}\mathbb{Z}} \mathrm{e}\left(\frac{2x_1}{p^{\frac{a_1}{2}}}\right).
\end{aligned}$$

By Lemma 4.3 and the fact that p is odd, we obtain

$$\sum_{\substack{v \pmod{p^k} \\ v^2 \equiv a \pmod{p^k}}} \mathrm{e}\left(\frac{2v}{p^k}\right) = \left(\mathrm{e}\left(\frac{2x_0}{p^{k-\frac{a_1}{2}}}\right) + \mathrm{e}\left(\frac{2(p^{k-a_1} - x_0)}{p^{k-\frac{a_1}{2}}}\right)\right) \mathbf{1}_{\{a_1=0\}}. \tag{4.39}$$

We take absolute values of both sides of (4.39) to obtain (4.33). □

Using Lemma 4.17, we obtain the following bound on Salié sums when q is a prime power.

Lemma 4.18. *Suppose that a and b are integers, k is a positive integer, ℓ is a nonnegative integer, p is an odd prime, and* $\gcd(a, p) = 1$. *Then*

$$|\kappa_{1,p^k}(ap^\ell, bp^\ell)| \le \tau(p^k)(\gcd(ap^\ell, bp^\ell, p^k))^{1/2}(p^k)^{1/2}. \tag{4.40}$$

Proof. By definition,

$$\kappa_{1,p^k}(ap^\ell, bp^\ell) = \sum_{d \pmod{p^k}} \left(\frac{d}{p^k}\right) \mathrm{e}\left(\frac{ap^\ell d + bp^\ell d^*}{p^k}\right).$$

Suppose that $\ell \ge k$. Then $\mathrm{e}\left(\frac{ap^\ell d + bp^\ell d^*}{p^k}\right)$ is always 1. Thus, since $\gcd(ap^\ell, bp^\ell, p^k) =$

p^k, when we apply a trivial bound for the sum, we obtain

$$|\kappa_{1,p^k}(ap^\ell, bp^\ell)| \le p^k \le \tau(p^k)(\gcd(ap^\ell, bp^\ell, p^k))^{1/2}(p^k)^{1/2}.$$

Now suppose that $\ell < k$. Then

$$\kappa_{1,p^k}(ap^\ell, bp^\ell) = \sum_{d \,\left(\operatorname{mod} p^k\right)} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right).$$

Now $\left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right)$ is periodic modulo $p^{k-\ell}$, so

$$\kappa_{1,p^k}(ap^\ell, bp^\ell) = p^\ell \sum_{d \,\left(\operatorname{mod} p^{k-\ell}\right)} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right). \tag{4.41}$$

If k is even, then $\sum_{d \,\left(\operatorname{mod} p^{k-\ell}\right)} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right)$ is a Kloosterman sum and we can apply Lemma 4.14 and see that

$$\begin{aligned}|\kappa_{1,p^k}(ap^\ell, bp^\ell)| &= p^\ell \tau(p^{k-\ell}) \gcd(a, b, p^{k-\ell})^{1/2}(p^{k-\ell})^{1/2} = \tau(p^{k-\ell})(p^\ell)^{1/2}(p^k)^{1/2} \\ &\le \tau(p^k)(p^\ell)^{1/2}(p^k)^{1/2}.\end{aligned}$$

Since $\gcd(ap^\ell, bp^\ell, p^k) = p^\ell$, we obtain (4.40) in the case that $\ell < k$ and k is even.

Now suppose that k is odd and ℓ is even. Then

$$\sum_{d \,\left(\operatorname{mod} p^{k-\ell}\right)} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) = \kappa_{1,p^{k-\ell}}(a, b),$$

so we can apply Lemma 4.15 and obtain

$$\kappa_{1,p^k}(ap^\ell, bp^\ell) = p^\ell \varepsilon_{p^{k-\ell}} (p^{k-\ell})^{1/2} \left(\frac{a}{p^{k-\ell}}\right) \sum_{\substack{v \pmod{p^{k-\ell}} \\ v^2 \equiv ab \pmod{p^{k-\ell}}}} \mathrm{e}\left(\frac{2v}{p^{k-\ell}}\right).$$

We take absolute values of both sides to obtain

$$|\kappa_{1,p^k}(ap^\ell, bp^\ell)| = (p^\ell)^{1/2}(p^k)^{1/2} \left| \sum_{\substack{v \pmod{p^{k-\ell}} \\ v^2 \equiv ab \pmod{p^{k-\ell}}}} \mathrm{e}\left(\frac{2v}{p^{k-\ell}}\right) \right|. \tag{4.42}$$

By applying Lemma 4.17 to (4.42), we find that

$$|\kappa_{1,p^k}(ap^\ell, bp^\ell)| \leq 2(p^\ell)^{1/2}(p^k)^{1/2}. \tag{4.43}$$

Since $\gcd(ap^\ell, bp^\ell, p^k) = p^\ell$ and $\tau(p^k) \geq 2$, we obtain (4.40) in the case that $\ell < k$, k is odd, and ℓ is even.

Now suppose that k and ℓ are odd. Then

$$\sum_{d \pmod{p^{k-\ell}}} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) = \sum_{d \pmod{p^{k-\ell}}} \left(\frac{d}{p}\right) \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) \tag{4.44}$$

and $k-\ell$ is even.

Let $d = d_1 + d_2 p^{\frac{k-\ell}{2}}$, where $0 \leq d_1 \leq p^{\frac{k-\ell}{2}} - 1$ and $d_2 \in \mathbb{Z}/p^{\frac{k-\ell}{2}}\mathbb{Z}$. If $\gcd(d_1, p) = 1$, then let $d_1^* \in \mathbb{Z}$ be such that $0 < d_1^* < p^{k-\ell}$ and $d_1 d_1^* \equiv 1 \pmod{p^{k-\ell}}$. Observe that $d^* \equiv d_1^* - (d_1^*)^2 d_2 p^{\frac{k-\ell}{2}} \pmod{p^{k-\ell}}$ since

$$\begin{aligned}(d_1 + d_2 p^{\frac{k-\ell}{2}})(d_1^* - (d_1^*)^2 d_2 p^{\frac{k-\ell}{2}}) &= d_1 d_1^* - d_1^* d_2 p^{\frac{k-\ell}{2}} + d_1^* d_2 p^{\frac{k-\ell}{2}} - (d_1^*)^2 d_2^2 p^{k-\ell} \\ &\equiv 1 \pmod{p^{k-\ell}}.\end{aligned}$$

Using this in (4.44), we see that

$$\begin{aligned}
&\sum_{d \pmod{p^{k-\ell}}} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) \\
&= \sum_{d_1=0}^{p^{\frac{k-\ell}{2}}-1} \sum_{d_2 \pmod{p^{\frac{k-\ell}{2}}}} \left(\frac{d_1+d_2p^{\frac{k-\ell}{2}}}{p}\right) \mathrm{e}\left(\frac{a(d_1+d_2p^{\frac{k-\ell}{2}})+b(d_1^*-(d_1^*)^2d_2p^{\frac{k-\ell}{2}})}{p^{k-\ell}}\right) \\
&= \sum_{d_1=0}^{p^{\frac{k-\ell}{2}}-1} \left(\frac{d_1}{p}\right) \mathrm{e}\left(\frac{ad_1+bd_1^*}{p^{k-\ell}}\right) \sum_{d_2 \pmod{p^{\frac{k-\ell}{2}}}} \mathrm{e}\left(\frac{(a-b(d_1^*)^2)d_2}{p^{\frac{k-\ell}{2}}}\right). \qquad (4.45)
\end{aligned}$$

Lemma 4.3 implies that

$$\sum_{d_2 \pmod{p^{\frac{k-\ell}{2}}}} \mathrm{e}\left(\frac{(a-b(d_1^*)^2)d_2}{p^{\frac{k-\ell}{2}}}\right) = p^{\frac{k-\ell}{2}} \mathbf{1}_{\left\{a \equiv b(d_1^*)^2 \pmod{p^{\frac{k-\ell}{2}}}\right\}}.$$

Substituting this into (4.45), we obtain

$$\begin{aligned}
&\sum_{d \pmod{p^{k-\ell}}} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) \\
&= p^{\frac{k-\ell}{2}} \sum_{d_1=0}^{p^{\frac{k-\ell}{2}}-1} \left(\frac{d_1}{p}\right) \mathrm{e}\left(\frac{ad_1+bd_1^*}{p^{k-\ell}}\right) \mathbf{1}_{\left\{a \equiv b(d_1^*)^2 \pmod{p^{\frac{k-\ell}{2}}}\right\}}.
\end{aligned}$$

Because $\left(\frac{d_1}{p}\right) = 0$ if $\gcd(d_1, p) \neq 1$, we have

$$\begin{aligned}
&\sum_{d \pmod{p^{k-\ell}}} \left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right) \\
&= p^{\frac{k-\ell}{2}} \sum_{d_1=0}^{p^{\frac{k-\ell}{2}}-1} \left(\frac{d_1}{p}\right) \mathrm{e}\left(\frac{ad_1+bd_1^*}{p^{k-\ell}}\right) \mathbf{1}_{\left\{\gcd(d_1,p)=1,\, a \equiv b(d_1^*)^2 \pmod{p^{\frac{k-\ell}{2}}}\right\}}.
\end{aligned}$$

We take absolute values of both sides to obtain

$$\begin{aligned}&\left|\sum_{d\,\left(\operatorname{mod} p^{k-\ell}\right)}\left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right)\right| \\ &\le p^{\frac{k-\ell}{2}}\left|\left\{d_1\in\mathbb{Z}: 0\le d_1\le p^{\frac{k-\ell}{2}}-1,\ \gcd(d_1,p)=1,\ a\equiv b(d_1^*)^2\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)\right\}\right|.\end{aligned} \tag{4.46}$$

Note that if $\gcd(d_1,p)=1$, then the condition $a\equiv b(d_1^*)^2\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)$ is equivalent to the condition $d_1^2\equiv a^*b\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)$. Thus, it suffices to count the number of d_1 such that $0\le d_1\le p^{\frac{k-\ell}{2}}-1$, $\gcd(d_1,p)=1$, and $d_1^2\equiv a^*b\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)$. Now the number of $d_0\ (\operatorname{mod} p)$ such that $\gcd(d_0,p)=1$ and $d_0^2\equiv a^*b\ (\operatorname{mod} p)$ is $1+\left(\frac{a^*b}{p}\right)$. By applying Hensel's lemma, we see that the number of $d_1\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)$ such that $\gcd(d_1,p)=1$ and $d_1^2\equiv a^*b\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)$ is also $1+\left(\frac{a^*b}{p}\right)$. (Here is the reason why Hensel's lemma is applicable in this instance: If there exists a solution $d_0\ (\operatorname{mod} p)$ such that $\gcd(d_0,p)=1$ and $d_0^2\equiv a^*b\ (\operatorname{mod} p)$, then $d_0^2-a^*b\equiv 0\ (\operatorname{mod} p)$ and $2d_0\not\equiv 0\ (\operatorname{mod} p)$.) Hence, we have

$$\begin{aligned}&\left|\left\{d_1\in\mathbb{Z}: 0\le d_1\le p^{\frac{k-\ell}{2}}-1,\ \gcd(d_1,p)=1,\ a\equiv b(d_1^*)^2\ \left(\operatorname{mod} p^{\frac{k-\ell}{2}}\right)\right\}\right| \\ &=1+\left(\frac{a^*b}{p}\right).\end{aligned}$$

Applying this to (4.46), we find that

$$\left|\sum_{d\,\left(\operatorname{mod} p^{k-\ell}\right)}\left(\frac{d}{p}\right)^k \mathrm{e}\left(\frac{ad+bd^*}{p^{k-\ell}}\right)\right| \le p^{\frac{k-\ell}{2}}\left(1+\left(\frac{a^*b}{p}\right)\right)\le 2p^{\frac{k-\ell}{2}}.$$

We apply this to (4.41) to see that

$$|\kappa_{1,p^k}(ap^\ell,bp^\ell)|\le 2p^\ell p^{\frac{k-\ell}{2}}=2(p^\ell)^{1/2}(p^k)^{1/2}\le\tau(p^k)(p^\ell)^{1/2}(p^k)^{1/2}.$$

Since $\gcd(ap^\ell, bp^\ell, p^k) = p^\ell$, we obtain (4.40) in the case that $\ell < k$ and k and ℓ are both odd. □

Because the expression $\tau(q)(\gcd(a, b, q))^{1/2}q^{1/2}$ is multiplicative as a function of q, we can repeatedly apply Lemmas 4.16 and 4.18 to obtain the following result.

Lemma 4.19. *If s is odd, a and b are integers, and q is a positive odd integer, then*

$$|\kappa_{s,q}(a, b)| \leq \tau(q)(\gcd(a, b, q))^{1/2}q^{1/2}. \tag{4.47}$$

Proof. First of all, notice that $\kappa_{s,q}(a, b) = \kappa_{1,q}(a, b)$ when s is odd.

If $q = 1$, then both sides of the inequality in (4.47) are equal to 1.

If q is a prime power, then Lemma 4.18 gives the result of this lemma.

Now suppose that $q > 1$ is not a prime power. Then $q = p^k t$, where p is an odd prime, k is a positive integer, t is a positive odd integer, and $\gcd(p^k, t) = 1$.

We will use the principle of mathematical induction to complete this proof. Assume that for each positive odd integer $v < q$ that

$$|\kappa_{s,v}(c, d)| \leq \tau(v)(\gcd(c, d, v))^{1/2}v^{1/2} \tag{4.48}$$

holds for any integers c and d.

Lemma 4.16 implies that

$$|\kappa_{s,q}(a, b)| = |\kappa_{s,p^k}(at^*, bt^*)||\kappa_{s,t}(a(p^k)^*, b(p^k)^*)|, \tag{4.49}$$

where $tt^* \equiv 1 \pmod{p^k}$ and $p^k(p^k)^* \equiv 1 \pmod{t}$. By applying Lemma 4.18 to $|\kappa_{s,p^k}(at^*, bt^*)|$ in (4.49), we find that

$$|\kappa_{s,q}(a, b)| \leq \tau(p^k)(\gcd(at^*, bt^*, p^k))^{1/2}(p^k)^{1/2}|\kappa_{s,t}(a(p^k)^*, b(p^k)^*)|. \tag{4.50}$$

Note that $t < q$ is a positive odd integer, so we apply the inductive hypothesis (4.48) with $v = t$, $c = at^*$, and $d = bt^*$ to (4.50) to obtain

$$
\begin{aligned}
|\kappa_{s,q}(a,b)| &\leq \tau(p^k)(\gcd(at^*, bt^*, p^k))^{1/2}(p^k)^{1/2}\tau(t)(\gcd(a(p^k)^*, b(p^k)^*, t))^{1/2}t^{1/2} \\
&= \tau(p^k)\tau(t)(\gcd(at^*, bt^*, p^k))^{1/2}(\gcd(a(p^k)^*, b(p^k)^*, t))^{1/2}q^{1/2}
\end{aligned}
$$

Because $\gcd(p^k, t) = 1$ and τ is multiplicative, we conclude that

$$
\begin{aligned}
|\kappa_{s,q}(a,b)| &\leq \tau(q)(\gcd(a, b, p^k))^{1/2}(\gcd(a, b, t))^{1/2}q^{1/2} \\
&= \tau(q)(\gcd(a, b, q))^{1/2}q^{1/2}.
\end{aligned}
$$

□

Combining the previous lemma with Lemma 4.14, we have the following bound on $\kappa_{s,q}(a,b)$ when q is odd.

Lemma 4.20. *If a, b, and s are integers and q is a positive odd integer, then*

$$
|\kappa_{s,q}(a,b)| \leq \tau(q)(\gcd(a, b, q))^{1/2}q^{1/2}. \tag{4.51}
$$

By applying Lemmas 4.13 and 4.20 to $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$, we obtain the following result.

Lemma 4.21. *Let $\alpha \in \mathbb{Z}$ be such that $\alpha A^{-1} \in M_s(\mathbb{Z})$ and $\gcd(\alpha, q_1) = 1$. The sum $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$ satisfies the following:*

$$
|K^{(q_0)}(\ell, n, \mathbf{r}; q_1)| \leq q_1^{(s+1)/2}\tau(q_1)(\gcd(\ell + 2^*\alpha^*2\alpha F^*(\mathbf{r}), n, q_1))^{1/2}.
$$

Proof. By applying Lemmas 4.13 and 4.20 to $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$, we see that

$$
|K^{(q_0)}(\ell, n, \mathbf{r}; q_1)| \leq q_1^{(s+1)/2}\tau(q_1)(\gcd(q_0^*(\ell + 2^*\alpha^*2\alpha F^*(\mathbf{r})), q_0^*n, q_1))^{1/2}.
$$

Because $\gcd(q_0^*, q_1) = 1$, we obtain the result of the lemma. □

4.5 Bounding the arithmetic part

We have now provided bounds for $\gamma(\ell)$, and $K^{(q_1)}(\ell, n, \mathbf{r}; q_0)$, and $K^{(q_0)}(\ell, n, \mathbf{r}; q_1)$. Therefore, we have what we need to compute an upper bound for the absolute value of $T_{\mathbf{r}}(q, n; x)$. Such an upper bound appears in the following lemma.

Lemma 4.22. *For $\mathbf{r} \in \mathbb{Z}^s$, $n \in \mathbb{Z}$, $x \in \mathbb{R}$, and positive $q \in \mathbb{Z}$, the sum $T_{\mathbf{r}}(q, n; x)$ is*

$$T_{\mathbf{r}}(q, n; x) \ll (\gcd(L, q_0))^{s/2} (\gcd(n, q_1))^{1/2} q_0^{1/2} q^{(s+1)/2} \tau(q) \log(2q). \tag{4.52}$$

The implied constant does not depend on F.

Proof. We use Lemmas 4.1 and 4.11 to obtain

$$T_{\mathbf{r}}(q, n; x) = \sum_{\ell \pmod q} \gamma(\ell) K^{(q_1)}(\ell, n, \mathbf{r}; q_0) K^{(q_0)}(\ell, n, \mathbf{r}; q_1). \tag{4.53}$$

We now apply Lemmas 4.4, 4.12, and 4.21 to see that

$$
\begin{aligned}
|T_{\mathbf{r}}(q,n;x)| \leq \sum_{-\frac{q}{2}<\ell\leq\frac{q}{2}} & (1+|\ell|)^{-1}(\gcd(L,q_0))^{s/2} q_0^{s/2+1} q_1^{(s+1)/2} \\
& \times \tau(q_1)(\gcd(\ell+2^*\alpha^*2\alpha F^*(\mathbf{r}),n,q_1))^{1/2} \\
= q^{(s+1)/2} & q_0^{1/2}\tau(q_1)(\gcd(L,q_0))^{s/2} \\
& \times \sum_{-\frac{q}{2}<\ell\leq\frac{q}{2}} (1+|\ell|)^{-1}(\gcd(\ell+2^*\alpha^*2\alpha F^*(\mathbf{r}),n,q_1))^{1/2} \\
\leq q^{(s+1)/2} & q_0^{1/2}\tau(q)(\gcd(L,q_0))^{s/2} \sum_{-\frac{q}{2}<\ell\leq\frac{q}{2}} (1+|\ell|)^{-1}(\gcd(n,q_1))^{1/2} \\
\leq q^{(s+1)/2} & q_0^{1/2}\tau(q)(\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2}\left(1+2\sum_{\ell=1}^{\lfloor\frac{q}{2}\rfloor}(1+\ell)^{-1}\right).
\end{aligned}
\tag{4.54}
$$

We use Euler's summation formula [Apo76, Theorem 3.1] to bound the sum in (4.54):

$$
\begin{aligned}
\sum_{\ell=1}^{\lfloor\frac{q}{2}\rfloor}(1+\ell)^{-1} &= \frac{1}{2} + \int_1^{\lfloor\frac{q}{2}\rfloor}(1+\ell)^{-1}\,d\ell - \int_1^{\lfloor\frac{q}{2}\rfloor}(\ell-\lfloor\ell\rfloor)(1+\ell)^{-2}\,d\ell \\
&= \frac{1}{2} + \log\left(1+\left\lfloor\frac{q}{2}\right\rfloor\right) - \log(2) - \int_1^{\lfloor\frac{q}{2}\rfloor}(\ell-\lfloor\ell\rfloor)(1+\ell)^{-2}\,d\ell.
\end{aligned}
\tag{4.55}
$$

Because $0\leq\ell-\lfloor\ell\rfloor\leq 1$, the integral in (4.55) is bounded (in absolute value) by

$$
\left|\int_1^{\lfloor\frac{q}{2}\rfloor}(1+\ell)^{-2}\,d\ell\right| = \left|\frac{1}{2}-\left(1+\left\lfloor\frac{q}{2}\right\rfloor\right)^{-1}\right| \leq \frac{1}{2}.
$$

Therefore,

$$
\sum_{\ell=1}^{\lfloor\frac{q}{2}\rfloor}(1+\ell)^{-1} = \log\left(1+\left\lfloor\frac{q}{2}\right\rfloor\right)+O(1) = \log\left(1+\frac{q}{2}\right)+O(1) \tag{4.56}
$$

since $\left|\log\left(1+\frac{q}{2}\right)-\log\left(1+\left\lfloor\frac{q}{2}\right\rfloor\right)\right| \leq \frac{1}{1+\lfloor q/2\rfloor} \leq 1$.

Substituting (4.56) into (4.54), we see that

$$\begin{aligned}
|T_{\mathbf{r}}(q,n;x)| &\leq q^{(s+1)/2} q_0^{1/2} \tau(q) (\gcd(L,q_0))^{s/2} (\gcd(n,q_1))^{1/2} \\
&\qquad \times \left(1 + 2\left(\log\left(1+\frac{q}{2}\right) + O(1)\right)\right) \\
&= q^{(s+1)/2} q_0^{1/2} \tau(q) (\gcd(L,q_0))^{s/2} (\gcd(n,q_1))^{1/2} \left(2\log\left(1+\frac{q}{2}\right) + O(1)\right).
\end{aligned}$$

We obtain the result of this lemma by noticing that

$$2\log\left(1+\frac{q}{2}\right) + O(1) \ll \log(2q)$$

for $q \geq 1$. $\square$

Chapter 5

Analyzing the archimedean part

In this chapter, we analyze the archimedean part $\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q)$ using a principle of nonstationary phase and other bounds on certain oscillatory integrals. An *oscillatory integral* is an integral of the form

$$\int_{\mathbb{R}^s} \mathrm{e}(f(\mathbf{m}))\, \psi(\mathbf{m})\ d\mathbf{m}, \tag{5.1}$$

where $f \in C^\infty(\mathbb{R}^s)$ and $\psi \in C_c^\infty(\mathbb{R}^s)$.

Typically, a principle of nonstationary phase says that the oscillatory integral in (5.1) is relatively small in absolute value if the gradient of f is nonzero for all $\mathbf{m} \in \mathrm{supp}(\psi)$. In Section 5.1, we describe a one-dimensional version of the principle of nonstationary phase.

In Section 5.2, we look at oscillatory integrals in which the function f in (5.1) is a quadratic polynomial. In that section, we give an upper bound for the absolute value of such an oscillatory integral.

In Section 5.3, we use the results of Sections 5.1 and 5.2 to obtain bounds for the archimedean part $\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q)$.

5.1 A principle of nonstationary phase

To analyze the archimedean part $\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q)$, we will use a one-dimensional principle of nonstationary phase. We will use a principle of nonstationary phase that is different than what is found in Proposition 1 in Chapter 8 of [Ste93], Proposition 6.1 of [Wol03], Lemma 2.6 in [Tao], and Proposition B.1 in [Gre13]. (The results in [Ste93], [Wol03], [Tao], and [Gre13] are not directly applicable for our purposes.)

