Appendix A

Michelle Bodnar, Andrew Lohr

August 27, 2017

Exercise A.1-1

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n(n+1) - n = n^2$$

Exercise A.1-2

Using the harmonic series formula we have that

$$\sum_{k=1}^{n} \frac{1}{(2k-1)} \leq 1 + \sum_{k=1}^{n} \frac{1}{2k} = 1 + \ln(\sqrt{n}) + O(1) = \ln(\sqrt{n}) + O(1).$$

Exercise A.1-3

First, we recall equation (A.8)

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

for $|x| < 1$. Then, we take a derivative of each side, taking the derivative of the left hand side term by term

$$\sum_{k=0}^{\infty} k \cdot kx^{k-1} = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{(1-x)+2x}{(1-x)^3} = \frac{(1+x)}{(1-x)^3}$$

Lastly, since we have a x^{k-1} instead of the x^k that we'd like, we'll multiply both sides of the equation by x to get the desired equality.

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}$$

Exercise A.1-4

Using formula A.8 we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k-1}{2^k} &= -1+\frac{1}{2}\sum_{k=0}^{\infty}k\left(\frac{1}{2}\right)^k\\ &= -1+\frac{1}{2}\cdot\frac{1/2}{1/4}\\ &= -1+1\\ &= 0.\end{aligned}$$

Exercise A.1-5

First, we'll start with the equation

$$\sum_{k=1}^{\infty}y^k=\frac{y}{1-y}$$

So long as $|y|<1$. Then, we'll let $y=x^2$ to get

$$\begin{aligned}\sum_{k=1}^{\infty}(x^2)^k &= \frac{x^2}{1-x^2}\\ \sum_{k=1}^{\infty}xx^{2k} &= \frac{x^3}{1-x^2}\\ \sum_{k=1}^{\infty}x^{2k+1} &= \frac{x^3}{1-x^2}\\ \sum_{k=1}^{\infty}(2k+1)x^{2k} &= \frac{3x^2(1-x^2)+2x^4}{(1-x^2)^2}\\ \sum_{k=1}^{\infty}(2k+1)x^{2k} &= \frac{3x^2-x^4}{(1-x^2)^2}\end{aligned}$$

so long as $|x|<1$.

Exercise A.1-6

Let $g_1, g_2, \ldots, g_n$ be any functions such that $g_k(i)=O(f_k(i))$. By the definition of big-oh there exist constant $c_1, c_2, \ldots, c_n$ such that $g_k(i)\le c_k f_k(i)$. Let $c=\max_{1\le k\le n} c_k$. Then we have

$$\sum_{k=1}^{n}g_k(i)\le\sum_{k=1}^{n}c_kf_k(i)\le c\sum_{k=1}^{n}f_k(i)=O\left(\sum_{k=1}^{n}f_k(i)\right).$$

Exercise A.1-7

$$
\begin{aligned}
&\lg\left(\prod_{k=1}^{n} 2\cdot 4^k\right)\\
&=\sum_{k=1}^{n}\lg(2\cdot 4^k)\\
&=\sum_{k=1}^{n}\lg(2)+k\lg(4)\\
&=\left(\lg(2)\sum_{k=1}^{n}1\right)+\left(\lg(4)\sum_{k=1}^{n}k\right)\\
&=n+2\frac{n(n+1)}{2}\\
&=n(n+2)
\end{aligned}
$$

This means that we need to raise 2 to this quantity to get the desired product, so out final answer is

$$2^{n(n+2)} = 2^{n^2}\cdot 4^n$$

Exercise A.1-8

We expand the product and cancel as follows:

$$
\begin{aligned}
\prod_{k=2}^{n} 1-1/k^2 &= \prod_{k=2}^{n}\frac{(k-1)(k+1)}{k^2}\\
&=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(n-1)\cdot(n+1)}{n\cdot n}\\
&=\frac{n+1}{2n}.
\end{aligned}
$$

Exercise A.2-1

Define a function $f_1 = \lceil \frac{1}{x^2} \rceil$ and $f_2 = 1 + \frac{1}{x^2}$. Note that we always have that $f_1 \leq f_2$. Then we have that the desired summation is exactly equal to $\int_1^\infty f_1$ because the graph of f_1 is a bunch of rectangles of width 1 and height equal to each of the terms in the sum. By monotonicity of integrals, we have that this is $\leq \int_1^\infty f_2 = 2$.