The next theorem is our one-dimensional version of the principle of nonstationary phase and is a slight generalization of the statement "Non-stationary phase" on p. 94 in [Zha18].

Theorem 5.1 (Principle of nonstationary phase in one variable)**.** *Let* $\psi \in C_c^\infty(\mathbb{R})$ *and let* $M \geq 0$*. Let* $f \in C^\infty(\mathbb{R})$ *be such that* $|f'(x)| \geq B > 0$ *and* $|f^{(j)}(x)| \leq |f'(x)|$ *for all* $x \in \operatorname{supp}(\psi)$ *and for each integer* j *satisfying* $2 \leq j \leq \lceil M \rceil$*. Then*

$$\int_{\mathbb{R}} \mathrm{e}(f(x))\, \psi(x)\ dx \ll_{\psi,M} B^{-M}.$$

Proof. Because of the monotonicity of B^{-M} as a function of M, it suffices to only consider when M is an integer. (If $B \geq 1$, then $B^{-M} \leq B^{-\lfloor M \rfloor}$ and we can use the implied constant for $\lfloor M \rfloor$. If $B \leq 1$, then $B^{-M} \leq B^{-\lceil M \rceil}$ and we can use the implied constant for $\lceil M \rceil$.) The theorem will be proven by using mathematical induction and integration by parts.

Let $\psi_0 = \psi$. For a positive integer j and a real number x, we define the function ψ_j recursively by

$$\psi_j(x) = \begin{cases} \left(\dfrac{\psi_{j-1}}{f'}\right)'(x) & \text{if } f'(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Induction proofs (which we omit) show that ψ_j is differentiable and that $\operatorname{supp}(\psi_j) \subseteq$

supp(ψ) for each nonnegative integer j. We first prove the following about the oscillatory integral that appears in Theorem 5.1.

Lemma 5.2. *For each nonnegative integer j,*

$$\int_{\mathbb{R}} \mathrm{e}(f(x))\,\psi(x)\;dx = (-2\pi i)^{-j} \int_{\mathrm{supp}(\psi)} \mathrm{e}(f(x))\,\psi_j(x)\;dx. \tag{5.2}$$

Proof of Lemma 5.2. Using definitions, we observe that

$$\int_{\mathbb{R}} \mathrm{e}(f(x))\,\psi(x)\;dx = (-2\pi i)^{-0} \int_{\mathrm{supp}(\psi)} \mathrm{e}(f(x))\,\psi_0(x)\;dx.$$

Thus, (5.2) holds for $j = 0$.

For the induction hypothesis, let $k \geq 0$ and assume that (5.2) holds for $j = k$. Observe that

$$\mathrm{e}(f(x)) = \frac{1}{2\pi i f'(x)} \frac{d}{dx}(\mathrm{e}(f(x))) \tag{5.3}$$

when $x \in \mathrm{supp}(\psi)$. The induction hypothesis and (5.3) imply that

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}(f(x))\,\psi(x)\;dx &= (-2\pi i)^{-k} \int_{\mathrm{supp}(\psi)} \mathrm{e}(f(x))\,\psi_k(x)\;dx \\ &= (-1)^k (2\pi i)^{-(k+1)} \int_{\mathrm{supp}(\psi)} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\;dx. \end{aligned} \tag{5.4}$$

Because $|f'(x)| \geq B > 0$ whenever $x \in \mathrm{supp}(\psi)$, the set $U = \{x \in \mathbb{R} : |f'(x)| > B/2\}$ is an open set such that $\mathrm{supp}(\psi) \subseteq U$. Because any fixed open set of $\mathbb{R}$ can be written as a countable union of disjoint open intervals, we have

$$U = \bigsqcup_{\ell=1}^{\infty} (c_\ell, d_\ell),$$

where $c_\ell \leq d_\ell \leq c_{\ell+1}$ for each $\ell \geq 1$. (Note that $(a, a) = \emptyset$ for any $a \in \mathbb{R}$. We consider the empty set to be an open interval.)

Because $\operatorname{supp}(\psi_k) \subseteq \operatorname{supp}(\psi) \subseteq U$ and $f'(x) \neq 0$ for all $x \in U$, we have

$$\begin{aligned}\int_{\operatorname{supp}(\psi)} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx &= \int_U \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx \\ &= \sum_{\ell=1}^{\infty}\int_{c_\ell}^{d_\ell} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx.\end{aligned}$$

Using integration by parts, we see that

$$\begin{aligned}\int_{\operatorname{supp}(\psi)} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx = \sum_{\ell=1}^{\infty}\Bigg(&\mathrm{e}(f(d_\ell))\,\frac{\psi_k(d_\ell)}{f'(d_\ell)} - \mathrm{e}(f(c_\ell))\,\frac{\psi_k(c_\ell)}{f'(c_\ell)} \\ &- \int_{c_\ell}^{d_\ell} \mathrm{e}(f(x))\left(\frac{\psi_k}{f'}\right)'(x)\ dx\Bigg).\end{aligned}$$

Observe that $\psi_k(c_\ell) = \psi_k(d_\ell) = 0$ for each $\ell \geq 1$ since $\operatorname{supp}(\psi_k)$ is a subset of U. For each $\ell \geq 1$, we also notice that $f'(c_\ell) \neq 0$ and $f'(d_\ell) \neq 0$, because f' is continuous and $|f'(x)| \geq B/2$ for $x \in (c_\ell, d_\ell)$. Therefore,

$$\int_{\operatorname{supp}(\psi)} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx = -\sum_{\ell=1}^{\infty}\int_{c_\ell}^{d_\ell} \mathrm{e}(f(x))\,\psi_{k+1}(x)\ dxt$$

Since $\operatorname{supp}(\psi_{k+1}) \subseteq \operatorname{supp}(\psi) \subseteq U = \bigsqcup_{\ell=1}^{\infty}(c_\ell, d_\ell)$, we deduce that

$$\int_{\operatorname{supp}(\psi)} \frac{d}{dx}(\mathrm{e}(f(x)))\frac{\psi_k(x)}{f'(x)}\ dx = -\int_{\operatorname{supp}(\psi)} \mathrm{e}(f(x))\,\psi_{k+1}(x)\ dx. \tag{5.5}$$

By substituting (5.5) into (5.4), we observe that (5.2) holds for $j = k+1$. By the principle of mathematical induction, (5.2) holds for each nonnegative integer j. $\square$

Using the previous lemma, we observe that for each nonnegative integer j that

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathrm{e}(f(x))\,\psi(x)\ dx\right| &= \left|(-2\pi i)^{-j}\int_{\mathrm{supp}(\psi)} \mathrm{e}(f(x))\,\psi_j(x)\ dx\right| \\ &\le (2\pi)^{-j}\int_{\mathrm{supp}(\psi)} \left|\mathrm{e}(f(x))\,\psi_j(x)\right|\ dx \\ &= (2\pi)^{-j}\int_{\mathrm{supp}(\psi)} \left|\psi_j(x)\right|\ dx.\end{aligned}$$

Therefore, to prove Theorem 5.1, it suffices to prove that

$$\int_{\mathrm{supp}(\psi)} \left|\psi_j(x)\right|\ dx \ll_{\psi,j} B^{-j} \tag{5.6}$$

for each nonnegative integer j.

To do this, we would like to write ψ_j as $\psi_j = P_j/(f')^{2^{j+1}-2}$, where P_j is a polynomial of functions in the set $\{\psi^{(k)} : 0 \le k \le j\} \cup \{f^{(k+1)} : 0 \le k \le j\}$. The next lemma allows us to write ψ_j in this way and states some useful properties of P_j.

Lemma 5.3. *For a nonnegative integer j, the function ψ_j can be written as*

$$\psi_j = \frac{P_j}{(f')^{2^{j+1}-2}}, \tag{5.7}$$

where P_j is a polynomial of functions in the set $\{\psi^{(k)} : 0 \le k \le j\} \cup \{f^{(k+1)} : 0 \le k \le j\}$.

The polynomial P_j is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(j+1)}$. Define $n_{j,f}$ to be the degree of P_j when viewed as a polynomial in $f^{(1)}, \ldots, f^{(j+1)}$. Then

$$n_{j,f} = 2^{j+1} - j - 2. \tag{5.8}$$

Proof of Lemma 5.3. We prove this lemma using the principle of mathematical induction.

By definition, we have $P_0 = \psi$, which is a polynomial in $\psi^{(0)}$ and $f^{(1)}$. Notice that P_0 is a homogeneous polynomial in $f^{(1)}$, and $n_{0,f} = 0$. Observe that $2^{0+1} - 0 - 2 = 0$, so (5.8) holds for $j = 0$.

Let ℓ be a nonnegative integer. Suppose that ψ_ℓ can be written as

$$\psi_\ell = \frac{P_\ell}{(f')^{2^{\ell+1}-2}}, \tag{5.9}$$

where P_ℓ is a polynomial of functions in the set $\{\psi^{(k)} : 0 \leq k \leq \ell\} \cup \{f^{(k+1)} : 0 \leq k \leq \ell\}$. Also, suppose that the polynomial P_ℓ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+1)}$ and that (5.8) holds for $j = \ell$.

We look at $\psi_{\ell+1}$ and see that

$$\begin{aligned}
\psi_{\ell+1} &= \left(\frac{\psi_k}{f'}\right)' \\
&= \left(\frac{P_\ell}{(f')^{2^{\ell+1}-1}}\right)' \\
&= \frac{(f')^{2^{\ell+1}-1}P_\ell' - (2^{\ell+1}-1)(f')^{2^{\ell+1}-2}f''P_\ell}{(f')^{2^{\ell+2}-2}}
\end{aligned} \tag{5.10}$$

by the quotient rule. Therefore,

$$P_{\ell+1} = (f')^{2^{\ell+1}-1}P_\ell' - (2^{\ell+1}-1)(f')^{2^{\ell+1}-2}f''P_\ell. \tag{5.11}$$

The induction hypothesis says that P_ℓ is a polynomial of functions in the set $\{\psi^{(k)} : 0 \leq k \leq \ell\} \cup \{f^{(k+1)} : 0 \leq k \leq \ell\}$. Because of how the product rule and the chain rule work, we observe that P_ℓ' is a polynomial of functions in the set $\{\psi^{(k)} : 0 \leq k \leq \ell+1\} \cup \{f^{(k+1)} : 0 \leq k \leq \ell+1\}$. Since $\ell + 1 \geq 1$, the functions f' and f'' are in the set $\{\psi^{(k)} : 0 \leq k \leq \ell+1\} \cup \{f^{(k+1)} : 0 \leq k \leq \ell+1\}$. Therefore, using (5.11), we conclude that $P_{\ell+1}$ is a polynomial of functions in the set $\{\psi^{(k)} : 0 \leq k \leq \ell+1\} \cup \{f^{(k+1)} : 0 \leq k \leq \ell+1\}$.

The induction hypothesis also states that P_ℓ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+1)}$ and that (5.8) holds for $j = \ell$. Again, because of how the product rule and the chain rule work, we find that P_ℓ' is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$ and the degree of P_ℓ' when viewed as a polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$ is equal to $n_{\ell,f}$. Therefore, the polynomial $(f')^{2^{\ell+1}-1}P_\ell'$ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$. Also, $(f')^{2^{\ell+1}-1}P_\ell'$ has degree $n_{\ell,f} + 2^{\ell+1} - 1$ when viewed as a polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$. Furthermore, the polynomial $(2^{\ell+1}-1)(f')^{2^{\ell+1}-2}f''P_\ell$ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$ since P_ℓ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+1)}$. The degree of $(2^{\ell+1}-1)(f')^{2^{\ell+1}-2}f''P_\ell$ is

$$2^{\ell+1} - 2 + 1 + n_{\ell,f} = n_{\ell,f} + 2^{\ell+1} - 1$$

when viewed as a polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$.

Therefore, using (5.11), we conclude that $P_{\ell+1}$ is a homogeneous polynomial in $f^{(1)}, \ldots, f^{(\ell+2)}$. Using the induction hypothesis, we also observe that $P_{\ell+1}$ has degree

$$\begin{aligned} n_{\ell,f} + 2^{\ell+1} - 1 &= 2^{\ell+1} - \ell - 2 + 2^{\ell+1} - 1 \\ &= 2^{\ell+2} - (\ell+1) - 2. \end{aligned}$$

Thus, (5.8) holds for $j = \ell + 1$. $\square$

Because of the previous lemma and the fact that $|f^{(k)}(x)| \leq |f'(x)|$ for all $x \in \operatorname{supp}(\psi)$ and for all $k \in \mathbb{Z}$ such that $2 \leq k \leq j$, we have for $x \in \operatorname{supp}(\psi)$ that

$$|\psi_j(x)| \leq \frac{|\tilde{P}_j(x)|}{|f'(x)|^j},$$

where $\tilde{P}_j$ is a polynomial of the functions in the set $\{|\psi^{(k)}| : 0 \leq k \leq j\}$. Since

$|f'(x)| \geq B > 0$, we have

$$|\psi_j(x)| \leq B^{-j}|\tilde{P}_j(x)|.$$

Therefore,

$$\int_{\mathrm{supp}(\psi)} |\psi_j(x)| \; dx \leq B^{-j} \int_{\mathrm{supp}(\psi)} |\tilde{P}_j(x)| \; dx.$$

Thus, we have (5.6) for each nonnegative integer j. □

5.2 Bounding an oscillatory integral

In this section, we give an upper bound for the absolute value of an oscillatory integral associated with a quadratic polynomial. The following theorem generalizes the statement "Stationary phase" on p. 95 of [Zha18] to quadratic polynomials in any positive number of variables.

Theorem 5.4. *Suppose that A is the Hessian matrix of a nonsingular quadratic form in s variables. Suppose that $\mathbf{b} \in \mathbb{R}^s$, $c \in \mathbb{R}$, and $\psi \in C_c^\infty(\mathbb{R}^s)$. Then*

$$\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c\right) \psi(\mathbf{x}) \; d\mathbf{x} \ll_\psi |\det(A)|^{-1/2}. \tag{5.12}$$

Remark 5.5. With the hypotheses of Theorem 5.4, one can prove that

$$\begin{aligned} &\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c\right) \psi(\mathbf{x}) \; d\mathbf{x} \\ &= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b}\right) \psi(-A^{-1}\mathbf{b}) e^{i\pi \,\mathrm{sgnt}(A)/4} |\det(A)|^{-1/2} \\ &\qquad + O_\psi(|\det(A)|^{-1/2}|\lambda_{min}|^{-1}), \end{aligned} \tag{5.13}$$

where λ_{min} is the eigenvalue of A that has the minimum absolute value. Because this is an asymptotic for an oscillatory integral that depends on when the gradient of $\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c$ is zero, this result could be called a principle of stationary phase for quadratic polynomials. However, we will not need this result for our purposes.

Before we prove Theorem 5.4, we prove a number of lemmas that will be used in our proof of Theorem 5.4. Our first lemma gives the Fourier transform of the one-dimensional Gaussian function e^{-ax^2} with $a > 0$.

Lemma 5.6. *For $a > 0$ and $y \in \mathbb{C}$, we have*

$$\int_{\mathbb{R}} e^{-ax^2} \mathrm{e}(-xy)\ dx = \sqrt{\frac{\pi}{a}} e^{-\pi^2 y^2/a}. \tag{5.14}$$

Proof. It is well-known that

$$\int_{\mathbb{R}} e^{-\pi x^2} \mathrm{e}(-xy)\ dx = e^{-\pi y^2}. \tag{5.15}$$

(For example, see Example 1 and Exercise 4 of Chapter 2 of [SS03].) Now

$$\int_{\mathbb{R}} e^{-ax^2} \mathrm{e}(-xy)\ dx = \int_{\mathbb{R}} e^{-\pi\left(\sqrt{\frac{a}{\pi}}x\right)^2} \mathrm{e}\left(-\left(\sqrt{\frac{a}{\pi}}x\right)\left(\sqrt{\frac{\pi}{a}}y\right)\right)\ dx.$$

Let $u = \sqrt{\frac{a}{\pi}}x$. Then

$$\int_{\mathbb{R}} e^{-ax^2} \mathrm{e}(-xy)\ dx = \sqrt{\frac{\pi}{a}} \int_{\mathbb{R}} e^{-\pi u^2} \mathrm{e}\left(-u\sqrt{\frac{\pi}{a}}y\right)\ du.$$

We use (5.15) to see that

$$\int_{\mathbb{R}} e^{-ax^2} \mathrm{e}(-xy)\ dx = \sqrt{\frac{\pi}{a}} e^{-\pi\left(\sqrt{\frac{\pi}{a}}y\right)^2} = \sqrt{\frac{\pi}{a}} e^{-\pi^2 y^2/a}. \qquad \square$$

Fourier transform of the one-dimensional Gaussian function allows us to prove the following lemma about the Fourier transform of a multi-dimensional Gaussian function.

Lemma 5.7. *Suppose that* $\mathbf{a} = (a_1, \ldots, a_s)^\top \in \mathbb{R}^s$ *be such that* $a_j > 0$ *for each* $1 \le j \le s$. *Then*

$$\int_{\mathbb{R}^s} e^{-\sum_{j=1}^s a_j x_j^2} \mathrm{e}(-\mathbf{x}\cdot\mathbf{y}) \ dx = e^{-\sum_{j=1}^s \pi^2 y_j^2/a_j} \prod_{j=1}^s \frac{\pi^{1/2}}{a_j^{1/2}} \tag{5.16}$$

for $\mathbf{y} \in \mathbb{C}^s$.

Proof. Using the Fubini-Tonelli theorem, we obtain

$$\int_{\mathbb{R}^s} e^{-\sum_{j=1}^s a_j x_j^2} \mathrm{e}(-\mathbf{x}\cdot\mathbf{y}) \ dx = \prod_{j=1}^s \int_{\mathbb{R}} e^{-a_j x_j^2} \mathrm{e}(-x_j y_j) \ dx_j.$$

By Lemma 5.6, the integral becomes

$$\int_{\mathbb{R}^s} e^{-\sum_{j=1}^s z_j x_j^2} \mathrm{e}(-\mathbf{x}\cdot\mathbf{y}) \ dx = \prod_{j=1}^s \sqrt{\frac{\pi}{a_j}} e^{-\pi^2 y_j^2/a_j},$$

which is equal to (5.16). $\square$

We can now apply Plancherel's theorem to obtain the following result.

Lemma 5.8. *Suppose that* D *be an* $s \times s$ *diagonal matrix with real entries and rank* s. *Suppose that* $\psi \in C_c^\infty(\mathbb{R}^s)$. *Then*

$$\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top D\mathbf{x}\right) \psi(\mathbf{x}) \ d\mathbf{x} = e^{i\pi\,\mathrm{sgnt}(D)/4} |\det(D)|^{-1/2} \int_{\mathbb{R}^s} \mathrm{e}\left(-\frac{1}{2}\mathbf{x}^\top D^{-1}\mathbf{x}\right) \widehat{\psi}(\mathbf{x}) \ d\mathbf{x}.$$

Proof of Lemma 5.8. Plancherel's theorem states that

$$\int_{\mathbb{R}^s} f(x)\overline{g(x)} \ dx = \int_{\mathbb{R}^s} \widehat{f}(y)\overline{\widehat{g}(y)} \ dy$$

if $f, g \in L^1(\mathbb{R}^s) \cap L^2(\mathbb{R}^s)$. Let $\mathbf{z} = (z_1, \ldots, z_s)^\top \in \mathbb{R}^s$ be such that $z_j > 0$ for each

$1 \le j \le s$. Plancherel's theorem with $f(\mathbf{x}) = \psi(\mathbf{x})$ and $g(\mathbf{x}) = e^{-\sum_{j=1}^{s} z_j x_j^2}$ gives us

$$\int_{\mathbb{R}^s} e^{-\sum_{j=1}^{s} z_j x_j^2} \psi(\mathbf{x})\ d\mathbf{x} = \int_{\mathbb{R}^s} e^{-\sum_{j=1}^{s} \pi^2 x_j^2 / z_j} \widehat{\psi}(\mathbf{x})\ d\mathbf{x} \prod_{j=1}^{s} \frac{\pi^{1/2}}{z_j^{1/2}}. \tag{5.17}$$

(We used Lemma 5.7 to compute the Fourier transform of $e^{-\sum_{j=1}^{s} z_j x_j^2}$.)

The left-hand side of (5.17) can be extended to a analytic function for all $\mathbf{z} \in \mathbb{C}^s$. (For each $\mathbf{x} \in \mathbb{R}^s$, we have $e^{-\sum_{j=1}^{s} z_j x_j^2} \psi(\mathbf{x})$ is analytic in $\mathbf{z}$. For each $\mathbf{z} \in \mathbb{C}^s$, the left-hand side of (5.17) is the integral of a continuous function over a compact set. We can then apply a theorem similar to Theorem 5.4 in Chapter 2 of [SS03].)

Using the Lebesgue dominated convergence theorem, the right-hand side can be extended to a continuous function on

$$S = \{\mathbf{z} \in \mathbb{C}^s : \mathbf{z} \neq \mathbf{0}, \operatorname{Re}(z_j) \ge 0 \text{ for all } 1 \le j \le s\}$$

that is analytic on the interior of S. (For all $\mathbf{z} \in S$ and $\mathbf{x} \in \mathbb{R}^s$, the inequality $\left| e^{-\sum_{j=1}^{s} \pi^2 x_j^2 / z_j} \widehat{\psi}(\mathbf{x}) \right| \le |\widehat{\psi}(\mathbf{x})|$ holds.)

It follows from the identity principle that (5.17) holds for all $\mathbf{z} \in S$. (For the square root function, we use the principal branch cut of the logarithm function along the nonpositive real axis.) In particular, (5.17) holds when each $z_j = -i\pi d_j$, where $D = \operatorname{diag}(d_1, \ldots, d_s)$. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top D\mathbf{x}\right) \psi(\mathbf{x})\ d\mathbf{x} &= \int_{\mathbb{R}^s} e^{-\sum_{j=1}^{s} \pi^2 x_j^2/(-i\pi d_j)} \widehat{\psi}(\mathbf{x})\ d\mathbf{x} \prod_{j=1}^{s} \frac{\pi^{1/2}}{(-i\pi d_j)^{1/2}} \\
&= \int_{\mathbb{R}^s} e^{-i\pi \sum_{j=1}^{s} x_j^2/d_j} \widehat{\psi}(\mathbf{x})\ d\mathbf{x} \prod_{j=1}^{s} (-id_j)^{-1/2} \\
&= e^{i\pi \operatorname{sgnt}(D)/4} |\det(D)|^{-1/2} \int_{\mathbb{R}^s} \mathrm{e}\left(-\frac{1}{2}\mathbf{x}^\top D^{-1}\mathbf{x}\right) \widehat{\psi}(\mathbf{x})\ d\mathbf{x}. \square
\end{aligned}$$

We are now in a position to give a proof for Theorem 5.4.

Proof of Theorem 5.4. Using the spectral theorem for symmetric matrices, we can write the symmetric matrix A as

$$A = P^\top D P,$$

where P is an orthogonal matrix and D is a diagonal matrix. Therefore, $P^\top = P^{-1}$, and

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top P^\top D P\mathbf{x} + \mathbf{b}^\top P^\top P\mathbf{x} + c\right)\psi(P^{-1}P\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}(c)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}(P\mathbf{x})^\top DP\mathbf{x} + (P\mathbf{b})^\top P\mathbf{x}\right)\psi(P^{-1}P\mathbf{x})\ d\mathbf{x}.
\end{aligned}$$

Let $\mathbf{y} = P\mathbf{x}$. Because P is orthogonal, we know that $|\det(P)| = 1$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}(c)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{y}^\top D\mathbf{y} + (P\mathbf{b})^\top\mathbf{y}\right)\psi(P^{-1}\mathbf{y})\frac{1}{|\det(P)|}\ d\mathbf{y} \\
&= \mathrm{e}(c)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{y}^\top D\mathbf{y} + (P\mathbf{b})^\top\mathbf{y}\right)\psi(P^{-1}\mathbf{y})\ d\mathbf{y}.
\end{aligned}$$

We now complete the square and see that

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}(c)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}(\mathbf{y} + D^{-1}P\mathbf{b})^\top D(\mathbf{y} + D^{-1}P\mathbf{b}) - \frac{1}{2}(P\mathbf{b})^\top D^{-1}P\mathbf{b}\right)\psi(P^{-1}\mathbf{y})\ d\mathbf{y} \\
&= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top P^\top D^{-1}P\mathbf{b}\right)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}(\mathbf{y} + D^{-1}P\mathbf{b})^\top D(\mathbf{y} + D^{-1}P\mathbf{b})\right)\psi(P^{-1}\mathbf{y})\ d\mathbf{y}.
\end{aligned}$$

Let $\mathbf{z} = \mathbf{y} + D^{-1}P\mathbf{b}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top P^\top D^{-1}P\mathbf{b}\right)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{z}^\top D\mathbf{z}\right)\psi(P^{-1}\mathbf{z} - P^{-1}D^{-1}P\mathbf{b})\ d\mathbf{z}. \qquad (5.18)
\end{aligned}$$

Now

$$\begin{aligned}
A^{-1} &= (P^\top DP)^{-1} = P^{-1}D^{-1}(P^\top)^{-1} = P^{-1}D^{-1}(P^{-1})^{-1} \\
&= P^{-1}D^{-1}P = P^\top D^{-1}P \qquad (5.19)
\end{aligned}$$

since $P^{-1} = P^\top$. We apply (5.19) to (5.18) and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b}\right)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{z}^\top D\mathbf{z}\right)\psi(P^{-1}\mathbf{z} - A^{-1}\mathbf{b})\ d\mathbf{z} \\
&= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b}\right)\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{z}^\top D\mathbf{z}\right)\varphi(\mathbf{z})\ d\mathbf{z}, \qquad (5.20)
\end{aligned}$$

where $\varphi(\mathbf{z}) = \psi(P^{-1}\mathbf{z} - A^{-1}\mathbf{b})$ for $\mathbf{z} \in \mathbb{R}^s$.

We apply Lemma 5.8 to (5.20) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x})\ d\mathbf{x} \\
&= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b}\right)e^{i\pi\,\mathrm{sgnt}(D)/4}|\det(D)|^{-1/2}\int_{\mathbb{R}^s} \mathrm{e}\left(-\frac{1}{2}\mathbf{z}^\top D^{-1}\mathbf{z}\right)\widehat{\varphi}(\mathbf{z})\ d\mathbf{z}. \qquad (5.21)
\end{aligned}$$

Before we bound (5.21), we compute $\widehat{\varphi}$ in terms of A and $\mathbf{b}$.

Lemma 5.9. *Suppose that* $\varphi(\mathbf{z}) = \psi(P^{-1}\mathbf{z} - A^{-1}\mathbf{b})$*, where* A*,* P*,* $\mathbf{b}$*, and* ψ *are as above. Then*

$$\widehat{\varphi}(\mathbf{w}) = \mathrm{e}\big(-(PA^{-1}\mathbf{b})^\top\mathbf{w}\big)\,\widehat{\psi}(P^\top\mathbf{w}) \qquad (5.22)$$

for $\mathbf{w} \in \mathbb{R}^s$.

Proof of Lemma 5.9. By definition, the Fourier transform of φ is

$$\widehat{\varphi}(\mathbf{w}) = \int_{\mathbb{R}^s} \psi(P^{-1}\mathbf{z} - A^{-1}\mathbf{b})\mathrm{e}(-\mathbf{z}\cdot\mathbf{w}) \ d\mathbf{z}.$$

Let $\mathbf{v} = P^{-1}\mathbf{z} - A^{-1}\mathbf{b}$. Then $\mathbf{z} = P\mathbf{v} + PA^{-1}\mathbf{b}$, and

$$\begin{aligned}\widehat{\varphi}(\mathbf{w}) &= \int_{\mathbb{R}^s} \psi(\mathbf{v})\mathrm{e}\big(-(P\mathbf{v} + PA^{-1}\mathbf{b})^\top\mathbf{w}\big)\,|\det(P)| \ d\mathbf{v} \\ &= \mathrm{e}\big(-(PA^{-1}\mathbf{b})^\top\mathbf{w}\big)\int_{\mathbb{R}^s} \psi(\mathbf{v})\mathrm{e}\big(-\mathbf{v}^\top P^\top\mathbf{w}\big) \ d\mathbf{v}\end{aligned}$$

since P is orthogonal and $|\det(P)| = 1$. We obtain (5.22) by noting that

$$\int_{\mathbb{R}^s} \psi(\mathbf{v})\mathrm{e}\big(-\mathbf{v}^\top P^\top\mathbf{w}\big) \ d\mathbf{v} = \widehat{\psi}(P^\top\mathbf{w}).$$

$\square$

We are now able to estimate $\int_{\mathbb{R}^s} \mathrm{e}\big(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\big)\,\psi(\mathbf{x}) \ d\mathbf{x}$. Using Lemma 5.9 in (5.21), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right)\psi(\mathbf{x}) \ d\mathbf{x} \\ &= \mathrm{e}\left(c - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b}\right) e^{i\pi\,\mathrm{sgnt}(D)/4}|\det(D)|^{-1/2} \\ &\qquad \times \int_{\mathbb{R}^s} \mathrm{e}\left(-\frac{1}{2}\mathbf{z}^\top D^{-1}\mathbf{z}\right)\mathrm{e}\big(-(PA^{-1}\mathbf{b})^\top\mathbf{z}\big)\,\widehat{\psi}(P^\top\mathbf{z}) \ d\mathbf{z}.\end{aligned}$$

We take absolute values of both sides and see that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right) \psi(\mathbf{x})\ d\mathbf{x}\right| \\
&= |\det(D)|^{-1/2} \left|\int_{\mathbb{R}^s} \mathrm{e}\left(-\frac{1}{2}\mathbf{z}^\top D^{-1}\mathbf{z}\right) \mathrm{e}\big(-(PA^{-1}\mathbf{b})^\top \mathbf{z}\big)\, \widehat{\psi}(P^\top \mathbf{z})\ d\mathbf{z}\right| \\
&\le |\det(D)|^{-1/2} \int_{\mathbb{R}^s} \left|\mathrm{e}\left(-\frac{1}{2}\mathbf{z}^\top D^{-1}\mathbf{z}\right) \mathrm{e}\big(-(PA^{-1}\mathbf{b})^\top \mathbf{z}\big)\, \widehat{\psi}(P^\top \mathbf{z})\right|\ d\mathbf{z} \\
&= |\det(D)|^{-1/2} \int_{\mathbb{R}^s} \left|\widehat{\psi}(P^\top \mathbf{z})\right|\ d\mathbf{z}. \qquad (5.23)
\end{aligned}$$

Let $\mathbf{w} = P^\top \mathbf{z}$. Then $\mathbf{z} = P\mathbf{w}$ (because $P^{-1} = P^\top$), and

$$\begin{aligned}
\int_{\mathbb{R}^s} \left|\widehat{\psi}(P^\top \mathbf{z})\right|\ d\mathbf{z} &= \int_{\mathbb{R}^s} \left|\widehat{\psi}(\mathbf{w})\right| |\det(P)|\ d\mathbf{w} \\
&= \int_{\mathbb{R}^s} \left|\widehat{\psi}(\mathbf{w})\right|\ d\mathbf{w} \qquad (5.24)
\end{aligned}$$

since $|\det(P)| = 1$. Substituting (5.24) into (5.23), we obtain

$$\left|\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right) \psi(\mathbf{x})\ d\mathbf{x}\right| \le |\det(D)|^{-1/2} \int_{\mathbb{R}^s} \left|\widehat{\psi}(\mathbf{w})\right|\ d\mathbf{w}.$$

Because $A = P^\top DP$ and $|\det(P)| = 1$, we know that $\det(A) = \det(D)$. Therefore,

$$\left|\int_{\mathbb{R}^s} \mathrm{e}\left(\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c\right) \psi(\mathbf{x})\ d\mathbf{x}\right| \le |\det(A)|^{-1/2} \int_{\mathbb{R}^s} \left|\widehat{\psi}(\mathbf{w})\right|\ d\mathbf{w},$$

which shows that (5.12) is true with an implied constant of $\int_{\mathbb{R}^s} \left|\widehat{\psi}(w)\right|\ dw$. □

5.3 Applying bounds for oscillatory integrals

In this section, we apply the results of Sections 5.1 and 5.2 to the archimedean part $\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q)$. To make the results of Sections 5.1 and 5.2 applicable to the archimedean part, we normalize our bump function so that it remains same regardless the value of X. We do this by applying a change a variables ($\mathbf{m} \mapsto X\mathbf{m}$) in (3.20) obtain the following:

$$\begin{aligned}\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) &= X^s \int_{\mathbb{R}^s} \mathrm{e}\left(xF(X\mathbf{m}) - \frac{1}{q}X\mathbf{m}\cdot\mathbf{r}\right)\psi_X(X\mathbf{m})\,d\mathbf{m} \\ &= X^s \int_{\mathbb{R}^s} \mathrm{e}\left(xF(X\mathbf{m}) - \frac{1}{q}X\mathbf{m}\cdot\mathbf{r}\right)\psi_1\left(\frac{X}{X}\mathbf{m}\right)\,d\mathbf{m} \\ &= X^s \int_{\mathbb{R}^s} \mathrm{e}\left(X^2xF(\mathbf{m}) - \frac{1}{q}X\mathbf{m}\cdot\mathbf{r}\right)\psi(\mathbf{m})\,d\mathbf{m}. \qquad (5.25)\end{aligned}$$

The following lemma gives a trivial bound for $\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)$.

Lemma 5.10. *Suppose that $x \in \mathbb{R}$, $X > 0$, $\mathbf{r} \in \mathbb{Z}^s$, and q is a positive integer. Then*

$$\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) \ll_\psi X^s. \qquad (5.26)$$

Proof. By (5.25), we know that

$$|\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)| \le X^s \int_{\mathbb{R}^s} |\psi(\mathbf{m})|\,d\mathbf{m} \ll_\psi X^s. \qquad \square$$

We will need less trivial bounds for $\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)$. We can apply Theorem 5.4 to the integral in (5.25) to obtain the following bound on $\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)$.

Theorem 5.11. *Suppose that $x \in \mathbb{R}$, $X > 0$, $\mathbf{r} \in \mathbb{Z}^s$, and q is a positive integer. Then*

$$\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) \ll_\psi \min\left\{X^s, |x|^{-s/2}(\det(A))^{-1/2}\right\}. \qquad (5.27)$$

Proof. By Theorem 5.4, the integral in (5.25) is

$$\begin{aligned}\int_{\mathbb{R}^s} \mathrm{e}\left(X^2xF(\mathbf{m}) - \frac{1}{q}X\mathbf{m}\cdot\mathbf{r}\right)\psi(\mathbf{m})\,d\mathbf{m} &\ll_\psi |\det(X^2xA)|^{-1/2} \\ &= |X^{2s}x^s\det(A)|^{-1/2}. \qquad (5.28)\end{aligned}$$

We notice that $X > 0$ and $\det(A) > 0$, so (5.28) implies that

$$\int_{\mathbb{R}^s} \mathrm{e}\left(X^2 x F(\mathbf{m}) - \frac{1}{q} X \mathbf{m} \cdot \mathbf{r}\right) \psi(\mathbf{m})\, d\mathbf{m} \ll_\psi X^{-s} |x|^{-s/2} (\det(A))^{-1/2}.$$

We multiply by X^s to obtain

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) \ll_\psi |x|^{-s/2} (\det(A))^{-1/2}. \tag{5.29}$$

The statement (5.27) is obtained by taking the minimum of (5.26) and (5.29). □

To obtain better estimates on sums and integrals involving $\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q)$, we need to determine when we can apply Theorem 5.1. To do this, we use partial derivatives and directional derivatives. We use the notation $\frac{\partial f}{\partial x_j}(\mathbf{m})$ to denote the the partial derivative in the jth coordinate evaluated at the point $\mathbf{m}$. We now define a directional derivative.

Definition 5.12 (Directional derivative)**.** For a unit vector $\mathbf{u} \in \mathbb{R}^s$ and a differentiable function $f \colon \mathbb{R}^s \to \mathbb{R}$, define the directional derivative $\nabla_{\mathbf{u}} f$ of f along $\mathbf{u}$ to be

$$\nabla_{\mathbf{u}} f = \mathbf{u} \cdot (\nabla f),$$

where ∇f is the gradient of f.

Remark 5.13. The quantity $\frac{\partial f}{\partial x_j}(\mathbf{m})$ is equal to $\nabla_{\mathbf{e}_j} f(\mathbf{m})$, where $\mathbf{e}_j \in \mathbb{R}^s$ is the unit vector whose jth entry is equal to 1 and is the only nonzero entry of $\mathbf{e}_j$.

Directional derivatives allow us to take repeatedly the derivative of a function f in a particular direction. This is needed for Theorem 5.1 to apply. We use directional derivatives, partial derivatives, and Theorem 5.1 to prove the following theorem.

Theorem 5.14. *Suppose that there exists an integer j with $1 \leq j \leq s$ such that*

$$|r_j| \geq qX|x|\lambda_s(\rho_\psi + 1). \tag{5.30}$$

Then

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) \ll_{M,\psi} \min\left\{X^s, X^{s-M}\left(\frac{1}{q(\rho_\psi + 1)\sqrt{s}}\|\mathbf{r}\|\right)^{-M}\right\} \tag{5.31}$$

for all $M \geq 0$.

The proof of Theorem 5.14 uses an upper bound for $\mathbf{u}^\top B\mathbf{u}$ when B is a symmetric matrix and $\mathbf{u}$ is a unit vector. Therefore, we prove the next lemma before proving Theorem 5.14.

Lemma 5.15. *If $B \in M_s(\mathbb{R})$ is a symmetric $s \times s$ matrix and $\mathbf{w} \in \mathbb{R}^s$, then*

$$|\mathbf{w}^\top B\mathbf{w}| \leq \sigma_s\|\mathbf{w}\|^2,$$

where σ_s is the largest singular value of B.

Remark 5.16. If B is positive definite, then $\sigma_s = \lambda_s$, where λ_s is the largest eigenvalue of B.

Proof of Lemma 5.15. Using the spectral theorem for symmetric matrices, we can write the matrix B as

$$B = P^\top DP,$$

where P is an orthogonal matrix and $D = \operatorname{diag}(d_1, \ldots, d_s)$ is a diagonal matrix. Note that $\{d_j\}_{j=1}^s$ is the set of eigenvalues of B. Then

$$\mathbf{w}^\top B\mathbf{w} = \mathbf{w}^\top P^\top DP\mathbf{w}. \tag{5.32}$$

Let $\mathbf{v} = P\mathbf{w}$. Note that $\|\mathbf{v}\| = \|\mathbf{w}\|$ since P is orthogonal. Substituting $\mathbf{v} = P\mathbf{w}$ into (5.32), we see that

$$\begin{aligned} \mathbf{w}^\top B\mathbf{w} &= \mathbf{v}^\top D\mathbf{v} \\ &= \sum_{j=1}^{s} d_j v_j^2. \end{aligned} \tag{5.33}$$

By taking absolute values, we find that

$$\begin{aligned} \left|\mathbf{w}^\top B\mathbf{w}\right| &= \left|\sum_{j=1}^{s} d_j v_j^2\right| \\ &\leq \sum_{j=1}^{s} |d_j| v_j^2. \end{aligned} \tag{5.34}$$

Without loss of generality, assume that $|d_1| \leq |d_2| \leq \cdots \leq |d_s|$. Therefore, (5.34) implies that

$$\begin{aligned} \left|\mathbf{w}^\top B\mathbf{w}\right| &\leq \sum_{j=1}^{s} |d_s| v_j^2 \\ &= |d_s| \sum_{j=1}^{s} v_j^2 \\ &= |d_s| \|\mathbf{v}\|^2. \end{aligned}$$

Because $\|\mathbf{v}\| = \|\mathbf{w}\|$ and the largest singular value of B is $|d_s|$, we obtain the result of this lemma. □

Now that we have Lemma 5.15, we prove Theorem 5.14.

Proof of Theorem 5.14. Suppose that $\mathbf{r}$ has an entry r_j that satisfies (5.30). Set j with $1 \leq j \leq s$ to be such that $r_j \geq r_k$ for all $1 \leq k \leq s$. Then r_j satisfies (5.30) and

$$|r_j| \geq \frac{1}{\sqrt{s}} \|\mathbf{r}\|. \tag{5.35}$$

(If r_j did not satisfy (5.35), then

$$\begin{aligned}\|\mathbf{r}\| &= \sqrt{\sum_{k=1}^{s} r_k^2} \leq \sqrt{\sum_{k=1}^{s} r_j^2} \\ &< \sqrt{\sum_{k=1}^{s} \frac{1}{s}\|\mathbf{r}\|^2} = \sqrt{\|\mathbf{r}\|^2} = \|\mathbf{r}\|,\end{aligned}$$

which contradicts the fact that $\|\mathbf{r}\| = \|\mathbf{r}\|$.)

Lemma 5.10 says that

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) \ll_\psi X^s,$$

so it suffices to prove that

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) \ll_{M,\psi} X^{s-M} \left(\frac{1}{q(\rho_\psi + 1)\sqrt{s}} \|\mathbf{r}\| \right)^{-M}. \tag{5.36}$$

From (5.25), we have

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) = X^s \int_{\mathbb{R}^s} \mathrm{e}(f(\mathbf{m}))\, \psi(\mathbf{m})\ d\mathbf{m},$$

where

$$f(\mathbf{m}) = X^2 x F(\mathbf{m}) - \frac{1}{q} X \mathbf{m} \cdot \mathbf{r}. \tag{5.37}$$

Therefore, to prove (5.36), it suffices to show that

$$\int_{\mathbb{R}^s} \mathrm{e}(f(\mathbf{m}))\, \psi(\mathbf{m})\ d\mathbf{m} \ll_{M,\psi} \left(\frac{X}{q(\rho_\psi + 1)\sqrt{s}} \|\mathbf{r}\| \right)^{-M}. \tag{5.38}$$

We will use partial derivatives, directional derivatives, and Theorem 5.1 to prove

(5.38). In order to apply Theorem 5.1, we show that for all $\mathbf{m} \in \operatorname{supp}(\psi)$, we have

$$\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right| \geq \frac{X}{q(\rho_\psi + 1)\sqrt{s}}\|\mathbf{r}\| \tag{5.39}$$

and

$$\left|\frac{\partial^k f}{\partial x_j^k}(\mathbf{m})\right| \leq \left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right|$$

for all $k \geq 2$. Because f is a quadratic polynomial in terms of the m_j, it suffices to show that for all $\mathbf{m} \in \operatorname{supp}(\psi)$, we have (5.39) holding and

$$\left|\frac{\partial^2 f}{\partial x_j^2}(\mathbf{m})\right| \leq \left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right|. \tag{5.40}$$

(Any higher partial derivatives of f equal 0.)

Let $\mathbf{u} \in \mathbb{R}^s$ be a unit vector. We would like to compute the directional derivative $\nabla_{\mathbf{u}} f$. In order to do this, we need to know the gradient of f, so we compute this in the next lemma.

Lemma 5.17. *The gradient of f is*

$$\nabla f(\mathbf{m}) = X^2 x A\mathbf{m} - \frac{1}{q}X\mathbf{r}. \tag{5.41}$$

In particular, the partial derivative $\frac{\partial f}{\partial x_j}(\mathbf{m})$ in the jth coordinate is

$$\frac{\partial f}{\partial x_j}(\mathbf{m}) = X^2 x \sum_{k=1}^{s} a_{jk} m_k - \frac{1}{q} X r_j. \tag{5.42}$$

Proof of Lemma 5.17. Now

$$\begin{aligned}
f(\mathbf{m}) &= \frac{1}{2}X^2x\mathbf{m}^\top A\mathbf{m} - \frac{1}{q}X\mathbf{m}\cdot\mathbf{r}\\
&= X^2x\left(\frac{1}{2}\sum_{\ell=1}^{s} a_{\ell\ell}m_\ell^2 + \sum_{\ell=1}^{s}\sum_{1\le k<\ell} a_{\ell k}m_\ell m_k\right) - \frac{1}{q}X\sum_{\ell=1}^{s} r_\ell m_\ell.
\end{aligned}$$

Thus, by using the fact that A is a symmetric, we find that the gradient of f is

$$\begin{aligned}
\nabla f(\mathbf{m}) &= \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{m}) & \cdots & \frac{\partial f}{\partial x_s}(\mathbf{m}) \end{pmatrix}^\top\\
&= \begin{pmatrix} X^2x\left(a_{11}m_1 + \sum_{\substack{1\le k\le s\\ k\ne 1}} a_{1k}m_k\right) - \frac{1}{q}Xr_1\\ \vdots \\ X^2x\left(a_{ss}m_s + \sum_{\substack{1\le k\le s\\ k\ne s}} a_{sk}m_k\right) - \frac{1}{q}Xr_s \end{pmatrix}\\
&= \begin{pmatrix} X^2x\sum_{k=1}^{s} a_{1k}m_k - \frac{1}{q}Xr_1\\ \vdots \\ X^2x\sum_{k=1}^{s} a_{sk}m_k - \frac{1}{q}Xr_s \end{pmatrix}\\
&= X^2xA\mathbf{m} - \frac{1}{q}X\mathbf{r}.
\end{aligned}$$

By looking at the jth entry of $\nabla f(\mathbf{m})$, we deduce (5.42). □

In light of Lemma 5.17, the directional derivative $\nabla_{\mathbf{u}}f$ is

$$\begin{aligned}
\nabla_{\mathbf{u}}f(\mathbf{m}) &= \mathbf{u}\cdot\left(X^2xA\mathbf{m} - \frac{1}{q}X\mathbf{r}\right)\\
&= X^2x\mathbf{u}^\top A\mathbf{m} - \frac{1}{q}X\mathbf{u}\cdot\mathbf{r}.
\end{aligned}$$

Notice that

$$\begin{aligned}\mathbf{u}^\top A\mathbf{m} &= \sum_{\ell=1}^{s}\sum_{k=1}^{s} a_{\ell k}u_\ell m_k \\ &= \sum_{\ell=1}^{s}\sum_{k=1}^{s} a_{k\ell}u_\ell m_k\end{aligned}$$

since A is symmetric. Thus, by the linearity of the gradient, we obtain

$$\nabla(\nabla_{\mathbf{u}} f)(\mathbf{m}) = X^2 x A\mathbf{u}.$$

Therefore,

$$(\nabla_{\mathbf{u}})^2 f(\mathbf{m}) = X^2 x\mathbf{u}^\top A\mathbf{u}. \tag{5.43}$$

Towards showing that (5.40) holds, we prove an upper bound for $|(\nabla_{\mathbf{u}})^2 f(\mathbf{m})|$ that does not depend on $\mathbf{u}$ or $\mathbf{m}$.