Exercise A.2-2

When $n = 2^m$ the sum becomes $n + n/2 + n/4 + \ldots + 1 = 2n - 1 = O(n)$. There always exists a power of 2 which lies between n and $2n$ for any choice of n, so let n' denote the smallest power of 2 which is greater than or equal to n. Then we have

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil \le \sum_{k=0}^{\lfloor \lg n' \rfloor} \lceil n'/2^k \rceil = 2n' - 1 \le 4n - 1 = O(n).$$

Exercise A.2-3
Similar to the derivation of (A.10), we split up the interval $[n]$ into $\lfloor \lg(n) \rfloor - 1$ pieces, with the ith starting at $1/2^i$ and going to $1/2^{i+1}$. So, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &\ge \sum_{i=0}^{\lg(n)-1} \sum_{j=0}^{2^i-1} \frac{1}{2^i + j} \\ &\ge \sum_{i=0}^{\lg(n)-1} \frac{1}{2^{i+1}} \\ &= \sum_{i=0}^{\lg(n)-1} \frac{1}{2} \\ &= \frac{1}{2} \lg(n)\end{aligned}$$

Which gets us that the nth harmonic number is $\Omega(\lg(n))$.

Exercise A.2-4

Since k^3 is monotonically increasing we use bound A.11. For the upper bound we have

$$\begin{aligned}\sum_{k=1}^{n} k^3 &\le \int_1^{n+1} x^3 dx \\ &= \left. \frac{x^4}{4} \right|_1^{n+1} \\ &= \frac{(n+1)^4 - 1}{4}.\end{aligned}$$

For the lower bound we have

$$\begin{aligned}\sum_{k=1}^{n} k^3 &\ge \int_0^{n} x^3 dx \\ &= \left. \frac{x^4}{4} \right|_0^{n} \\ &= \frac{n^4}{4}.\end{aligned}$$

Exercise A.2-5
If we were to apply the integral approximation given in (A.12) directly to the sum, then we would be trying to evaluate the integral

$$\int_0^n \frac{dx}{x}$$

Which is an improper integral that doesn't have a finite value.

Problem A-1

a. Applying the integral approximation to this, we get that

$$\int_0^n x^r dx \le \sum_{k=1}^n k^r \le \int_1^{n+1} x^r dx$$
$$\frac{n^{r+1}}{r+1} \le \sum_{k=1}^n k^r \le \frac{(n+1)^{r+1}-1}{r+1}$$

So, the given sum is $n^{r+1}(\frac{1}{r+1} + o(1))$.

b. We will first show that we can approximate the integral well by $g(x) = x\ln(x)^s(\frac{1}{\ln(2)^s} + f(x))$ where $f(x)$ is $O(1/\ln(x))$. The derivative of the RHS is $\ln(x)^s(\frac{1}{\ln(2)^s} + f(x)) + \ln(x)^{r-1}(\frac{1}{\ln(2)^s} + f(x)) + x\ln(x)^s f'(x)$. Since $f(x)$ is going to zero at least as fast as $\frac{1}{\ln(x)}$, we have that $f'(x) \in O(\frac{1}{x\ln(x)^2})$, this means that $\frac{d}{dx}g(x) = \lg(x)^s(1 + O(1/\ln(x)))$. So, since the integral of the derivative is the original function,

$$\begin{aligned} g(x) &= \int_1^x \lg(x)^s(1 + O(1/\ln(x)))dx \\ &= \int_1^x \lg(x)^s dx + \int_1^x O(\ln(x)^{r-1}))dx \\ &= \int_1^x \lg(x)^s dx + O(x\ln(x)^{r-1})) \end{aligned}$$

but this second term is insignificant enough to be absorbed into our $f(x)$ in how we defined $g(x)$. Then, since we can bound the sum above and below by the integral, just with shifted endpoints, and the derivative of the main term of the integral is small enough to fit in the remainder term, shifting it by one will not cause it's value to change asymptotically. So, we have that the sum is $n\ln(n)^s(\frac{1}{\ln(2)^s} + O(\frac{1}{\ln(n)})) = n\lg(n)^s(1 + O(\frac{1}{\ln(n)})$ which is asymptotically tight.

c. For this problem we show that the integral is well approximated by $g(n) = n^{r+1}/(r+1)\lg(n)^s(1+O(\frac{1}{\ln(n)}))$. Then, we have $g'(x) = n^r \lg(n)^s(1 + O(\frac{1}{\ln(n)})) + n^r/(r+1)\lg(n)^{s-1}(\frac{1}{\ln(2)} + O(\frac{1}{\ln(n)})) + n^r \lg(n)^s O(\frac{1}{n\ln(n)^2}) = n^r \lg(n)^s + O(n^r \lg(n)^{s-1})$. So, the exact same approximation goes through as in the previous problem. So, we get that the sum is asymptotically equal to

$$\frac{n^{r+1}\lg(n)^s}{n+1}\left(1+O\left(\frac{1}{\ln(n)}\right)\right)$$