Lemma 5.18. *For f as in* (5.37)*, we have*

$$\left|(\nabla_{\mathbf{u}})^2 f(\mathbf{m})\right| \leq X^2|x|\lambda_s. \tag{5.44}$$

In particular,

$$\left|\frac{\partial^2 f}{\partial x_j^2}(\mathbf{m})\right| \leq X^2|x|\lambda_s. \tag{5.45}$$

Remark 5.19. A similar result can be shown if F is any (not just a positive definite) quadratic form. Instead of λ_s, the upper bound would have the largest singular value of the Hessian matrix associated with the quadratic form.

Proof of Lemma 5.18. An application of Lemma 5.15 and Remark 5.16 to (5.43) proves (5.44) since $\mathbf{u}$ is a unit vector. We obtain (5.45) by noting that $\frac{\partial^2 f}{\partial x_j^2}(\mathbf{m})$ equals

$(\nabla_{\mathbf{e}_j})^2 f(\mathbf{m})$. $\square$

We now begin to compute an lower bound for $\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right|$. (A lower bound for $\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right|$ is needed to show (5.39) and (5.40) hold for all $\mathbf{m} \in \operatorname{supp}(\psi)$.) By using the triangle inequality with (5.42) in Lemma 5.17, we obtain

$$\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right| \geq \frac{1}{q} X|r_j| - X^2|x| \left|\sum_{k=1}^{s} a_{jk} m_k\right|. \tag{5.46}$$

To effectively use this lower bound for $\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right|$, we need an upper bound for $|\sum_{k=1}^{s} a_{jk} m_k|$. The following lemma is a step towards finding such an upper bound.

Lemma 5.20. *For* $\mathbf{m} \in \operatorname{supp}(\psi)$, *then*

$$\|A\mathbf{m}\| \leq \lambda_s \rho_\psi. \tag{5.47}$$

Proof of Lemma 5.20. By definition of the Eurclidean norm, we have

$$\begin{aligned} \|A\mathbf{m}\|^2 &= \mathbf{m}^\top A^\top A\mathbf{m} \\ &= \mathbf{m}^\top A^2 \mathbf{m} \end{aligned}$$

since A is symmetric.

Because A^2 is a positive definite symmetric matrix, Lemma 5.15 and Remark 5.16 apply, and we obtain

$$\|A\mathbf{m}\|^2 \leq \|\mathbf{m}\|^2 \lambda_s^2. \tag{5.48}$$

(Note that each eigenvalue of A^2 is the square of an eigenvalue of A. Therefore, λ_s^2 is the largest eigenvalue of A^2.) By taking square roots of both sides of (5.48), we obtain

$$\|A\mathbf{m}\| \leq \lambda_s \|\mathbf{m}\|. \tag{5.49}$$

Because $\mathbf{m} \in \operatorname{supp}(\psi)$, we know that $\|\mathbf{m}\| \leq \rho_\psi$. Applying this to (5.49), we obtain (5.47). □

Since $\sum_{k=1}^{s} a_{jk} m_k$ is the jth entry of the vector $A\mathbf{m}$, we use the previous lemma to give an upper bound for $|\sum_{k=1}^{s} a_{jk} m_k|$. We state this upper bound in the next lemma.

Lemma 5.21. *For* $\mathbf{m} \in \operatorname{supp}(\psi)$, *then*

$$\left| \sum_{k=1}^{s} a_{jk} m_k \right| \leq \lambda_s \rho_\psi. \tag{5.50}$$

Proof of Lemma 5.21. Observe that

$$\begin{aligned} \left| \sum_{k=1}^{s} a_{jk} m_k \right| &\leq \sqrt{\sum_{\ell=1}^{s} \left(\sum_{k=1}^{s} a_{\ell k} m_k \right)^2} \\ &= \|A\mathbf{m}\| \\ &\leq \lambda_s \rho_\psi \end{aligned}$$

by Lemma 5.20. □

We now apply Lemma 5.21 to (5.46) to obtain

$$\left| \frac{\partial f}{\partial x_j}(\mathbf{m}) \right| \geq \frac{1}{q} X |r_j| - X^2 |x| \lambda_s \rho_\psi \tag{5.51}$$

for all $\mathbf{m} \in \operatorname{supp}(\psi)$. We are now in a position to prove that (5.39) and (5.40) hold for all $\mathbf{m} \in \operatorname{supp}(\psi)$.

Lemma 5.22. *For all* $\mathbf{m} \in \operatorname{supp}(\psi)$*, the statement* (5.39) *holds.*

Proof of Lemma 5.22. The inequality (5.30) can be rewritten as

$$-X|x|\lambda_s \geq -\frac{1}{q(\rho_\psi + 1)}|r_j|.$$

Using this in (5.51), we obtain

$$\begin{aligned}\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right| &\geq \frac{1}{q}X|r_j| - \frac{\rho_\psi}{q(\rho_\psi + 1)}X|r_j| \\ &= \frac{1}{q(\rho_\psi + 1)}X|r_j|. \end{aligned} \tag{5.52}$$

By applying (5.35) to (5.52), we obtain (5.39). $\square$

We now prove that (5.40) holds for all $\mathbf{m} \in \operatorname{supp}(\psi)$.

Lemma 5.23. *For all* $\mathbf{m} \in \operatorname{supp}(\psi)$*, the statement* (5.40) *holds.*

Proof of Lemma 5.22. By applying (5.30) to (5.51), we find that

$$\begin{aligned}\left|\frac{\partial f}{\partial x_j}(\mathbf{m})\right| &\geq X^2|x|(\rho_\psi + 1)\lambda_s - X^2|x|\lambda_s\rho_\psi \\ &= X^2|x|\lambda_s.\end{aligned}$$

Now (5.45) in Lemma 5.18 says that $\left|\frac{\partial^2 f}{\partial x_j^2}(\mathbf{m})\right| \leq X^2|x|\lambda_s$, so we obtain (5.40). $\square$

Lemmas 5.22 and 5.23 state that (5.39) and (5.40) hold for all $\mathbf{m} \in \operatorname{supp}(\psi)$. This is sufficient to apply Theorem 5.1 and obtain for all $M \geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^s} \mathrm{e}(f(\mathbf{m}))\, \psi(\mathbf{m})\ d\mathbf{m} \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathrm{e}(f(\mathbf{m}))\, \psi(\mathbf{m})\ dm_j\ dm_1\ dm_2 \cdots\ dm_{j-1}\ dm_{j+1}\ dm_{j+2} \cdots\ dm_s \\
&\ll_{M,\psi} \int_{S_{\psi,j}} \left(\frac{X}{q(\rho_\psi + 1)\sqrt{s}} \|\mathbf{r}\| \right)^{-M} dm_1\ dm_2 \cdots\ dm_{j-1}\ dm_{j+1}\ dm_{j+2} \cdots\ dm_s,
\end{aligned} \tag{5.53}$$

where $S_{\psi,j} \subseteq \mathbb{R}^{s-1}$ is the set of $(m_1, m_2, \ldots, m_{j-1}, m_{j+1}, m_{j+2}, \ldots, m_s)^\top$ in which there exists $m_j \in \mathbb{R}$ such that $(m_1, m_2, \ldots, m_s)^\top \in \operatorname{supp}(\psi)$. Because ψ has compact support, the set $S_{\psi,j}$ is bounded and (5.53) implies (5.38). $\square$

For our purposes, we will want to apply the principle of nonstationary phase outside of an s-dimensional ball as opposed to outside of an s-dimensional cube. Therefore, we have the following corollary.

Corollary 5.24. *If*

$$\|\mathbf{r}\| \geq qX|x|\lambda_s(\rho_\psi + 1)\sqrt{s}, \tag{5.54}$$

then

$$\mathcal{I}_{F,\psi}(x, X, \mathbf{r}, q) \ll_{M,\psi} \min\left\{ X^s, X^{s-M} \left(\frac{1}{q(\rho_\psi + 1)\sqrt{s}} \|\mathbf{r}\| \right)^{-M} \right\} \tag{5.55}$$

for all $M \geq 0$.

Proof. If $\mathbf{r}$ satisfies (5.54), then there exists an integer j with $1 \leq j \leq s$ such that r_j satisfies (5.30). (If not, then $\|\mathbf{r}\|$ would be less than $qX|x|\lambda_s(\rho_\psi + 1)\sqrt{s}$, which would contradict (5.54).) An application of Theorem 5.14 gives the result of this corollary. $\square$

Chapter 6

Putting estimates together

In this chapter, we use results from previous chapters to prove Theorem 1.1 and Corollaries 1.4 and 1.5. We begin this chapter by splitting up the weighted representation number $R_{F,\psi,X}(n)$ into a main term and some error terms. From (3.21), we see that

$$R_{F,\psi,X}(n) = M_{F,\psi,X}(n) + E_{F,\psi,X,1}(n) + E_{F,\psi,X,2}(n) + E_{F,\psi,X,3}(n), \tag{6.1}$$

where

$$M_{F,\psi,X}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, \mathcal{I}_{F,\psi}(x,X,\mathbf{0},q) T_{\mathbf{0}}(q,n;x)\ dx \right), \tag{6.2}$$

$$\begin{aligned} E_{F,\psi,X,1}(n) = 2\,\mathrm{Re}\Bigg(&\sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx) \\ &\times \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ 0<\|\mathbf{r}\|\le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} \mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) T_{\mathbf{r}}(q,n;x)\ dx \Bigg), \end{aligned} \tag{6.3}$$

$$E_{F,\psi,X,2}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \frac{1}{q^s} \int_{\frac{1}{q(q+Q)}}^{\frac{1}{qQ}} \mathrm{e}(-nx) \right.$$
$$\left. \times \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\|\le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} \mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) T_{\mathbf{r}}(q,n;x)\ dx \right), \quad (6.4)$$

and

$$E_{F,\psi,X,3}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \frac{1}{q^s} \int_{0}^{\frac{1}{qQ}} \mathrm{e}(-nx) \right.$$
$$\left. \times \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\|> qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} \mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) T_{\mathbf{r}}(q,n;x)\ dx \right). \quad (6.5)$$

We call $M_{F,\psi,X}(n)$ the main term of $R_{F,\psi,X}(n)$. We call $E_{F,\psi,X,1}(n)$, $E_{F,\psi,X,2}(n)$, and $E_{F,\psi,X,3}(n)$ the error terms of $R_{F,\psi,X}(n)$.

In this chapter, we will provide an asymptotic for the main term $M_{F,\psi,X}(n)$ and upper bounds for the absolute values of the error terms. To do this, we will first prove some more results that will help us prove Theorem 1.1 and Corollaries 1.4 and 1.5.

6.1 Stating some supporting results

In this section, we state some lemmas that will be used to provide an asymptotic for the main term $M_{F,\psi,X}(n)$ and upper bounds for the absolute values of the error terms.

6.1.1 An upper bound for the absolute value of a particular sum

The sum in the following lemma will come up multiple times in our estimates. (The sum is related to our estimate of the sum $T_{\mathbf{r}}(q,n;x)$.) The lemma provides provides an upper bound for the absolute value of this sum.

Lemma 6.1. *Let $Q \geq 1$ and let C and n be nonzero integers. For an integer q, we split q into $q = q_0 q_1$ such that q_0 is the largest factor of q having all of its prime divisors dividing C so that $\gcd(q_1, C) = 1$. Then*

$$\sum_{1 \leq q \leq Q} (\gcd(n, q_1))^{1/2} q_1^{-1/2} \tau(q) \log(2q) \ll_\varepsilon Q^{1/2+\varepsilon} \tau(n) \prod_{p|C} (1 - p^{-1/2})^{-1} \tag{6.6}$$

for any $\varepsilon > 0$.

Remark 6.2. In our applications of Lemma 6.1, the integer C is equal to $2\det(A)$.

The proof of Lemma 6.1 requires another lemma, which we now state and prove.

Lemma 6.3. *Let $Q \geq 1$ and let C and n be nonzero integers. For an integer q, we split q into $q = q_0 q_1$ such that q_0 is the largest factor of q having all of its prime divisors dividing C so that $\gcd(q_1, C) = 1$. Then*

$$\sum_{1 \leq q \leq Q} q_1^{-1/2} (\gcd(n, q_1))^{1/2} \ll Q^{1/2} \tau(n) \prod_{p|C} (1 - p^{-1/2})^{-1}. \tag{6.7}$$

Proof. Throughout this proof, we have q_0 such that if p divides q_0 then p divides C. Observe that

$$\begin{aligned} \sum_{1 \leq q \leq Q} q_1^{-1/2} (\gcd(n, q_1))^{1/2} &\leq \sum_{1 \leq q \leq Q} q_1^{-1/2} \sum_{\substack{d|n \\ d|q_1}} d^{1/2} \\ &\leq \sum_{1 \leq q_0 \leq Q} \sum_{1 \leq q_1 \leq Q/q_0} q_1^{-1/2} \sum_{\substack{d|n \\ d|q_1}} d^{1/2} \\ &= \sum_{d|n} d^{1/2} \sum_{1 \leq q_0 \leq Q} \sum_{\substack{1 \leq q_1 \leq Q/q_0 \\ q_1 \equiv 0 \pmod d}} q_1^{-1/2} \end{aligned} \tag{6.8}$$

by switching the order of summation.

Let q_2 be q_1/d. Then

$$\sum_{d|n} d^{1/2} \sum_{1\le q_0\le Q} \sum_{\substack{1\le q_1\le Q/q_0\\ q_1\equiv 0 \pmod d}} q_1^{-1/2} = \sum_{d|n} d^{1/2} \sum_{1\le q_0\le Q} \sum_{1\le q_2\le Q/(q_0 d)} (q_2 d)^{-1/2}$$
$$= \sum_{d|n} \sum_{1\le q_0\le Q} \sum_{1\le q_2\le Q/(q_0 d)} q_2^{-1/2}.$$

Substituting this into (6.8), we have

$$\sum_{1\le q\le Q} q_1^{-1/2}(\gcd(n,q_1))^{1/2} \le \sum_{d|n} \sum_{1\le q_0\le Q} \sum_{1\le q_2\le Q/(q_0 d)} q_2^{-1/2}$$
$$\le \sum_{d|n} \sum_{1\le q_0\le Q} \sum_{1\le q_2\le Q/q_0} q_2^{-1/2}.$$

Using part (b) Theorem 3.2 of [Apo76], we obtain

$$\begin{aligned}\sum_{1\le q\le Q} q_1^{-1/2}(\gcd(n,q_1))^{1/2} &\ll \sum_{d|n} \sum_{1\le q_0\le Q} \left(\frac{Q}{q_0}\right)^{1/2}\\ &= Q^{1/2} \sum_{d|n} \sum_{1\le q_0\le Q} q_0^{-1/2}\\ &\le Q^{1/2} \sum_{d|n} \sum_{q_0>0} q_0^{-1/2}\\ &= Q^{1/2}\tau(n) \sum_{q_0>0} q_0^{-1/2}. \end{aligned} \tag{6.9}$$

We see that (6.7) follows from (6.9), because

$$\begin{aligned}\sum_{q_0>0} q_0^{-1/2} &= \prod_{p|C} \sum_{j=0}^{\infty} (p^{-1/2})^j\\ &= \prod_{p|C} (1-p^{-1/2})^{-1}.\end{aligned}$$

$\square$

The proof of Lemma 6.1 follows quickly from Lemma 6.3

Proof of Lemma 6.1. For any $\varepsilon > 0$, we have $\tau(q) \ll_\varepsilon q^\varepsilon \le Q^\varepsilon$ and $\log(2q) \ll_\varepsilon (2q)^\varepsilon \le (2Q)^\varepsilon \ll_\varepsilon Q^\varepsilon$. Thus,

$$\sum_{1 \le q \le Q} (\gcd(n, q_1))^{1/2} q_1^{-1/2} \tau(q) \log(2q) \ll_\varepsilon Q^\varepsilon \sum_{1 \le q \le Q} (\gcd(n, q_1))^{1/2} q_1^{-1/2}.$$

By applying Lemma 6.3, we obtain the result of Lemma 6.1. □

6.1.2 The volume of an s-dimensional ball, integer lattice point counting, and sums over integer lattice points

This subsection contains information about the volume of an s-dimensional ball, integer lattice point counting, and sums over integer lattice points.

We begin by stating (without proof) the following result about the volume of an s-dimensional ball that can be found in a number of sources (including, for example, Section 2.C of Chapter 21 of [CS99]).

Lemma 6.4. *The volume of an s-dimensional ball of radius R is*

$$\frac{\pi^{s/2}}{\Gamma(s/2+1)} R^s. \tag{6.10}$$

We are primarily concerned with s-dimensional balls centered at the origin. Let $B_s(R)$ be the closed s-dimensional ball centered at the origin with radius R. This s-dimensional ball is defined by

$$B_s(R) = \{\mathbf{x} \in \mathbb{R}^s : \|\mathbf{x}\| \le R\}. \tag{6.11}$$

Let $B_s^o(R)$ be the open s-dimensional ball centered at the origin with radius R. This

s-dimensional ball is defined by

$$B_s^o(R) = \{\mathbf{x} \in \mathbb{R}^s : \|\mathbf{x}\| < R\}. \tag{6.12}$$

For a Lebesgue measurable subset W of $\mathbb{R}^s$, let $\mathrm{Vol}_s(W)$ be the s-dimensional volume of W. Then Lemma 6.4 implies that

$$\mathrm{Vol}_s(B_s(R)) = \mathrm{Vol}_s(B_s^o(R)) = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)} R^s \tag{6.13}$$

since the the boundary of an s-dimensional ball has zero volume.

We will also need to count the number of integer lattices points in an s-dimensional ball centered at the origin. We first state a result that can be proven with a geometric argument (originally due to Gauss [Gau11] for counting lattice points inside a circle).

Lemma 6.5. *Let s be a positive integer and $R \geq 1$. Then the number of integer lattice points in $B_s(R)$ is*

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)} R^s + O_s(R^{s-1}). \tag{6.14}$$

Remark 6.6. The error term in (6.14) is not the best possible error term for $s \geq 2$. For example, for $s = 2$, Huxley [Hux02] obtained an error term of $O(R^{131/208})$. (This is not necessarily the best result for $s = 2$; there is a preprint by Bourgain and Watt [BW17] that claims a better error term for $s = 2$.) For $s = 3$, Heath-Brown [HB99] obtained an error term of $O(R^{21/16})$. For $s \geq 4$, the best known error term is $O_s(R^{s-2})$, which can be proved using a classical formula for the number of representations of an integer as the sum of four squares. (See, for instance, [Fri82].) Given all of this, the error term in (6.14) is sufficient for our purposes.

Remark 6.7. Lemma 6.5 requires that $R \geq 1$. A lower bound like this is required, because for fixed s, we have

$$\lim_{R\to 0^+} R^s = \lim_{R\to 0^+} R^{s-1} = 0. \tag{6.15}$$

However, since the zero vector is contained in the set $\{\mathbf{m}\in\mathbb{Z}^s : \|\mathbf{m}\| \le R\}$, we have

$$|\{\mathbf{m}\in\mathbb{Z}^s : \|\mathbf{m}\| \le R\}| \ge 1 \tag{6.16}$$

for all positive R. Therefore, we cannot have (6.14) be true for all $R > 0$. (The implied constant would have to become larger and larger as R becomes closer and closer to zero.)

Proof of Lemma 6.5. For each $\mathbf{m}\in\mathbb{Z}^s$, place a unit s-dimensional cube centered at $\mathbf{m}$. Orient this unit s-dimensional cube so that each edge of the cube is parallel to a coordinate axis. (Here we are assuming that our coordinate axes are orthogonal.) We use $K_{\mathbf{m}}$ to denote this unit s-dimensional cube centered at $\mathbf{m}$ in this particular orientation.

Notice that $\mathrm{Vol}_s(K_{\mathbf{m}}) = 1$ for each $\mathbf{m}\in\mathbb{Z}^s$. We also note that if $\mathbf{x},\mathbf{y}\in\mathbb{Z}^s$ and $\mathbf{x}\neq\mathbf{y}$, then $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$ have disjoint interiors. These facts allow us to turn our integer lattice point counting problem into a question about the volume of a set. Namely, we notice that

$$|\{\mathbf{m}\in\mathbb{Z}^s : \|\mathbf{m}\| \le R\}| = \mathrm{Vol}_s\left(\bigcup_{\mathbf{m}\in\mathbb{Z}^s\cap B_s(R)} K_{\mathbf{m}}\right). \tag{6.17}$$

We now suppose that $R \ge \max\left\{1, \frac{\sqrt{s}}{2}\right\}$. Because the longest diagonal of a unit s-dimensional cube is $\sqrt{s}$, we have

$$B_s\left(R - \frac{\sqrt{s}}{2}\right) \subseteq \bigcup_{\mathbf{m}\in\mathbb{Z}^s\cap B_s(R)} K_{\mathbf{m}} \subseteq B_s\left(R + \frac{\sqrt{s}}{2}\right). \tag{6.18}$$

Therefore,

$$\mathrm{Vol}_s\left(B_s\left(R-\frac{\sqrt{s}}{2}\right)\right) \le \mathrm{Vol}_s\left(\bigcup_{\mathbf{m}\in\mathbb{Z}^s\cap B_s(R)} K_{\mathbf{m}}\right) \le \mathrm{Vol}_s\left(B_s\left(R+\frac{\sqrt{s}}{2}\right)\right). \quad (6.19)$$

Using (6.13), we obtain

$$\mathrm{Vol}_s\left(B_s\left(R-\frac{\sqrt{s}}{2}\right)\right) = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)}\left(R-\frac{\sqrt{s}}{2}\right)^s \quad (6.20)$$

and

$$\mathrm{Vol}_s\left(B_s\left(R+\frac{\sqrt{s}}{2}\right)\right) = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)}\left(R+\frac{\sqrt{s}}{2}\right)^s. \quad (6.21)$$

By the binomial theorem, we have

$$(R+h)^s = R^s + \sum_{j=0}^{s-1}\binom{s}{j}R^j h^{s-j} \quad (6.22)$$

for $h \in \mathbb{R}$. Thus, since $R \ge 1$, we have

$$\mathrm{Vol}_s\left(B_s\left(R-\frac{\sqrt{s}}{2}\right)\right) = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)}R^s + O_s(R^{s-1}) \quad (6.23)$$

and

$$\mathrm{Vol}_s\left(B_s\left(R+\frac{\sqrt{s}}{2}\right)\right) = \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)}R^s + O_s(R^{s-1}). \quad (6.24)$$

Therefore, if $R \ge \max\left\{1, \frac{\sqrt{s}}{2}\right\}$, then (6.14) follows from (6.17), (6.19), (6.23), and (6.24).

Now suppose that $1 \leq R < \frac{\sqrt{s}}{2}$. We still have

$$\bigcup_{\mathbf{m} \in \mathbb{Z}^s \cap B_s(R)} K_{\mathbf{m}} \subseteq B_s\left(R + \frac{\sqrt{s}}{2}\right) \tag{6.25}$$

and

$$\operatorname{Vol}_s\left(\bigcup_{\mathbf{m} \in \mathbb{Z}^s \cap B_s(R)} K_{\mathbf{m}}\right) \leq \operatorname{Vol}_s\left(B_s\left(R + \frac{\sqrt{s}}{2}\right)\right). \tag{6.26}$$

Furthermore, (6.24) still holds. Thus, using (6.17), we still have

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| \leq \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)} R^s + O_s(R^{s-1}). \tag{6.27}$$

Thus, $|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}|$ is bounded for $R \in \left[1, \frac{\sqrt{s}}{2}\right)$. Therefore, the expression

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| - \frac{\pi^{s/2}}{\Gamma\left(s/2+1\right)} R^s$$

is bounded for $R \in \left[1, \frac{\sqrt{s}}{2}\right)$. Because $R^{s-1} \geq 1$ for $R \in \left[1, \frac{\sqrt{s}}{2}\right)$, we have (6.14) for $R \in \left[1, \frac{\sqrt{s}}{2}\right)$. □

As noted in Remark 6.7, we cannot have (6.14) be true for all $R > 0$. However, we do want to have an upper bound for the number of points in $B_s(R)$ that is true for all $R > 0$. We first state a result that gives an exact count for the number of points in $B_s(R)$ if $0 < R < 1$.

Lemma 6.8. *Let s be a positive integer and $0 < R < 1$. Then the number of integer lattice points in $B_s(R)$ is*

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| = 1.$$

Proof. The zero vector is in $\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}$. If $\mathbf{m} \in \mathbb{Z}^s$ is a nonzero vector, then $\|\mathbf{m}\| \geq 1 > R$. Therefore, the only vector in $\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}$ is the zero vector, and $|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| = 1$. □

We use Lemmas 6.5 and 6.8 to obtain the following upper bound for the number of points in $B_s(R)$.

Theorem 6.9. *Let s be a positive integer and $R > 0$. Then the number of integer lattice points in $B_s(R)$ is*

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| \ll_s R^s + 1.$$

Proof. If $R \geq 1$, then

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| \ll_s R^s \leq R^s + 1$$

by Lemma 6.5.

If $0 < R < 1$, then

$$|\{\mathbf{m} \in \mathbb{Z}^s : \|\mathbf{m}\| \leq R\}| = 1 \leq R^s + 1$$

by Lemma 6.8. □

For $R > 0$, sometimes we will want to have an upper bound for the number of integer lattice points $\mathbf{m} \in \mathbb{Z}^s$ satisfying $0 < \|\mathbf{m}\| \leq R$. Such an upper bound is stated in the next corollary.

Corollary 6.10. *Let s be a positive integer and $R > 0$. Then the number of nonzero integer lattice points in $B_s(R)$ is*

$$|\{\mathbf{m} \in \mathbb{Z}^s : 0 < \|\mathbf{m}\| \leq R\}| \ll_s R^s.$$

Proof. If $R \geq 1$, then the result of the corollary follows from Lemma 6.5.

If $0 < R < 1$, then

$$|\{\mathbf{m} \in \mathbb{Z}^s : 0 < \|\mathbf{m}\| \leq R\}| = 0 \leq R^s. \qquad \square$$

We will have summations involving the Euclidean norm of vectors. The following theorem provides an upper bound for such a sum.

Theorem 6.11. *Suppose that $B \geq 1$ and $M > s$. Then*

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ \|\mathbf{r}\| > B}} \|\mathbf{r}\|^{-M} \ll_s \left(1 + \frac{4}{M(M-s)}\right) B^{s-M}.$$

Proof. Define $\mathcal{L}$ to be the lattice sum

$$\mathcal{L} = \sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ \|\mathbf{r}\| > B}} \|\mathbf{r}\|^{-M} = \sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ \|\mathbf{r}\|^2 > B^2}} (\|\mathbf{r}\|^2)^{-M/2}.$$

For a nonnegative integer j, let

$$a(j) = \left|\{\mathbf{r} \in \mathbb{Z}^s : \|\mathbf{r}\|^2 = j\}\right|.$$

Then

$$\mathcal{L} = \sum_{j > B^2} a(j) j^{-M/2}.$$

For $R \geq 1$, let $\mathcal{L}(R)$ be

$$\mathcal{L}(R) = \sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ B < \|\mathbf{r}\| \leq R}} \|\mathbf{r}\|^{-M} = \sum_{B^2 < j \leq R^2} a(j) j^{-M/2}$$

so that $\lim_{R \to \infty} \mathcal{L}(R) = \mathcal{L}$.

We will use summation by parts. Therefore, it is useful to define the following sum: For $y \geq 0$, let

$$S(y) = \sum_{j=0}^{\lfloor y \rfloor} a(j). \tag{6.28}$$

The definition of $a(j)$ implies that

$$\begin{aligned} S(y) &= \left|\{\mathbf{r} \in \mathbb{Z}^s : \|\mathbf{r}\|^2 \leq y\}\right| \\ &= \left|\{\mathbf{r} \in \mathbb{Z}^s : \|\mathbf{r}\| \leq \sqrt{y}\}\right|. \end{aligned} \tag{6.29}$$

Therefore, if $y \geq 1$, we know from Lemma 6.5 that

$$S(y) \ll_s y^{s/2}. \tag{6.30}$$

Let $\tilde{B} = \lfloor B^2 \rfloor$ and $\tilde{R} = \lfloor R^2 \rfloor$. Then

$$\mathcal{L}(R) = \sum_{\tilde{B} < j \leq \tilde{R}} (S(j) - S(j-1))j^{-M/2} = \sum_{j=\tilde{B}+1}^{\tilde{R}} (S(j) - S(j-1))j^{-M/2}$$

since $a(j) = S(j) - S(j-1)$. By summation by parts,

$$\begin{aligned} \mathcal{L}(R) &= S(\tilde{R})\tilde{R}^{-M/2} - S(\tilde{B})(\tilde{B}+1)^{-M/2} - \sum_{j=\tilde{B}+1}^{\tilde{R}-1} S(j)((j+1)^{-M/2} - j^{-M/2}) \\ &= S(\tilde{R})\tilde{R}^{-M/2} - S(\tilde{B})(\tilde{B}+1)^{-M/2} + \frac{2}{M} \sum_{j=\tilde{B}+1}^{\tilde{R}-1} S(j) \int_j^{j+1} k^{-M/2-1}\, dk \\ &= S(\tilde{R})\tilde{R}^{-M/2} - S(\tilde{B})(\tilde{B}+1)^{-M/2} + \frac{2}{M} \sum_{j=\tilde{B}+1}^{\tilde{R}-1} \int_j^{j+1} S(k)k^{-M/2-1}\, dk \\ &= S(\tilde{R})\tilde{R}^{-M/2} - S(\tilde{B})(\tilde{B}+1)^{-M/2} + \frac{2}{M} \int_{\tilde{B}+1}^{\tilde{R}} S(k)k^{-M/2-1}\, dk. \end{aligned} \tag{6.31}$$

Using (6.30) in (6.31), we find that

$$\begin{aligned}\mathcal{L}(R) &\ll_s \tilde{R}^{(s-M)/2} + \tilde{B}^{s/2}(\tilde{B}+1)^{-M/2} + \frac{2}{M}\int_{\tilde{B}+1}^{\tilde{R}} k^{(s-M)/2-1}\, dk \\ &= \tilde{R}^{(s-M)/2} + \tilde{B}^{s/2}(\tilde{B}+1)^{-M/2} + \frac{4}{M(M-s)}\left((\tilde{B}+1)^{(s-M)/2} - \tilde{R}^{(s-M)/2}\right).\end{aligned} \tag{6.32}$$

Taking the limits in (6.32) as $R \to \infty$, we find that

$$\mathcal{L} \ll_s \tilde{B}^{s/2}(\tilde{B}+1)^{-M/2} + \frac{4}{M(M-s)}(\tilde{B}+1)^{(s-M)/2}$$

since $s < M$ and $\tilde{R} \to \infty$ as $R \to \infty$. Now because $\tilde{B} = \lfloor B^2 \rfloor$, we have

$$\begin{aligned}\mathcal{L} &\ll_s \lfloor B^2 \rfloor^{s/2}(\lfloor B^2 \rfloor + 1)^{-M/2} + \frac{4}{M(M-s)}(\lfloor B^2 \rfloor + 1)^{(s-M)/2} \\ &\leq B^{s-M} + \frac{4}{M(M-s)} B^{s-M}\end{aligned}$$

since $M > s > 0$. $\square$

Instead of using the previous theorem, we will use a corollary of it for ease of use.

Corollary 6.12. *Suppose that $B \geq 1$ and $M \geq s+1$. Then*

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ \|\mathbf{r}\| > B}} \|\mathbf{r}\|^{-M} \ll_s B^{s-M}. \tag{6.33}$$

Proof. Because $M \geq s+1$, we have

$$1 + \frac{4}{M(M-s)} \leq 1 + \frac{4}{s+1},$$

so

$$\left(1+\frac{4}{M(M-s)}\right)B^{s-M} \ll_s B^{s-M}.$$

The result of the corollary follows from Theorem 6.11. $\square$

Because such a sum will arise, we provide an upper bound for $\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ \|\mathbf{r}\|>B}} \|\mathbf{r}\|^{-M}$ when $B \geq 0$.

Corollary 6.13. *Suppose that $B \geq 0$ and $M \geq s+1$. Then*

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ \|\mathbf{r}\|>B}} \|\mathbf{r}\|^{-M} \ll_s 1. \tag{6.34}$$

Proof. If $B \geq 1$, then it follows from Corollary 6.12 that

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ \|\mathbf{r}\|>B}} \|\mathbf{r}\|^{-M} \ll_s B^{s-M} \leq 1$$

since $M \geq s+1$.

Now suppose that $0 \leq B < 1$. Then

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ \|\mathbf{r}\|>B}} \|\mathbf{r}\|^{-M} = |\{\mathbf{r}\in\mathbb{Z}^s : \|\mathbf{r}\| = 1\}| + \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ \|\mathbf{r}\|>1}} \|\mathbf{r}\|^{-M}. \tag{6.35}$$

Now

$$|\{\mathbf{r}\in\mathbb{Z}^s : \|\mathbf{r}\| = 1\}| = 2s \tag{6.36}$$

since the vectors $\mathbf{r}\in\mathbb{Z}^s$ with $\|\mathbf{r}\| = 1$ are the vectors with exactly one nonzero entry and that nonzero entry is either 1 or -1. From Corollary 6.12, we know that

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\|>1}} \|\mathbf{r}\|^{-M} \ll_s 1. \tag{6.37}$$

Substituting (6.36) and (6.37) into (6.35), we obtain (6.34). □

6.1.3 The compactness of the preimage of a positive definite quadratic form

In this subsection, we prove that the set $V = \{\mathbf{m} \in \mathbb{R}^s : F(\mathbf{m}) = n\}$ is compact when F is a positive definite quadratic form and n is a real number. Before we do this, we prove that if $\mathbf{m}$ is in V, then there are bounds on $\|\mathbf{m}\|$.

Lemma 6.14. *Suppose that F is a positive definite quadratic form in s variables. Suppose that n is a real number. Let $A \in M_s(\mathbb{R})$ be the Hessian matrix of F. Let λ_1 be the smallest eigenvalue of A, and let λ_s be the largest eigenvalue of A. Suppose that $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = n$. Then*

$$\sqrt{\frac{2n}{\lambda_s}} \le \|\mathbf{m}\| \le \sqrt{\frac{2n}{\lambda_1}}. \tag{6.38}$$

Remark 6.15. Other bounds for the size of a real solution of $F(\mathbf{m}) = n$ have been found when F is a positive definite integral quadratic form. For example, let f_{jk} be the coefficient of $m_j m_k$ in F, where $1 \le j \le k \le s$. Kornhauser [Kor90, Lemma 10] showed that if $F(\mathbf{m}) = n$, then

$$\|\mathbf{m}\| < 2(4sH)^{(s-1)/2} n^{1/2},$$

where $H = \max_{1\le j\le k\le s} |f_{jk}|$. (For a proof of this result, see the proof of Lemma 10 in [Kor89].)

Remark 6.16. The condition that F is positive definite is crucial in Lemma 6.14. For example, the set $\{(m_1, m_2)^\top \in \mathbb{R}^2 : m_1^2 - m_2^2 = 1\}$ is a hyperbola and is not bounded.

Proof of Lemma 6.14. We first use Lemma 5.15 and Remark 5.16 to prove the first inequality in (6.38). By Lemma 5.15 and Remark 5.16, we have

$$n = F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m} \leq \frac{1}{2}\lambda_s \|\mathbf{m}\|^2. \tag{6.39}$$

Solving for $\|\mathbf{m}\|$ in (6.39), we obtain the first inequality in (6.38).

We now prove the second inequality in (6.38). Using the spectral theorem for symmetric matrices, we can write the matrix A as

$$A = P^\top DP,$$

where P is an orthogonal matrix and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_s)$ is a diagonal matrix. Note that $\{\lambda_j\}_{j=1}^s$ is the set of eigenvalues of A. Then

$$\begin{aligned} F(\mathbf{m}) &= \frac{1}{2}\mathbf{m}^\top A\mathbf{m} \\ &= \frac{1}{2}\mathbf{m}^\top P^\top DP\mathbf{m}. \end{aligned} \tag{6.40}$$

Let $\mathbf{v} = P\mathbf{m}$. Note that $\|\mathbf{v}\| = \|\mathbf{m}\|$ since P is orthogonal. Substituting $\mathbf{v} = P\mathbf{m}$ into (6.40), we see that

$$\begin{aligned} F(\mathbf{m}) &= \frac{1}{2}\mathbf{v}^\top D\mathbf{v} \\ &= \frac{1}{2}\sum_{j=1}^{s} \lambda_j v_j^2. \end{aligned} \tag{6.41}$$

Since A is positive definite, each eigenvalue λ_j is positive. Without loss of generality, we assume that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s$. Therefore, (6.41) implies that

$$F(\mathbf{m}) \geq \frac{1}{2}\sum_{j=1}^{s} \lambda_1 v_j^2 = \frac{\lambda_1}{2}\|\mathbf{v}\|^2. \tag{6.42}$$

Solving (6.42) for $\|\mathbf{v}\|$, we find that

$$\|\mathbf{v}\| \leq \sqrt{\frac{2F(\mathbf{m})}{\lambda_1}}.$$

Because $\|\mathbf{v}\| = \|\mathbf{m}\|$ and $F(\mathbf{m}) = n$, we obtain the second inequality in (6.38). □

Lemma 6.14 shows that V is bounded. We now use the Heine-Borel theorem to show that V compact.

Theorem 6.17. *Suppose that F is a positive definite quadratic form in s variables. Suppose that n is a real number. Then the set*

$$V = \{\mathbf{m} \in \mathbb{R}^s : F(\mathbf{m}) = n\} \tag{6.43}$$

is compact.

Proof. Because F is continuous and $\{n\}$ is a closed set, the preimage $F^{-1}(\{n\}) = V$ is a closed set. Lemma 6.14 says that V is bounded. Therefore, since V is closed and bounded, the Heine-Borel theorem tells us that V is compact. □

6.2 Analyzing the main term

Given the previous section, we now analyze the main term $M_{F,\psi,X}(n)$ of our weighted representation number. Using (3.22), (3.19), and (5.25), we expand (6.2) to discover that

$$M_{F,\psi,X}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\; d\mathbf{m} \right.$$
$$\left. \times \sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right)\; dx \right) \tag{6.44}$$

(We are integrating x only in the region that $qdx < 1$, so we can drop the condition $qdx < 1$ from the sum over d.) We use the fact that $2\,\mathrm{Re}(z) = z + \overline{z}$ for any $z \in \mathbb{C}$ to expand (6.44) and obtain

$$\begin{aligned}
M_{F,\psi,X}(n) = & \sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\; d\mathbf{m} \\
& \quad \times \sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right)\; dx \\
& + \overline{\sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}(X^2 x F(\mathbf{m}))\, \psi(\mathbf{m})\; d\mathbf{m}} \\
& \quad \times \overline{\sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right)\; dx} \\
= & \sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\; d\mathbf{m} \\
& \quad \times \sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right)\; dx \\
& + \sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(-X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\; d\mathbf{m} \\
& \quad \times \sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q}F(\mathbf{h})\right)\; dx.
\end{aligned} \tag{6.45}$$

Now, by mapping d to $-d$, we have

$$\begin{aligned}
&\sum_{\substack{Q<d\leq q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right)\\
&= \sum_{\substack{-Q>d\geq-(q+Q)\\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q}F(\mathbf{h})\right). \qquad (6.46)
\end{aligned}$$

Because $\mathrm{e}\left(\frac{\cdot}{q}\right)$ is periodic modulo q, we can sum d over any reduced residue system and obtain the same value for (6.46). Therefore, we rewrite (6.46) as

$$\sum_{\substack{Q<d\leq q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right) = \sum_{\substack{Q<d\leq q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q}F(\mathbf{h})\right). \qquad (6.47)$$

Substituting this into (6.45), we obtain

$$\begin{aligned}
M_{F,\psi,X}(n) = &\sum_{1\leq q\leq Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2xF(\mathbf{m})\big)\, \psi(\mathbf{m})\ d\mathbf{m}\\
&\quad \times \sum_{\substack{Q<d\leq q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q}F(\mathbf{h})\right)\ dx\\
&+ \sum_{1\leq q\leq Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(-X^2xF(\mathbf{m})\big)\, \psi(\mathbf{m})\ d\mathbf{m}\\
&\quad \times \sum_{\substack{Q<d\leq q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q}F(\mathbf{h})\right)\ dx. \qquad (6.48)
\end{aligned}$$

By mapping x to $-x$ in the second integral in x and simplifying, we see that

$$\begin{aligned}
M_{F,\psi,X}(n) &= \sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\, d\mathbf{m} \\
&\qquad \times \sum_{\substack{Q<d\le q+Q \\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q} F(\mathbf{h})\right)\, dx \\
&\quad - \sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{-\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\, d\mathbf{m} \\
&\qquad \times \sum_{\substack{Q<d\le q+Q \\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q} F(\mathbf{h})\right)\, dx \\
&= \sum_{1\le q\le Q} \frac{1}{q^s} \sum_{\substack{Q<d\le q+Q \\ \gcd(d,q)=1}} \mathrm{e}\left(-n\frac{d^*}{q}\right) \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d^*}{q} F(\mathbf{h})\right) \\
&\qquad \times \int_{-\frac{1}{q(q+Q)}}^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\, d\mathbf{m}\, dx. \qquad (6.49)
\end{aligned}$$

Because the sum over d is a complete sum that only depends on d modulo q, we can map d^* to d in (6.49) and obtain

$$\begin{aligned}
M_{F,\psi,X}(n) &= \sum_{1\le q\le Q} \frac{1}{q^s} \sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\,(F(\mathbf{h})-n)\right) \\
&\qquad \times \int_{-\frac{1}{q(q+Q)}}^{\frac{1}{q(q+Q)}} \mathrm{e}(-nx)\, X^s \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\, d\mathbf{m}\, dx. \qquad (6.50)
\end{aligned}$$

To provide an asymptotic for $M_{F,\psi,X}(n)$ with an appropriate error term, we will need an upper bound for the absolute value of the sum

$$\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\,(F(\mathbf{h})-n)\right).$$

The next lemma gives such an upper bound.

Lemma 6.18. *If q is a positive integer and n is an integer, then*

$$\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right)$$
$$\ll (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2}q_0^{1/2}q^{(s+1)/2}\tau(q)\log(2q). \tag{6.51}$$

Proof. If $x<\frac{1}{q(q+Q)}$, then

$$T_{\mathbf{0}}(q,n;x)=\sum_{\substack{Q<d\le q+Q\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right)\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\mathrm{e}\left(\frac{-d^*}{q}F(\mathbf{h})\right) \tag{6.52}$$
$$=\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right). \tag{6.53}$$

(We obtained (6.53) from (6.52) by mapping d to $-d^*$ and noticing that the sum over d is a complete sum modulo q.) Therefore, we can use Lemma 4.22 to provide an upper bound for the absolute value of $\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s}\mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right)$ and obtain (6.51). □

The rest of this section is devoted to providing an asymptotic for $M_{F,\psi,X}(n)$.

6.2.1 Extending to the singular integral

For a bump function $\psi\in C_c^\infty(\mathbb{R}^s)$, positive real numbers X and B, and a real number n, we define the *truncated singular integral* $J_{F,\psi}(n,X;B)$ to be

$$J_{F,\psi}(n,X;B)=\int_{-B}^{B}\mathrm{e}(-nx)\,X^s\int_{\mathbb{R}^s}\mathrm{e}\big(X^2xF(\mathbf{m})\big)\,\psi(\mathbf{m})\,d\mathbf{m}\,dx. \tag{6.54}$$

Notice that the truncated singular integral $J_{F,\psi}\left(n, X; \frac{1}{q(q+Q)}\right)$ appears in (6.50) so that

$$M_{F,\psi,X}(n) = \sum_{1\leq q\leq Q} \frac{1}{q^s} \sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right) J_{F,\psi}\left(n, X; \frac{1}{q(q+Q)}\right). \tag{6.55}$$

The first step towards providing an asymptotic for $M_{F,\psi,X}(n)$ is to extend (up to some acceptable error term) the truncated singular integral $J_{F,\psi}\left(n, X; \frac{1}{q(q+Q)}\right)$ to the *singular integral*

$$J_{F,\psi}(n, X) = X^s \int_{-\infty}^{\infty} \mathrm{e}(-nx) \int_{\mathbb{R}^s} \mathrm{e}\big(X^2 x F(\mathbf{m})\big)\, \psi(\mathbf{m})\, d\mathbf{m}\, dx, \tag{6.56}$$

where $\psi \in C_c^\infty(\mathbb{R}^s)$, $X > 0$, and $n \in \mathbb{R}$. We do this in the following lemma.

Lemma 6.19. *For a bump function $\psi \in C_c^\infty(\mathbb{R}^s)$, positive real numbers X and B, and a real number n, we have*

$$J_{F,\psi}(n, X; B) = J_{F,\psi}(n, X) + O_{\psi,s}\left((\det(A))^{-1/2} B^{1-\frac{s}{2}}\right) \tag{6.57}$$

Proof. By (5.25), the difference between the singular integral and the truncated singular integral $J_{F,\psi}(n, X; B)$ is equal to

$$J_{F,\psi}(n, X) - J_{F,\psi}(n, X; B) = \int_{|x|>B} \mathrm{e}(-nx)\, \mathcal{I}_{F,\psi}(x, X, \mathbf{0}, q)\, dx. \tag{6.58}$$

We apply Theorem 5.11 to this difference and obtain

$$\begin{aligned} J_{F,\psi}(n,X) - J_{F,\psi}(n,X;B) &\ll_\psi \int_{|x|>B} |x|^{-s/2}(\det(A))^{-1/2}\, dx \\ &= 2(\det(A))^{-1/2}\int_B^\infty x^{-s/2}\, dx \\ &= \frac{2}{1-\frac{s}{2}}(\det(A))^{-1/2}B^{1-\frac{s}{2}}. \end{aligned}$$

This gives (6.57) with an implied constant of $\frac{2}{1-\frac{s}{2}}$. $\square$

We use Lemma 6.19 with $B = \frac{1}{q(q+Q)}$ to conclude that

$$J_{F,\psi}\left(n, X; \frac{1}{q(q+Q)}\right) = J_{F,\psi}(n,X) + O_{\psi,s}\left((\det(A))^{-1/2}(q(q+Q))^{s/2-1}\right). \quad (6.59)$$

Substituting this into (6.55), we see that

$$\begin{aligned} M_{F,\psi,X}(n) = &\sum_{1\le q\le Q}\frac{1}{q^s}\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right) J_{F,\psi}(n,X) \\ &+ O_{\psi,s}\left(\left|\sum_{1\le q\le Q}\frac{1}{q^s}\sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right)\right.\right. \\ &\qquad\left.\left. \times (\det(A))^{-1/2}(q(q+Q))^{s/2-1}\right|\right). \quad (6.60) \end{aligned}$$

By Lemma 6.18 and the fact that $s \geq 4$, we have

$$\begin{aligned}
&\sum_{1 \leq q \leq Q} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right) (\det(A))^{-1/2} (q(q+Q))^{s/2-1} \\
&\ll \sum_{1 \leq q \leq Q} (\gcd(L, q_0))^{s/2} (\gcd(n, q_1))^{1/2} q_0^{1/2} q^{(1-s)/2} \\
&\qquad \times \tau(q) \log(2q) (\det(A))^{-1/2} (q(q+Q))^{s/2-1} \\
&\leq L^{s/2} (\det(A))^{-1/2} (2Q)^{s/2-1} \sum_{1 \leq q \leq Q} (\gcd(n, q_1))^{1/2} q_0^{1/2} q^{-1/2} \tau(q) \log(2q) \\
&= L^{s/2} (\det(A))^{-1/2} (2Q)^{s/2-1} \sum_{1 \leq q \leq Q} (\gcd(n, q_1))^{1/2} q_1^{-1/2} \tau(q) \log(2q). \qquad (6.61)
\end{aligned}$$

We apply Lemma 6.1 to (6.61) to obtain

$$\begin{aligned}
&\sum_{1 \leq q \leq Q} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right) (\det(A))^{-1/2} (q(q+Q))^{s/2-1} \\
&\ll_\varepsilon L^{s/2} (\det(A))^{-1/2} 2^{s/2-1} Q^{(s-1)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \qquad (6.62)
\end{aligned}$$

for any $\varepsilon > 0$.

Substituting (6.62) into (6.60), we conclude that

$$\begin{aligned}
M_{F,\psi,X}(n) = &\sum_{1 \leq q \leq Q} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right) J_{F,\psi}(n, X) \\
&+ O_{\psi,s,\varepsilon}\left(L^{s/2} (\det(A))^{-1/2} Q^{(s-1)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right).
\end{aligned} \qquad (6.63)$$

6.2.2 Evaluating the singular integral

In this subsection, we evaluate the singular integral under certain conditions. Only in this subsection, the positive definite form F might not be integral. All quantities used in this subsection that involve F make sense if F is not integral.

We first apply to (6.56) the change of variables $x \mapsto x/X^2$ to obtain

$$J_{F,\psi}(n,X) = X^{s-2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^s} \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big)\, \psi(\mathbf{m})\, d\mathbf{m}\, dx. \tag{6.64}$$

Let $\tilde{\sigma}_{F,\psi,\infty}(n,X)$ be the quantity

$$\tilde{\sigma}_{F,\psi,\infty}(n,X) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^s} \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big)\, \psi(\mathbf{m})\, d\mathbf{m}\, dx \tag{6.65}$$

so that

$$J_{F,\psi}(n,X) = \tilde{\sigma}_{F,\psi,\infty}(n,X) X^{s-2}. \tag{6.66}$$

We now prove the following theorem about $\tilde{\sigma}_{F,\psi,\infty}(n,X)$.

Theorem 6.20. *For a bump function $\psi \in C_c^{\infty}(\mathbb{R})$, a real number n, and a positive real number X, we have*

$$\tilde{\sigma}_{F,\psi,\infty}(n,X) = \sigma_{F,\psi,\infty}(n,X), \tag{6.67}$$

where $\sigma_{F,\psi,\infty}(n,X)$ is the real factor defined in (1.5).

Proof. We use tent functions to create continuous approximations to the indicator function $\mathbf{1}_{\{|x|<\varepsilon\}}$, where $\varepsilon > 0$.

For $x \in \mathbb{R}$, define the tent function $\mathfrak{t}$ by

$$\mathfrak{t}(x) = \max\{0, 1 - |x|\}. \tag{6.68}$$

For nonzero $x \in \mathbb{R}$, define the sinc^2 function by

$$\operatorname{sinc}^2(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2. \tag{6.69}$$

Also, set $\operatorname{sinc}^2(0) = 1$ so that sinc^2 function is a continuous function on $\mathbb{R}$.

It is well-known that the tent function $\mathfrak{t}$ is the Fourier transform of the sinc^2 function and that the sinc^2 function is the Fourier transform of the tent function $\mathfrak{t}$. (See, for example, Appendix 2 of [Kam08].)

For $\eta > 0$, we define the function $\mathfrak{t}_\eta$ by

$$\mathfrak{t}_\eta(x) = \max\left\{0, 1 - \frac{|x|}{\eta}\right\}. \tag{6.70}$$

Using part b of Theorem 8.22 in [Fol99] about a scaling property for the Fourier transform, we find that the Fourier transform of $\mathfrak{t}_\eta$ is

$$w_\eta(x) = \eta \left(\frac{\sin(\pi\eta x)}{\pi\eta x}\right)^2 \tag{6.71}$$

and the Fourier transform of w_η is $\mathfrak{t}_\eta$.

If $\eta, \delta > 0$, then we define the function $\mathfrak{T}_{\eta,\delta}$ by

$$\mathfrak{T}_{\eta,\delta}(x) = \left(1 + \frac{\eta}{\delta}\right) \mathfrak{t}_{\eta+\delta}(x) - \frac{\eta}{\delta}\mathfrak{t}_\eta(x) \tag{6.72}$$

for $x \in \mathbb{R}$. After some manipulations, we find that

$$\mathfrak{T}_{\eta,\delta}(x) = \begin{cases} 1 & \text{if } |x| \le \eta, \\ 1 - \frac{|x|-\eta}{\delta} & \text{if } \eta < |x| < \eta + \delta, \\ 0 & \text{if } |x| \ge \eta + \delta. \end{cases} \tag{6.73}$$

Let $0 < \varepsilon < 1$. For $x \in \mathbb{R}$, define $\mathfrak{T}_\varepsilon^+$ and $\mathfrak{T}_\varepsilon^-$ to be

$$\mathfrak{T}_\varepsilon^- = \mathfrak{T}_{\varepsilon-\varepsilon^2,\varepsilon^2} = \varepsilon^{-1}\mathfrak{t}_\varepsilon + \left(1 - \varepsilon^{-1}\right) \mathfrak{t}_{\varepsilon-\varepsilon^2} \tag{6.74}$$

and

$$\mathfrak{T}_{\varepsilon}^{+} = \mathfrak{T}_{\varepsilon,\varepsilon^2} = \left(1+\varepsilon^{-1}\right)\mathfrak{t}_{\varepsilon+\varepsilon^2} - \varepsilon^{-1}\mathfrak{t}_{\varepsilon}. \tag{6.75}$$

Using (6.73), we observe that

$$\mathfrak{T}_{\varepsilon}^{-}(x) \leq \mathbf{1}_{\{|x|<\varepsilon\}} \leq \mathfrak{T}_{\varepsilon}^{+}(x) \tag{6.76}$$

for all $x \in \mathbb{R}$. Therefore, $\mathfrak{T}_{\varepsilon}^{-}(x)$ provides a lower bound for $\mathbf{1}_{\{|x|<\varepsilon\}}$, and $\mathfrak{T}_{\varepsilon}^{+}(x)$ provides a upper bound for $\mathbf{1}_{\{|x|<\varepsilon\}}$.

Let

$$\nu_{F,\psi,n,X}(\varepsilon) = \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} \psi(\mathbf{m})\ d\mathbf{m}.$$

Since

$$\int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} \psi(\mathbf{m})\ d\mathbf{m} = \int_{\mathbb{R}^s} \mathbf{1}_{\left\{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon\right\}} \psi(\mathbf{m})\ d\mathbf{m},$$

the inequalities in (6.76) imply that

$$\int_{\mathbb{R}^s} \mathfrak{T}_{\varepsilon}^{-}\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\psi(\mathbf{m})\ d\mathbf{m} \leq \nu_{F,\psi,n,X}(\varepsilon) \leq \int_{\mathbb{R}^s} \mathfrak{T}_{\varepsilon}^{+}\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\psi(\mathbf{m})\ d\mathbf{m}. \tag{6.77}$$

Using (6.74) and (6.75), we manipulate some of the integrals that appear in (6.77) and see that

$$\int_{\mathbb{R}^s} \mathfrak{T}_{\varepsilon}^{-}\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\psi(\mathbf{m})\ d\mathbf{m} = U_{F,n,X}(\varepsilon) - (\varepsilon-1)^2 U_{F,n,X}(\varepsilon-\varepsilon^2) \tag{6.78}$$

and

$$\int_{\mathbb{R}^s} \mathfrak{T}_\varepsilon^+ \left(F(\mathbf{m}) - \frac{n}{X^2}\right) \psi(\mathbf{m})\ d\mathbf{m} = (\varepsilon+1)^2 U_{F,n,X}(\varepsilon+\varepsilon^2) - U_{F,n,X}(\varepsilon), \tag{6.79}$$

where

$$U_{F,n,X}(\eta) = \int_{\mathbb{R}^s} \eta^{-1} \mathfrak{t}_\eta \left(F(\mathbf{m}) - \frac{n}{X^2}\right) \psi(\mathbf{m})\ d\mathbf{m} \tag{6.80}$$

for $\eta > 0$.

In light of (6.78) and (6.79), we provide an asymptotic for $U_{F,n,X}(\eta)$ when η is sufficiently close to ε.

Lemma 6.21. *Let $\varepsilon > 0$. For $\eta \in \mathbb{R}$ with $|\eta - \varepsilon| < \varepsilon$, we have*

$$U_{F,n,X}(\eta) = \tilde{\sigma}_{F,\psi,\infty}(n,X) + O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right). \tag{6.81}$$

Proof of Lemma 6.21. Using the inverse Fourier transform and the fact that $\widehat{\mathfrak{t}}_\eta = w_\eta$, we find that

$$U_{F,n,X}(\eta) = \int_{\mathbb{R}^s} \int_{-\infty}^{\infty} \eta^{-1} w_\eta(x) \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m})\ dx\ d\mathbf{m}.$$

Applying the Fubini-Tonelli theorem, we see that

$$U_{F,n,X}(\eta) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^s} \eta^{-1} w_\eta(x) \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m})\ d\mathbf{m}\ dx. \tag{6.82}$$

Therefore,

$$\begin{aligned} &U_{F,n,X}(\eta) - \tilde{\sigma}_{F,\psi,\infty}(n,X) \\ &= \int_{-\infty}^{\infty} \left(\eta^{-1} w_\eta(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m})\ d\mathbf{m}\ dx. \end{aligned} \tag{6.83}$$

In order to prove (6.81), it suffices to prove (6.83) is $O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right)$. To do this, we split up the integral over x into two regions: one in which $|x| \le \varepsilon^{\delta}$ and the other in which $|x| > \varepsilon^{\delta}$, where $\delta \in \mathbb{R}$ will be chosen later. Let $\mathfrak{D} = \left[-\varepsilon^{\delta}, \varepsilon^{\delta}\right]$ so that the two regions under consideration for x are $\mathfrak{D}$ and $\mathbb{R} \setminus \mathfrak{D}$.

Observe that

$$0 \le 1 - \eta^{-1} w_{\eta}(x) \le 1. \tag{6.84}$$

From a Taylor series approximation of $w_{\eta}(x)$, we see that

$$0 \le 1 - \eta^{-1} w_{\eta}(x) \ll \min\{1, \eta^2 x^2\} \ll \min\{1, \varepsilon^2 x^2\} \tag{6.85}$$

since $0 < \eta < 2\varepsilon$.

Now

$$\begin{aligned}
&\left| \int_{\mathfrak{D}} \left(\eta^{-1} w_{\eta}(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m}) \, d\mathbf{m} \, dx \right| \\
&\le \int_{\mathfrak{D}} \left|\eta^{-1} w_{\eta}(x) - 1\right| \int_{\mathbb{R}^s} |\psi(\mathbf{m})| \, d\mathbf{m} \, dx.
\end{aligned} \tag{6.86}$$

Using (6.85), we obtain

$$\begin{aligned}
&\left| \int_{\mathfrak{D}} \left(\eta^{-1} w_{\eta}(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x\left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) |\psi(\mathbf{m})| \, d\mathbf{m} \, dx \right| \\
&\ll \int_{-\varepsilon^{\delta}}^{\varepsilon^{\delta}} \varepsilon^2 x^2 \int_{\mathbb{R}^s} |\psi(\mathbf{m})| \, d\mathbf{m} \, dx \\
&\ll_{\psi} \varepsilon^2 \int_{-\varepsilon^{\delta}}^{\varepsilon^{\delta}} x^2 \, dx = \frac{2}{3} \varepsilon^{2+3\delta} \\
&\ll \varepsilon^{2+3\delta}.
\end{aligned} \tag{6.87}$$

Now we look at the region in which $|x| > \varepsilon^{\delta}$. By (6.84), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}\setminus\mathfrak{D}} \left(\eta^{-1} w_{\eta}(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x \left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m}) \; d\mathbf{m} \; dx \right| \\
&\leq \int_{\mathbb{R}\setminus\mathfrak{D}} \left| \int_{\mathbb{R}^s} \mathrm{e}\Big(x \left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m}) \; d\mathbf{m} \right| \; dx. \qquad (6.88)
\end{aligned}$$

We apply Theorem 5.4 to (6.88) to obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}\setminus\mathfrak{D}} \left(\eta^{-1} w_{\eta}(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x \left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m}) \; d\mathbf{m} \; dx \right| \\
&\ll_{\psi} \int_{\mathbb{R}\setminus\mathfrak{D}} |x|^{-s/2} |\det(A)|^{-1/2} \; dx. \qquad (6.89)
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\mathbb{R}\setminus\mathfrak{D}} |x|^{-s/2} |\det(A)|^{-1/2} \; dx &= 2|\det(A)|^{-1/2} \int_{\varepsilon^{\delta}}^{\infty} x^{-s/2} \; dx \\
&= 2|\det(A)|^{-1/2} \frac{1}{s/2-1} \varepsilon^{\delta(1-s/2)} \\
&\ll_{F,s} \varepsilon^{\delta(1-s/2)}. \qquad (6.90)
\end{aligned}$$

We substitute (6.90) into (6.89) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\mathfrak{D}} \left(\eta^{-1} w_{\eta}(x) - 1\right) \int_{\mathbb{R}^s} \mathrm{e}\Big(x \left(F(\mathbf{m}) - \frac{n}{X^2}\right)\Big) \psi(\mathbf{m}) \; d\mathbf{m} \; dx \\
&\ll_{F,\psi,s} \varepsilon^{\delta(1-s/2)}. \qquad (6.91)
\end{aligned}$$

Combining (6.87) and (6.91) with (6.83), we see that

$$U_{F,n,X}(\eta) - \tilde{\sigma}_{F,\psi,\infty}(n, X) \ll_{F,\psi,s} \varepsilon^{2+3\delta} + \varepsilon^{\delta(1-s/2)}. \qquad (6.92)$$

We now choose $\delta \in \mathbb{R}$ so that the terms on the right-hand side of (6.92) are equal to each other. We do this by solving

$$\varepsilon^{2+3\delta} = \varepsilon^{\delta(1-s/2)} \tag{6.93}$$

for δ. We find that $\delta = -4/(s+4)$ solves (6.93). By taking $\delta = -4/(s+4)$ in (6.92), we obtain (6.81). $\square$

Because $\varepsilon^2 < \varepsilon$ when $0 < \varepsilon < 1$, Lemma 6.21 applies to all instances of $U_{F,n,X}$ in (6.78) and (6.79). By applying Lemma 6.21 to (6.78) and (6.79), we find that

$$\int_{\mathbb{R}^s} \mathfrak{T}_\varepsilon^- \left(F(\mathbf{m}) - \frac{n}{X^2}\right) \psi(\mathbf{m})\, d\mathbf{m} = (2\varepsilon - \varepsilon^2)\left(\tilde{\sigma}_{F,\psi,\infty}(n,X) + O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right)\right) \tag{6.94}$$

and

$$\int_{\mathbb{R}^s} \mathfrak{T}_\varepsilon^+ \left(F(\mathbf{m}) - \frac{n}{X^2}\right) \psi(\mathbf{m})\, d\mathbf{m} = (2\varepsilon + \varepsilon^2)\left(\tilde{\sigma}_{F,\psi,\infty}(n,X) + O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right)\right) \tag{6.95}$$

By substituting (6.94) and (6.95) into (6.77) and then dividing by 2ε, we obtain

$$\begin{aligned}
&\left(1 - \frac{\varepsilon}{2}\right)\left(\tilde{\sigma}_{F,\psi,\infty}(n,X) + O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right)\right) \\
&\leq \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m}) - \frac{n}{X^2}\right| < \varepsilon} \psi(\mathbf{m})\, d\mathbf{m} \\
&\leq \left(1 + \frac{\varepsilon}{2}\right)\left(\tilde{\sigma}_{F,\psi,\infty}(n,X) + O_{F,\psi,s}\left(\varepsilon^{(2s-4)/(s+4)}\right)\right).
\end{aligned} \tag{6.96}$$

Because $s \geq 4$, we observe that $\frac{2s-4}{s+4} \geq \frac{1}{2}$, so $\lim_{\varepsilon \to 0^+} \varepsilon^{(2s-4)/(s+4)} = 0$. Thus, by taking limits in (6.96) as $\varepsilon \to 0^+$, we conclude that $\tilde{\sigma}_{F,\psi,\infty}(n,X) \leq \sigma_{F,\psi,\infty}(n,X) \leq \tilde{\sigma}_{F,\psi,\infty}(n,X)$. This implies (6.67). $\square$

Because we have determined that $\tilde{\sigma}_{F,\psi,\infty}(n,X) = \sigma_{F,\psi,\infty}(n,X)$ with Theorem 6.20, we now turn our attention to evaluating $\sigma_{F,\psi,\infty}(n, X)$. In order to do this, we compute the density of real solutions to $F(\mathbf{m}) = n$ in the following theorem.

Theorem 6.22. *Suppose that F is a positive definite quadratic form in s variables. Let A be the Hessian matrix of F. Suppose that n is a real positive number and that c is a real number. Then*

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{|F(\mathbf{m})-n|<\varepsilon} c \; d\mathbf{m} = \frac{(2\pi)^{s/2} c}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1}. \tag{6.97}$$

Proof. The integral on the left-hand side of (6.97) is

$$\begin{aligned} \int_{|F(\mathbf{m})-n|<\varepsilon} c \; d\mathbf{m} &= c \int_{F(\mathbf{m})<n+\varepsilon} 1 \; d\mathbf{m} - c \int_{F(\mathbf{m})\leq n-\varepsilon} 1 \; d\mathbf{m} \\ &= c \int_{\mathbf{m}^\top A\mathbf{m}<2(n+\varepsilon)} 1 \; d\mathbf{m} - c \int_{\mathbf{m}^\top A\mathbf{m}\leq 2(n-\varepsilon)} 1 \; d\mathbf{m}. \end{aligned} \tag{6.98}$$

Using the spectral theorem for symmetric matrices, we can write the symmetric matrix A as

$$A = P^\top D P,$$

where P is an orthogonal matrix and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_s)$ is a diagonal matrix with the eigenvalues of A as diagonal entries.

Let $B = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_s})P$. (There are no issues with taking the square roots here since all of the eigenvalues of a positive definite matrix are positive.) Then

$$A = B^\top B$$

and

$$\det(B) = \sqrt{\det(A)} > 0.$$

We perform the change of variables of $\mathbf{m} \mapsto B^{-1}\mathbf{m}$ in (6.98) to obtain

$$\begin{aligned}\int_{|F(\mathbf{m})-n|<\varepsilon} c\ d\mathbf{m} &= \frac{c}{|\det(B)|}\int_{(B^{-1}\mathbf{m})^\top(B^\top B)B^{-1}\mathbf{m}<2(n+\varepsilon)} 1\ d\mathbf{m} \\ &\quad - \frac{c}{|\det(B)|}\int_{(B^{-1}\mathbf{m})^\top(B^\top B)B^{-1}\mathbf{m}\le 2(n-\varepsilon)} 1\ d\mathbf{m} \\ &= \frac{c}{|\det(B)|}\left(\int_{\mathbf{m}^\top\mathbf{m}<2(n+\varepsilon)} 1\ d\mathbf{m} - \int_{\mathbf{m}^\top\mathbf{m}\le 2(n-\varepsilon)} 1\ d\mathbf{m}\right). \qquad (6.99)\end{aligned}$$

Because $\det(B) = \sqrt{\det(A)} > 0$ and $\mathbf{m}^\top\mathbf{m} = \|\mathbf{m}\|^2$, we find that (6.99) is equivalent to

$$\begin{aligned}\int_{|F(\mathbf{m})-n|<\varepsilon} c\ d\mathbf{m} &= \frac{c}{\sqrt{\det(A)}}\left(\int_{\|\mathbf{m}\|^2<2(n+\varepsilon)} 1\ d\mathbf{m} - \int_{\|\mathbf{m}\|^2\le 2(n-\varepsilon)} 1\ d\mathbf{m}\right) \\ &= \frac{c}{\sqrt{\det(A)}}\left(\int_{\|\mathbf{m}\|<\sqrt{2(n+\varepsilon)}} 1\ d\mathbf{m} - \int_{\|\mathbf{m}\|\le\sqrt{2(n-\varepsilon)}} 1\ d\mathbf{m}\right) \\ &= \frac{c}{\sqrt{\det(A)}}\left(\mathrm{Vol}_s\left(B_s^o\left(\sqrt{2(n+\varepsilon)}\right)\right) - \mathrm{Vol}_s\left(B_s\left(\sqrt{2(n-\varepsilon)}\right)\right)\right).\end{aligned} \tag{6.100}$$

By (6.13), we have

$$\int_{|F(\mathbf{m})-n|<\varepsilon} c\ d\mathbf{m} = \frac{c}{\sqrt{\det(A)}}\left(\frac{\pi^{s/2}}{\Gamma(s/2+1)}(2(n+\varepsilon))^{s/2} - \frac{\pi^{s/2}}{\Gamma(s/2+1)}(2(n-\varepsilon))^{s/2}\right) \tag{6.101}$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{|F(\mathbf{m})-n|<\varepsilon} c\ d\mathbf{m} = \frac{\pi^{s/2}c}{\Gamma(s/2+1)\sqrt{\det(A)}}\lim_{\varepsilon\to 0^+}\frac{(2(n+\varepsilon))^{s/2} - (2(n-\varepsilon))^{s/2}}{2\varepsilon}. \tag{6.102}$$

We recognize that the limit on the right-hand side of (6.102) is the (symmetric) derivative of $(2x)^{s/2}$ evaluated at $x = n$. The derivative of $(2x)^{s/2}$ is $s(2x)^{s/2-1}$, so

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{|F(\mathbf{m})-n|<\varepsilon} c \, d\mathbf{m} = \frac{\frac{s}{2}(2\pi)^{s/2} c}{\Gamma\left(s/2+1\right)\sqrt{\det(A)}} n^{s/2-1}.$$

By noticing that $\Gamma(s/2+1) = \frac{s}{2}\Gamma(s/2)$ and simplifying, we obtain (6.97). $\square$

We will need an upper bound for the absolute value of the singular integral $J_{F,\psi}(n, X)$. To do this, we first provide an upper bound for the absolute value of $\sigma_{F,\psi,\infty}(n, X)$.

Corollary 6.23. *Suppose that F is a positive definite quadratic form in s variables. Let A be the Hessian matrix of F. Suppose that n and X are positive real numbers. Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$. Then*

$$|\sigma_{F,\psi,\infty}(n, X)| \leq \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} X^{2-s} \max_{\mathbf{m}\in\mathbb{R}^s} |\psi(\mathbf{m})|. \tag{6.103}$$

Remark 6.24. Because $\psi \in C_c^\infty(\mathbb{R}^s)$, the quantity $\max_{\mathbf{m}\in\mathbb{R}^s} |\psi(\mathbf{m})|$ exists and is finite.

Proof of Corollary 6.23. By taking absolute values of both sides of (1.5), we obtain

$$\begin{aligned} |\sigma_{F,\psi,\infty}(n, X)| &\leq \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} |\psi(\mathbf{m})| \, d\mathbf{m} \\ &\leq \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} \max_{\mathbf{m}\in\mathbb{R}^s} |\psi(\mathbf{m})| \, d\mathbf{m}. \end{aligned} \tag{6.104}$$

We apply Theorem 6.22 to (6.104) to obtain (6.103). $\square$

The next corollary gives an upper bound for the absolute value of the singular integral $J_{F,\psi}(n, X)$. It follows immediately from (6.66), Theorem 6.20, and Corollary 6.23.

Corollary 6.25. *Suppose that F is a positive definite quadratic form in s variables. Let A be the Hessian matrix of F. Suppose that n and X are real positive numbers.*

Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$. Then

$$|J_{F,\psi}(n,X)| \le \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} \max_{\mathbf{m}\in\mathbb{R}^s} |\psi(\mathbf{m})|.$$

We would like to obtain an equality for the singular integral $J_{F,\psi}(n,X)$ under certain conditions. One such condition is requiring that ψ is equal to 1 on the set $V_X = \{\mathbf{m} \in \mathbb{R}^s : F(\mathbf{m}) = n/X^2\}$. Before we can give an equality under such a condition, we prove a lemma that allows us to give an upper bound on how close V_1 is to a point $\mathbf{m} \in \mathbb{R}^s$ satisfying $|F(\mathbf{m}) - n| < \varepsilon$.

Lemma 6.26. *Suppose that F is a positive definite quadratic form in s variables. Suppose that $\mathbf{m} \in \mathbb{R}^s$. Suppose that n is a real positive number and $0 < \varepsilon < n$. Let $V = \{\mathbf{v} \in \mathbb{R}^s : F(\mathbf{v}) = n\}$. Let λ_1 be the smallest eigenvalue of the Hessian matrix of F. Then the inequality $|F(\mathbf{m}) - n| < \varepsilon$ implies that there exists $\mathbf{v} \in V$ such that*

$$\|\mathbf{m} - \mathbf{v}\| < \frac{\varepsilon}{n-\varepsilon}\sqrt{\frac{n+\varepsilon}{2\lambda_1}}. \tag{6.105}$$

Proof. Suppose that $|F(\mathbf{m}) - n| < \varepsilon$. This is equivalent to the statement

$$n - \varepsilon < F(\mathbf{m}) < n + \varepsilon. \tag{6.106}$$

Using Lemma 6.14, we find that

$$\|\mathbf{m}\| < \sqrt{\frac{2(n+\varepsilon)}{\lambda_1}}. \tag{6.107}$$

Let

$$\mathbf{v} = \sqrt{\frac{n}{F(\mathbf{m})}}\,\mathbf{m}.$$

Because F is a quadratic form,

$$\begin{aligned} F(\mathbf{v}) &= F\left(\sqrt{\frac{n}{F(\mathbf{m})}}\,\mathbf{m}\right) \\ &= \frac{n}{F(\mathbf{m})}F(\mathbf{m}) \\ &= n. \end{aligned}$$

Therefore, $\mathbf{v}$ is in V.

We will prove the lemma by showing that (6.105) is satisfied by $\mathbf{m}$ and this particular choice of $\mathbf{v}$. Now

$$\begin{aligned} \|\mathbf{m}-\mathbf{v}\| &= \left\|\mathbf{m} - \sqrt{\frac{n}{F(\mathbf{m})}}\,\mathbf{m}\right\| \\ &= \left|1 - \sqrt{\frac{n}{F(\mathbf{m})}}\right| \|\mathbf{m}\| \\ &= \left|\frac{\sqrt{F(\mathbf{m})}-\sqrt{n}}{\sqrt{F(\mathbf{m})}}\right| \|\mathbf{m}\| \\ &= \left|\frac{F(\mathbf{m})-n}{\sqrt{F(\mathbf{m})}(\sqrt{F(\mathbf{m})}+\sqrt{n})}\right| \|\mathbf{m}\|. \end{aligned} \tag{6.108}$$

By (6.106), (6.107), and the fact that $|F(\mathbf{m})-n| < \varepsilon$, we have

$$\begin{aligned} \left|\frac{F(\mathbf{m})-n}{\sqrt{F(\mathbf{m})}(\sqrt{F(\mathbf{m})}+\sqrt{n})}\right| \|\mathbf{m}\| &< \frac{\varepsilon}{\sqrt{n-\varepsilon}(\sqrt{n-\varepsilon}+\sqrt{n})}\sqrt{\frac{2(n+\varepsilon)}{\lambda_1}} \\ &< \frac{\varepsilon}{2(n-\varepsilon)}\sqrt{\frac{2(n+\varepsilon)}{\lambda_1}} \end{aligned}$$

Combining this with (6.108), we obtain (6.105). □

We use the previous lemma to prove the following theorem that gives an equality for $\sigma_{F,\psi,\infty}(n, X)$ when ψ is equal to 1 on the set V_X.

Theorem 6.27. *Suppose that F is a positive definite quadratic form in s variables.*

Let A be the Hessian matrix of F. Suppose that n and X are real positive numbers. Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$. Suppose that $\psi(\mathbf{m}) = 1$ whenever $F(\mathbf{m}) = n/X^2$. Then

$$\sigma_{F,\psi,\infty}(n, X) = \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} X^{2-s}. \tag{6.109}$$

Proof. Let $V_X = \{\mathbf{m} \in \mathbb{R}^s : F(\mathbf{m}) = n/X^2\}$. Therefore, $\psi(\mathbf{m}) = 1$ if $\mathbf{m} \in V_X$. From Theorem 6.17, we know that V is compact.

Let $\eta > 0$. Because ψ is continuous and V_X is compact, there exists $\delta > 0$ such that if $\operatorname{dist}(\mathbf{m}, V_X) < \delta$, then $|\psi(\mathbf{m}) - 1| < \eta$. (Because ψ is continuous, for every $\mathbf{v} \in V_X$, there exists $\delta_{\mathbf{v}} > 0$ such that $|\psi(\mathbf{m}) - 1| < \eta$ for all $\mathbf{m} \in \mathbb{R}^s$ satisfying $\|\mathbf{m} - \mathbf{v}\| < \delta_{\mathbf{v}}$. For $\mathbf{v} \in V_X$, let $U_{\mathbf{v}} = \{\mathbf{m} \in \mathbb{R}^s : \|\mathbf{m} - \mathbf{v}\| < \delta_{\mathbf{v}}\}$. For a given $\mathbf{v} \in V_X$, note that $|\psi(\mathbf{m}) - 1| < \eta$ for any $\mathbf{m} \in U_{\mathbf{v}}$. Therefore, the collection $\{U_{\mathbf{v}}\}_{\mathbf{v} \in V_X}$ is an open cover of V_X. Thus, the set $\mathbb{R}^s \setminus \left(\bigcup_{\mathbf{v} \in V_X} U_{\mathbf{v}}\right)$ is closed. Because V_X is compact and $\mathbb{R}^s \setminus \left(\bigcup_{\mathbf{v} \in V_X} U_{\mathbf{v}}\right)$ is closed, there exists $\delta > 0$ such $\operatorname{dist}(\mathbf{m}, V_X) \geq \delta$ for every $\mathbf{m} \in \mathbb{R}^s \setminus \left(\bigcup_{\mathbf{v} \in V_X} U_{\mathbf{v}}\right)$. Therefore, if $\operatorname{dist}(\mathbf{m}, V_X) < \delta$, then $\mathbf{m} \in \bigcup_{\mathbf{v} \in V_X} U_{\mathbf{v}}$ and $|\psi(\mathbf{m}) - 1| < \eta$.)

Observe that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\frac{n}{X^2} - \varepsilon} \sqrt{\frac{\frac{n}{X^2} + \varepsilon}{2\lambda_1}} = 0,$$

so there exist infinitely many $\varepsilon > 0$ such that

$$\frac{\varepsilon}{\frac{n}{X^2} - \varepsilon} \sqrt{\frac{\frac{n}{X^2} + \varepsilon}{2\lambda_1}} \leq \delta. \tag{6.110}$$

Choose $\varepsilon > 0$ so that (6.110) holds and $\varepsilon < n/X^2$. Then Lemma 6.26 implies that $\operatorname{dist}(\mathbf{m}, V_X) < \delta$ for all $\mathbf{m} \in \mathbb{R}^s$ satisfying $\left|F(\mathbf{m}) - \frac{n}{X^2}\right| < \varepsilon$. Therefore, for all $\mathbf{m} \in \mathbb{R}^s$

satisfying $\left|F(\mathbf{m}) - \frac{n}{X^2}\right| < \varepsilon$, we have $|\psi(\mathbf{m}) - 1| < \eta$. Thus,

$$\frac{1}{2\varepsilon}\int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon}(1-\eta)\ d\mathbf{m} \leq \frac{1}{2\varepsilon}\nu_{F,\psi,n,X}(\varepsilon) \leq \frac{1}{2\varepsilon}\int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon}(1+\eta)\ d\mathbf{m}, \quad (6.111)$$

where

$$\nu_{F,\psi,n,X}(\varepsilon) = \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon}\psi(\mathbf{m})\ d\mathbf{m}.$$

We have shown that for sufficiently small $\varepsilon > 0$ that if $\mathbf{m} \in \mathbb{R}^s$ satisfies $|F(\mathbf{m}) - n| < \varepsilon$, then (6.111) holds. Taking limits in (6.111) as $\varepsilon \to 0^+$, we obtain

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon}(1-\eta)\ d\mathbf{m} \leq \sigma_{F,\psi,\infty}(n,X) \leq \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon}(1+\eta)\ d\mathbf{m} \quad (6.112)$$

since

$$\sigma_{F,\psi,\infty}(n,X) = \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\nu_{F,\psi,n,X}(\varepsilon).$$

We apply Theorem 6.22 to (6.112) to obtain

$$\frac{(2\pi)^{s/2}(1-\eta)}{\Gamma(s/2)\sqrt{\det(A)}}n^{s/2-1}X^{2-s} \leq \sigma_{F,\psi,\infty}(n,X) \leq \frac{(2\pi)^{s/2}(1+\eta)}{\Gamma(s/2)\sqrt{\det(A)}}n^{s/2-1}X^{2-s}. \quad (6.113)$$

Because $\eta > 0$ was arbitrary, we obtain (6.109) from (6.113). □

The following corollary gives an equality for the singular integral $J_{F,\psi}(n,X)$ if $\psi(\mathbf{m}) = 1$ whenever $F(\mathbf{m}) = n/X^2$. It follows immediately from (6.66), Theorem 6.20, and Theorem 6.27.

Corollary 6.28. *Suppose that F is a positive definite quadratic form in s variables. Let A be the Hessian matrix of F. Suppose that n and X are real positive numbers.*

Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$. *Suppose that* $\psi(\mathbf{m}) = 1$ *whenever* $F(\mathbf{m}) = n/X^2$. *Then*

$$J_{F,\psi}(n, X) = \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1}.$$

6.2.3 Extending to the singular series

From (6.63), we know that

$$\begin{aligned} M_{F,\psi,X}(n) &= \mathfrak{S}_F(n; Q) J_{F,\psi}(n, X) \\ &\quad + O_{\psi,s,\varepsilon}\left(L^{s/2} (\det(A))^{-1/2} Q^{(s-1)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right), \end{aligned} \tag{6.114}$$

where $\mathfrak{S}_F(n; Q)$ is the *truncated singular series*

$$\mathfrak{S}_F(n; Q) = \sum_{1 \le q \le Q} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q} \left(F(\mathbf{h}) - n \right) \right). \tag{6.115}$$

In the following lemma, we extend (up to some acceptable error term) the truncated singular series to the singular series $\mathfrak{S}_F(n)$ as defined in (1.4).

Lemma 6.29. *For* $Q \ge 1$, *the truncated singular series* $\mathfrak{S}_F(n; Q)$ *is*

$$\mathfrak{S}_F(n; Q) = \mathfrak{S}_F(n) + O_{s,\varepsilon}\left(L^{s/2} Q^{(3-s)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right) \tag{6.116}$$

for any $\varepsilon > 0$.

Proof. We apply Lemma 6.18 to find that

$$\begin{aligned}
\mathfrak{S}_F(n) - \mathfrak{S}_F(n;Q) &= \sum_{q>Q} \frac{1}{q^s} \sum_{d\in(\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h})-n\right)\right) \\
&\ll \sum_{q>Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2} q_0^{1/2} q^{(1-s)/2}\tau(q)\log(2q) \\
&= L^{s/2}\sum_{q>Q}(\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2}\tau(q)\log(2q). \qquad (6.117)
\end{aligned}$$

We now dyadically decompose the sum to obtain

$$\begin{aligned}
&\sum_{q>Q}(\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2}\tau(q)\log(2q) \\
&= \sum_{k=0}^{\infty} \sum_{2^kQ<q\le 2^{k+1}Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2}\tau(q)\log(2q).
\end{aligned}$$

Because $s \geq 4$, we find that

$$\begin{aligned}
&\sum_{q>Q}(\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2}\tau(q)\log(2q) \\
&\le \sum_{k=0}^{\infty} (2^kQ)^{1-s/2} \sum_{2^kQ<q\le 2^{k+1}Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2}\tau(q)\log(2q) \\
&\le \sum_{k=0}^{\infty} (2^kQ)^{1-s/2} \sum_{1\le q\le 2^{k+1}Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2}\tau(q)\log(2q). \qquad (6.118)
\end{aligned}$$

We apply Lemma 6.1 to (6.118) to obtain

$$\begin{aligned}
&\sum_{q>Q}(\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2}\tau(q)\log(2q) \\
&\ll_\varepsilon \sum_{k=0}^{\infty} (2^kQ)^{1-s/2}(2^{k+1}Q)^{1/2+\varepsilon}\tau(n) \prod_{p\mid 2\det(A)} (1-p^{-1/2})^{-1} \\
&= 2^{1/2+\varepsilon} Q^{(3-s)/2+\varepsilon}\tau(n)\left(\prod_{p\mid 2\det(A)} (1-p^{-1/2})^{-1}\right)\sum_{k=0}^{\infty}\left(2^{(3-s)/2+\varepsilon}\right)^k \qquad (6.119)
\end{aligned}$$

for any $\varepsilon > 0$. The series $\sum_{k=0}^{\infty} \left(2^{(3-s)/2+\varepsilon}\right)^k$ is a geometric series and converges if $\varepsilon < (s-3)/2$. If $\varepsilon < (s-3)/2$, then

$$\sum_{k=0}^{\infty} \left(2^{(3-s)/2+\varepsilon}\right)^k = \left(1 - 2^{(3-s)/2+\varepsilon}\right)^{-1}. \tag{6.120}$$

Substituting (6.120) into (6.119), we obtain

$$\begin{aligned}
&\sum_{q>Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2} q^{1-s/2} \tau(q) \log(2q) \\
&\ll_\varepsilon 2^{1/2+\varepsilon} Q^{(3-s)/2+\varepsilon} \tau(n) \left(1 - 2^{(3-s)/2+\varepsilon}\right)^{-1} \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \\
&\ll_{s,\varepsilon} Q^{(3-s)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1}
\end{aligned} \tag{6.121}$$

for any $\varepsilon \in \mathbb{R}$ satisfying $0 < \varepsilon < (s-3)/2$.

Applying (6.121) to (6.117), we conclude that (6.116) holds for any $\varepsilon \in \mathbb{R}$ satisfying $0 < \varepsilon < (s-3)/2$. Because $Q \geq 1$, we notice that (6.116) is true for all $\varepsilon > 0$, so we only require that $\varepsilon > 0$. $\square$

Applying Lemma 6.29 to (6.114), we find that

$$\begin{aligned}
M_{F,\psi,X}(n) = \mathfrak{S}_F(n;Q) J_{F,\psi}(n,X) \\
&+ O_\varepsilon\left(|J_{F,\psi}(n,X)| L^{s/2} Q^{(3-s)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\
&+ O_{\psi,s,\varepsilon}\left(L^{s/2} (\det(A))^{-1/2} Q^{(s-1)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right)
\end{aligned}$$

for any $\varepsilon > 0$. We apply Corollary 6.25 to obtain

$$\begin{aligned} M_{F,\psi,X}(n) = {} & \mathfrak{S}_F(n) J_{F,\psi}(n,X) \\ & + O_{\psi,s,\varepsilon}\left(L^{s/2}(\det(A))^{-1/2} n^{s/2-1} Q^{(3-s)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\ & + O_{\psi,s,\varepsilon}\left(L^{s/2}(\det(A))^{-1/2} Q^{(s-1)/2+\varepsilon} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \end{aligned} \tag{6.122}$$

for any $\varepsilon > 0$.

We will not further analyze the singular series in this dissertation. We do mention that the singular series is a product of p-adic densities, so the singular series is said to contain local information for the representation number. For more information about the singular series, please see Section 11.5 in [Iwa97].

6.3 Analyzing the error term $E_{F,\psi,X,1}(n)$

We now analyze the error term $E_{F,\psi,X,1}(n)$. We apply Lemma 4.22 and Theorem 5.11 to (6.3) to obtain

$$\begin{aligned} E_{F,\psi,X,1}(n) \ll_\psi & \sum_{1 \le q \le Q} (\gcd(L,q_0))^{s/2} (\gcd(n,q_1))^{1/2} q_0^{1/2} q^{(1-s)/2} \tau(q) \log(2q) \\ & \times \int_0^{\frac{1}{q(q+Q)}} \min\left\{X^s, |x|^{-s/2}(\det(A))^{-1/2}\right\} \sum_{\substack{\mathbf{r} \in \mathbb{Z}^s \\ 0 < \|\mathbf{r}\| \le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} 1 \, dx. \end{aligned} \tag{6.123}$$

Using Corollary 6.10, we obtain

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^s\\ 0<\|\mathbf{r}\|\le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} 1 = \left|\{\mathbf{r}\in\mathbb{Z}^s : 0<\|\mathbf{r}\|\le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}\}\right|$$
$$\ll_s (qX|x|\lambda_s(\rho_\psi+1)\sqrt{s})^s.$$

Using this in (6.123), we find that

$$\begin{aligned}
E_{F,\psi,X,1}(n) &\ll_{\psi,s} (X\lambda_s(\rho_\psi+1)\sqrt{s})^s \sum_{1\le q\le Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2}q_0^{1/2}q^{(s+1)/2}\\
&\qquad \times \tau(q)\log(2q)\int_0^{\frac{1}{q(q+Q)}} \min\left\{X^s|x|^s, |x|^{s/2}(\det(A))^{-1/2}\right\}\, dx\\
&\le (X\lambda_s(\rho_\psi+1)\sqrt{s})^s L^{s/2} \sum_{1\le q\le Q} (\gcd(n,q_1))^{1/2}q_0^{1/2}q^{(s+1)/2}\tau(q)\log(2q)\\
&\qquad \times (\det(A))^{-1/2}\int_0^{\frac{1}{q(q+Q)}} |x|^{s/2}\, dx\\
&= (X\lambda_s(\rho_\psi+1)\sqrt{s})^s L^{s/2} \sum_{1\le q\le Q} (\gcd(n,q_1))^{1/2}q_0^{1/2}q^{(s+1)/2}\tau(q)\log(2q)\\
&\qquad \times (\det(A))^{-1/2}\frac{1}{\frac{s}{2}+1}(q(q+Q))^{-s/2-1}\\
&= (X\lambda_s(\rho_\psi+1)\sqrt{s})^s L^{s/2} \sum_{1\le q\le Q} (\gcd(n,q_1))^{1/2}q_0^{1/2}q^{-1/2}\tau(q)\log(2q)\\
&\qquad \times (\det(A))^{-1/2}\frac{1}{\frac{s}{2}+1}(q+Q)^{-s/2-1}\\
&= (X\lambda_s(\rho_\psi+1)\sqrt{s})^s L^{s/2} \sum_{1\le q\le Q} (\gcd(n,q_1))^{1/2}q_1^{-1/2}\tau(q)\log(2q)\\
&\qquad \times (\det(A))^{-1/2}\frac{1}{\frac{s}{2}+1}(q+Q)^{-s/2-1}.
\end{aligned} \tag{6.124}$$

Since $q>0$, we know that $(q+Q)^{-s/2-1} < Q^{-s/2-1}$. Therefore,

$$\begin{aligned}
E_{F,\psi,X,1}(n) &\ll_{\psi,s} (X\lambda_s(\rho_\psi+1))^s L^{s/2}(\det(A))^{-1/2}Q^{-s/2-1}\\
&\qquad \times \sum_{1\le q\le Q} (\gcd(n,q_1))^{1/2}q_1^{-1/2}\tau(q)\log(2q).
\end{aligned} \tag{6.125}$$

By applying Lemma 6.1 to (6.125), we conclude that

$$E_{F,\psi,X,1}(n) \ll_{\psi,s,\varepsilon} X^s \lambda_s^s L^{s/2} (\det(A))^{-1/2} Q^{-s/2-1/2+\varepsilon} \tau(n) \prod_{p | 2\det(A)} (1-p^{-1/2})^{-1} \tag{6.126}$$

for any $\varepsilon > 0$.

6.4 Analyzing the error term $E_{F,\psi,X,2}(n)$

We now analyze the error term $E_{F,\psi,X,2}(n)$. We apply Lemma 4.22 and Theorem 5.11 to (6.4) to obtain

$$\begin{aligned} E_{F,\psi,X,2}(n) \ll_\psi & \sum_{1\le q\le Q} (\gcd(L,q_0))^{s/2} (\gcd(n,q_1))^{1/2} q_0^{1/2} q^{(1-s)/2} \tau(q) \log(2q) \\ & \times \int_{\frac{1}{q(q+Q)}}^{\frac{1}{qQ}} |x|^{-s/2} (\det(A))^{-1/2} \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\| \le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} 1 \; dx. \end{aligned} \tag{6.127}$$

By Theorem 6.9, we know that

$$\begin{aligned} \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\| \le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} 1 \; dx &= \left| \{ \mathbf{r} \in \mathbb{Z}^s : \|\mathbf{r}\| \le qX|x|\lambda_s(\rho_\psi+1)\sqrt{s} \} \right| \\ &\ll_s (qX|x|\lambda_s(\rho_\psi+1)\sqrt{s})^s + 1. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{\frac{1}{q(q+Q)}}^{\frac{1}{qQ}} |x|^{-s/2} \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\|\leq qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} 1 \ dx \\
&\ll_s \int_{\frac{1}{q(q+Q)}}^{\frac{1}{qQ}} \left((qX\lambda_s(\rho_\psi+1)\sqrt{s})^s|x|^{s/2} + |x|^{-s/2}\right) \ dx \\
&= (qX\lambda_s(\rho_\psi+1)\sqrt{s})^s \frac{1}{\frac{s}{2}+1}\left((qQ)^{-s/2-1} - (q(q+Q))^{-s/2-1}\right) \\
&\qquad + \frac{1}{1-\frac{s}{2}}\left((qQ)^{s/2-1} - (q(q+Q))^{s/2-1}\right) \\
&\ll_s q^{s/2-1}\left((X\lambda_s(\rho_\psi+1))^s Q^{-s/2-1} + Q^{s/2-1}\right)
\end{aligned}
\tag{6.128}
$$

since $q \leq Q$.

Substituting (6.128) into (6.127), we find that

$$
\begin{aligned}
E_{F,\psi,X,2}(n) &\ll_{\psi,s} \sum_{1\leq q\leq Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2} q_0^{1/2} q^{-1/2}\tau(q)\log(2q) \\
&\qquad\qquad \times (\det(A))^{-1/2}\left((X\lambda_s(\rho_\psi+1))^s Q^{-s/2-1} + Q^{s/2-1}\right) \\
&\leq L^{s/2}(\det(A))^{-1/2}\left(X^s\lambda_s^s(\rho_\psi+1)^s Q^{-s/2-1} + Q^{s/2-1}\right) \\
&\qquad \times \sum_{1\leq q\leq Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2}\tau(q)\log(2q).
\end{aligned}
\tag{6.129}
$$

By applying Lemma 6.1 to (6.129), we conclude that By applying Lemma 6.1 to (6.129), we conclude that

$$
\begin{aligned}
E_{F,\psi,X,2}(n) &\ll_{\psi,s,\varepsilon} L^{s/2}(\det(A))^{-1/2}\left(X^s\lambda_s^s Q^{-s/2-1/2+\varepsilon} + Q^{(s-1)/2+\varepsilon}\right) \\
&\qquad \times \tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}.
\end{aligned}
\tag{6.130}
$$

for any $\varepsilon > 0$.

6.5 Analyzing the error term $E_{F,\psi,X,3}(n)$

We now analyze the error term $E_{F,\psi,X,3}(n)$. Applying Lemma 4.22 and Corollary 5.24 to (6.5), we obtain for any $M \geq 0$ that

$$E_{F,\psi,X,3}(n) \ll_{M,\psi} \sum_{1\leq q\leq Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2} q_0^{1/2} q^{(1-s)/2}\tau(q)\log(2q) \\ \times \int_0^{\frac{1}{qQ}} \sum_{\substack{\mathbf{r}\in\mathbb{Z}^s \\ \|\mathbf{r}\|>qX|x|\lambda_s(\rho_\psi+1)\sqrt{s}}} X^{s-M}\left(\frac{1}{q(\rho_\psi+1)\sqrt{s}}\|\mathbf{r}\|\right)^{-M} dx. \tag{6.131}$$

In order to apply Corollary 6.13, we suppose that $M \geq s+1$. Applying Corollary 6.13 to (6.131), we obtain

$$\begin{aligned} E_{F,\psi,X,3}(n) &\ll_{M,\psi,s} \sum_{1\leq q\leq Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2} q_0^{1/2} q^{(1-s)/2}\tau(q)\log(2q) \\ &\qquad \times X^{s-M}(q(\rho_\psi+1)\sqrt{s})^M \int_0^{\frac{1}{qQ}} 1\ dx. \\ &= X^{s-M}(\rho_\psi+1)^M s^{M/2} Q^{-1} \\ &\qquad \times \sum_{1\leq q\leq Q} (\gcd(L,q_0))^{s/2}(\gcd(n,q_1))^{1/2} q_0^{1/2} q^{M-1/2-s/2}\tau(q)\log(2q) \\ &\leq X^{s-M}(\rho_\psi+1)^M s^{M/2} Q^{M-s/2-1} L^{s/2} \\ &\qquad \times \sum_{1\leq q\leq Q} (\gcd(n,q_1))^{1/2} q_1^{-1/2}\tau(q)\log(2q). \end{aligned} \tag{6.132}$$

We apply Lemma 6.1 to obtain

$$E_{F,\psi,X,3}(n) \ll_{M,\psi,s,\varepsilon} X^{s-M} Q^{M-s/2-1/2+\varepsilon} L^{s/2}\tau(n) \prod_{p|2\det(A)} (1-p^{-1/2})^{-1}. \tag{6.133}$$

for any $\varepsilon > 0$.

6.6 Choosing parameters

By putting together estimates from previous sections and by choosing some parameters in this section, we prove Theorem 1.1 and Corollary 1.4. We substitute (6.122), (6.126), (6.130), and (6.133) into (6.1) to obtain

$$\begin{aligned}
R_{F,\psi,X}(n) &= \mathfrak{S}_F(n)J_{F,\psi}(n,X) \\
&\quad + O_{\psi,s,\varepsilon}\left(L^{s/2}(\det(A))^{-1/2}n^{s/2-1}Q^{(3-s)/2+\varepsilon}\tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\right) \\
&\quad + O_{\psi,s,\varepsilon}\left(L^{s/2}(\det(A))^{-1/2}Q^{(s-1)/2+\varepsilon}\tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\right) \\
&\quad + O_{\psi,s,\varepsilon}\left(X^s\lambda_s^sL^{s/2}(\det(A))^{-1/2}Q^{-s/2-1/2+\varepsilon}\tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\right) \\
&\quad + O_{\psi,s,\varepsilon}\Bigg(L^{s/2}(\det(A))^{-1/2}\left(X^s\lambda_s^sQ^{-s/2-1/2+\varepsilon}+Q^{(s-1)/2+\varepsilon}\right) \\
&\qquad\qquad \times\tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\Bigg) \\
&\quad + O_{M,\psi,s,\varepsilon}\left(X^{s-M}Q^{M-s/2-1/2+\varepsilon}L^{s/2}\tau(n)\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\right) \qquad (6.134)
\end{aligned}$$

for any $\varepsilon > 0$.

We would like to choose Q and X that somewhat balance all of the error terms in (6.134). To do this, we find a value of Q that practically minimizes the right-hand side of (6.130), which is an upper bound for the absolute value of $E_{F,\psi,X,2}(n)$. By solving

$$X^s\lambda_s^sQ^{-s/2-1/2+\varepsilon} = Q^{(s-1)/2+\varepsilon}$$

for Q, we find that

$$Q = \lambda_s X$$

will give a bound within a factor of 2 of the optimal bound for the right-hand side of (6.130). (See the commentary after Theorem 2.3 in [MV06] for an explanation as to why this choice of Q will give a bound within a factor of 2 of the optimal bound for the right-hand side of (6.130). We also note that $\lambda_s X$ is not necessarily an integer, which is why we purposefully allowed Q to be not an integer in Section 3.1.)

In order to set $Q = \lambda_s X$, we need to place an additional restriction on X. Recall that $Q \geq 1$. Thus, to guarantee that $Q = \lambda_s X \geq 1$, we require that $X \geq 1/\lambda_s$.

By setting $Q = \lambda_s X$ in (6.134), we obtain

$$\begin{aligned}
&R_{F,\psi,X}(n) \\
&= \mathfrak{S}_F(n) J_{F,\psi}(n, X) \\
&\quad + O_{\psi,s,\varepsilon}\left(n^{s/2-1} X^{(3-s)/2+\varepsilon} \lambda_s^{(3-s)/2+\varepsilon} L^{s/2} (\det(A))^{-1/2} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right) \\
&\quad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon} \lambda_s^{(s-1)/2+\varepsilon} L^{s/2} (\det(A))^{-1/2} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right) \\
&\quad + O_{M,\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon} \lambda_s^{M-s/2-1/2+\varepsilon} L^{s/2} \tau(n) \prod_{p \mid 2\det(A)} (1 - p^{-1/2})^{-1} \right) \qquad (6.135)
\end{aligned}$$

for any $\varepsilon > 0$.

We notice that the only place that M occurs in (6.135) is as an exponent for λ_s. Now because F is a positive definite integral quadratic form, the determinant of A is at least one, so at least one of the eigenvalues of A must be at least one. Since λ_s is the largest eigenvalue of A, we must have $\lambda_s \geq 1$. Therefore, we want to set M to be as small as possible. The only condition that we have on M is that $M \geq s + 1$, so we choose to set $M = s+1$. With this choice of M, we find that (6.135) becomes

$$\begin{aligned}
&R_{F,\psi,X}(n) \\
&= \mathfrak{S}_F(n) J_{F,\psi}(n,X) \\
&\qquad + O_{\psi,s,\varepsilon}\left(n^{s/2-1} X^{(3-s)/2+\varepsilon} \lambda_s^{(3-s)/2+\varepsilon} L^{s/2} (\det(A))^{-1/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\
&\qquad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon} \lambda_s^{(s-1)/2+\varepsilon} L^{s/2} (\det(A))^{-1/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\
&\qquad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon} \lambda_s^{(s+1)/2+\varepsilon} L^{s/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \qquad (6.136)
\end{aligned}$$

for any $\varepsilon > 0$. (Because M now is completely determined by s, we have $O_{M,\psi,s,\varepsilon}(f) = O_{\psi,s,\varepsilon}(f)$ for any function f.)

Recall that $\det(A) \geq 1$ and $\lambda_s \geq 1$, so $(\det(A))^{-1/2} \leq \lambda_s$. Therefore,

$$\lambda_s^{(s-1)/2+\varepsilon} (\det(A))^{-1/2} \leq \lambda_s^{(s+1)/2+\varepsilon}$$

for any $\varepsilon > 0$. Thus, we conclude that

$$\begin{aligned}
&R_{F,\psi,X}(n) \\
&= \mathfrak{S}_F(n) J_{F,\psi}(n,X) \\
&\qquad + O_{\psi,s,\varepsilon}\left(n^{s/2-1} X^{(3-s)/2+\varepsilon} \lambda_s^{(3-s)/2+\varepsilon} (\det(A))^{-1/2} L^{s/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\
&\qquad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon} \lambda_s^{(s+1)/2+\varepsilon} L^{s/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \qquad (6.137)
\end{aligned}$$

for any $\varepsilon > 0$. Because of (6.66) and Theorem 6.20, we deduce Theorem 1.1 from (6.137).

We now want to choose X to minimize the error terms in (6.137). By setting the error terms equal to each other and canceling like expressions, we obtain

$$n^{s/2-1}X^{(3-s)/2+\varepsilon}\lambda_s^{(3-s)/2+\varepsilon}(\det(A))^{-1/2} = X^{(s-1)/2+\varepsilon}\lambda_s^{(s+1)/2+\varepsilon}. \tag{6.138}$$

Solving (6.138) for X, we find that

$$X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}$$

will give a bound within a factor of 2 of the optimal bound for the error terms in (6.137). In order to set $X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}$, we need to place an additional restriction on n. Recall that $X \geq 1/\lambda_s$. Thus, to guarantee that $X \geq 1/\lambda_s$, we require that $n \geq \lambda_s^{2/(s-2)}(\det(A))^{1/(s-2)}$.

By setting $X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}$ in (6.137), we deduce that

$$\begin{aligned}
&R_{F,\psi,n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}}(n)\\
&= \mathfrak{S}_F(n)J_{F,\psi}(n, n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)})\\
&\quad + O_{\psi,s,\varepsilon}\Bigg(n^{(s-1)/4+\varepsilon/2}\lambda_s^{(s-3)/(2s-4)-\varepsilon/(s-2)}(\det(A))^{(1-s)/(4s-8)-\varepsilon/(2s-4)}\\
&\qquad\qquad \times L^{s/2}\tau(n)\prod_{p\mid 2\det(A)}(1-p^{-1/2})^{-1}\Bigg)
\end{aligned} \tag{6.139}$$

for any $\varepsilon > 0$. Because $\tau(n) \ll_\varepsilon n^{\varepsilon/2}$ for any $\varepsilon > 0$, we conclude Corollary 1.4 from (6.139), (6.66), and Theorem 6.20.

6.7 Proof of Corollary 1.5

In this section, we complete a proof of Corollary 1.5. To prove Corollary 1.5, we choose a bump function ψ that satisfies certain conditions. Notice that

$$|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}| = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}}.$$

Because

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}} \psi_X(\mathbf{m}),$$

we find that

$$|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}| = R_{F,\psi,X}(n) \tag{6.140}$$

if $\psi_X(\mathbf{m}) = 1$ for each $\mathbf{m} \in \mathbb{Z}^s$ satisfying $F(\mathbf{m}) = n$. Because scaling by $X > 0$ does not necessarily preserve integrality of solutions to $F(\mathbf{m}) = n$, we require that $\psi_X(\mathbf{x}) = 1$ whenever $\mathbf{x} \in \mathbb{R}^s$ satisfies $F(\mathbf{x}) = n$. Using the invertible mapping $\mathbf{x} \mapsto X\mathbf{m}$ and the fact that

$$\psi_X(\mathbf{m}) = \psi\left(\frac{1}{X}\mathbf{m}\right),$$

we deduce that $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = n/X^2$ if and only if $\psi_X(\mathbf{x}) = 1$ whenever $\mathbf{x} \in \mathbb{R}^s$ satisfies $F(\mathbf{x}) = n$. (To prove this, use the fact that F is a quadratic form. Thus, if $\mathbf{x} = X\mathbf{m}$, then $F(\mathbf{x}) = X^2 F(\mathbf{m})$.)

To remove any dependence of ψ on n or X, we set

$$X = n^{1/2} \tag{6.141}$$

so that the only condition on $\psi \in C_c^\infty(\mathbb{R}^s)$ is that $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = 1$. We note that it is possible for ψ to satisfy this condition because of Theorem 6.17.

Observe that ψ now depends on F, so we now consider any implied constants dependent on ψ to now be dependent on ψ and F. Therefore, by applying (6.141) to (6.137), we obtain

$$R_{F,\psi,X}(n) = \mathfrak{S}_F(n) J_{F,\psi}(n, X) + O_{F,\psi,s,\varepsilon}\left(n^{(s-1)/4+\varepsilon/2}\right) \tag{6.142}$$

for any $\varepsilon > 0$. Because $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = n/X^2$, we deduce that Corollary 6.28, (6.140), and (6.142) imply that

$$|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}| = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} + O_{F,\psi,s,\varepsilon}\left(n^{(s-1)/4+\varepsilon/2}\right) \tag{6.143}$$

for any $\varepsilon > 0$. Because $|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}|$ is a number that does not depend on ψ, we can remove the dependency on ψ for the implied constant in (6.143) and obtain

$$|\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}| = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} + O_{F,s,\varepsilon}\left(n^{(s-1)/4+\varepsilon/2}\right) \tag{6.144}$$

for any $\varepsilon > 0$. By replacing $\varepsilon/2$ with ε in (6.144), we conclude Corollary 1.5.

Remark 6.30. In the proof of Corollary 1.5, we could have set X to be $cn^{1/2}$ for any fixed $c > 0$. (We decided to choose c to be 1 in (6.141).) If $X = cn^{1/2}$, then we would require that $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = 1/c^2$.

If we had set c to be $\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}$, then Corollary 1.5 would follow from Corollary 1.4.

Chapter 7

A strong asymptotic local-global principle for certain Kleinian sphere packings

In this dissertation, we have developed a version of the Kloosterman circle method with a bump function. A potential application of this version of the Kloosterman circle method is a proof of a strong asymptotic local-global principle for certain integral Kleinian sphere packings.

To state a conjectured strong asymptotic local-global principle for certain integral Kleinian sphere packings, we first need to state some definitions. The *bend* of a $(d-1)$-sphere is the reciprocal of the radius of the $(d-1)$-sphere. An $(d-1)$-sphere packing is called *integral* if the bend of each $(d-1)$-sphere in the packing is an integer. An integral $(d-1)$-sphere packing is called *primitive* if the greatest common divisor of all of the bends in the packing is 1. An $(d-1)$-sphere packing is *Kleinian* if its limit set is that of a geometrically finite group Γ of isometries of $(d+1)$-dimensional hyperbolic space. Kontorovich and Nakamura [KN19] and Kontorovich and Kapovich [KK21] proved that there are infinitely many conformally inequivalent integral Kleinian sphere packings.

There may be local or congruence restrictions on the bends in a fixed Kleinian sphere packing. This motivates the following definition of admissibility.

Definition 7.1 (Admissible integers for sphere packings)**.** Let $\mathcal{P}$ be an integral Kleinian sphere packing. An integer m is *admissible* (or *locally represented*) if, for every $q \geq 1$, we have

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Before we state a conjectured strong asymptotic local-global principle for certain Kleinian sphere packings, we briefly discuss orientation-preserving isometries of $(d+1)$-dimensional hyperbolic space. The group of orientation-preserving isometries of $(d+1)$-dimensional hyperbolic space can be identified with the group of orientation-preserving Möbius transformations acting on $\mathbb{R}^d \cup \{\infty\}$. These groups can be identified with a certain group of 2×2 matrices with entries in a Clifford algebra. (See [Vah02], [Ahl85], or [Wat93] for how this can be done.) This matrix group contains $\mathrm{PSL}_2(\mathbb{C})$ if $d \geq 2$.

The following is a conjectured strong asymptotic local-global principle for certain Kleinian sphere packings.

Conjecture 7.2. *Let $\mathcal{P}$ be a primitive integral Kleinian $(d-1)$-sphere packing in $\mathbb{R}^d \cup \{\infty\}$ with an orientation-preserving automorphism group Γ of Möbius transformations.*

1. *Suppose that there exists a $(d-1)$-sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field and $\mathcal{O}_K$ is the ring of integers of K. This condition implies that $d \geq 3$.*

2. *Suppose that there is a $(d-1)$-sphere $S_1 \in \mathcal{P}$ that is tangent to S_0.*

Then every sufficiently large admissible integer is a bend of a $(d-1)$*-sphere in* $\mathcal{P}$*. That is, there exists an* $N_0 = N_0(\mathcal{P})$ *such that if* m *is admissible and* $m > N_0$*, then* m *is the bend of a* $(d-1)$*-sphere in* $\mathcal{P}$*.*

There has been work towards Conjecture 7.2. Examples of integral Kleinian sphere packings are integral Soddy sphere packings and integral orthoplicial Apollonian sphere packings. Kontorovich [Kon19] proved the strong asymptotic local-global principle for integral Soddy sphere packings. Dias [Dia14] and Nakamura [Nak14] independently did work towards proving the strong asymptotic local-global principle for integral orthoplicial Apollonian sphere packings.

When this was written, the author did not know of a proof of a strong asymptotic local-global principle that applied to multiple conformally inequivalent integral Kleinian sphere packings. The author is making progress towards proving Conjecture 7.2, which would apply to multiple conformally inequivalent integral Kleinian sphere packings. By using Möbius transformations on $\mathbb{R}^d \cup \{\infty\}$ and inversive coordinates of $(d-1)$-spheres, one can obtain a family of integral quadratic polynomials in four variables with a coprimality condition on the variables. Potentially, the version of the Kloosterman circle method discussed in this dissertation could be then used to prove a result towards Conjecture 7.2 that applies to multiple conformally inequivalent integral Kleinian sphere packings.

Bibliography

[Ahl85] L. V. Ahlfors, *Möbius transformations and Clifford numbers*, Differential Geometry and Complex Analysis (I. Chavel and H. M. Farkas, eds.), Springer-Verlag, 1985, pp. 65 – 73.

[Apo76] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, 1976.

[BEW98] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi sums*, John Wiley & Sons, 1998.

[BW17] J. Bourgain and N. Watt, *Mean square of zeta function, circle problem and divisor problem revisited*, preprint, arXiv:1709.04340, 2017.

[CS99] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices, and groups*, 3rd ed., Grundlehren der mathematischen Wissenschaften, vol. 290, Springer, 1999.

[Dia14] D. Dias, *The local-global principle for integral generalized Apollonian sphere packings*, preprint, arXiv:1401.4789, 2014.

[Die03] R. Dietmann, *Small solutions of quadratic Diophantine equations*, Proc. London Math. Soc. (3) **86** (2003), no. 3, 545–582.

[Fol99] G. B. Folland, *Real analysis: Modern techniques and their applications*, 2nd ed., Wiley-Interscience Publication, 1999.

[Fri82] F. Fricker, *Einführung in die gitterpunktlehre*, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, vol. 73, Birkhäuser Basel, 1982.

[Gau11] C. F. Gauss, *Werke*, vol. 2, ch. DE NEXU INTER MULTITUDINEM CLASSIUM, IN QUAS FORMAE BINARIAE SECUNDI GRADUS DISTRIBUUNTUR, EARUMQUE DETERMINANTEM, pp. 269–291, Cambridge University Press, 2011.

[Gre13] B. Green, *Restriction and Kakeya phenomena*, `http://people.maths.ox.ac.uk/greenbj/papers/rkp.pdf`, 2013.

[HB96] D. R. Heath-Brown, *A new form of the circle method, and its application to quadratic forms*, J. Reine Angew. Math. **481** (1996), 149–206.

[HB99] ———, *Lattice points in the sphere*, Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 883–892.

[Hux02] M. N. Huxley, *Integer points, exponential sums and the Riemann zeta function*, Number theory for the millennium, II (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 275–290.

[HW08] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 6th ed., Oxford University Press, 2008.

[IK04] H. Iwaniec and E. Kowalski, *Analytic number theory*, Colloquium Publications, vol. 53, American Mathematical Society, 2004.

[Iwa97] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, 1997.

[Kam08] D. W. Kammler, *A first course in fourier analysis*, 2nd ed., Cambridge University Press, 2008.

[KK21] M. Kapovich and A. Kontorovich, *On superintegral Kleinian sphere packings, bugs, and arithmetic groups*, Preprint, `https://arxiv.org/abs/2104.13838`, 2021.

[Klo24] H. D. Kloosterman, *Over het splitsen van geheele positieve getallen in een som van kwadraten*, Ph.D. thesis, Universiteit Leiden, 1924.

[Klo26] ———, *On the representation of numbers in the form* $ax^2+by^2+cz^2+dt^2$, Acta Mathematica **49** (1926), no. 3–4, 407–464.

[KN19] A. Kontorovich and K. Nakamura, *Geometry and arithmetic of crystallographic sphere packings*, Proc. Natl. Acad. Sci. USA **116** (2019), no. 2, 436–441, doi:10.1073/pnas.1721104116.

[Kon19] A. Kontorovich, *The local-global principle for integral Soddy sphere packings*, J. Mod. Dyn. **15** (2019), 209–236, doi:10.3934/jmd.2019019.

[Kor89] D. M. Kornhauser, *Bounds for the smallest integer solution of general quadratic equations*, Ph.D. thesis, University of Michigan, 1989.

[Kor90] ———, *On small solutions of the general nonsingular quadratic Diophantine equation in five or more unknowns*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 197–211.

[MV06] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory I: Classical theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2006.

[Nak14] K. Nakamura, *The local-global principle for integral bends in orthoplicial Apollonian sphere packings*, preprint, arXiv:1401.2980, 2014.

[SS03] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. II, Princeton University Press, 2003.

[Ste93] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 63, Princeton University Press, 1993.

[Tao] T. Tao, *Lecture notes 8 for 247b*, `https://www.math.ucla.edu/~tao/247b.1.07w/notes8.pdf`.

[Vah02] K. T. Vahlen, *Ueber bewegungen und complexe zahlen*, Math. Ann. **55** (1902), 585–593.

[Vau97] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd ed., Cambridge Tracts in Mathematics, Cambridge University Press, 1997.

[Wat60] G. L. Watson, *Integral quadratic forms*, Cambridge University Press, 1960.

[Wat93] P. L. Waterman, *Möbius transformations in several dimensions*, Adv. Math. **101** (1993), no. 1, 87 –113.

[Wol03] T. H. Wolff, *Lectures in harmonic analysis*, University Lecture Series, vol. 29, American Mathematical Society, 2003.

[Woo20] T. D. Wooley, *Arithmetic harmonic analysis, 2020*, `https://www.math.purdue.edu/~twooley/2020aha/2020ahanotes.pdf`, 2020.

[Zha18] X. Zhang, *On the local-global principle for integral Apollonian 3-circle packings*, Journal für die reine und angewandte Mathematik (Crelle's Journal) **2018** (2018), no. 737, 71–110.