Theory of Bessel Functions of High Rank

Dissertation

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Abstract

In this thesis, we shall study fundamental Bessel functions for $\mathrm{GL}_n(\mathbb{F})$ arising from the Voronoĭ summation formula as well as Bessel functions for $\mathrm{GL}_2(\mathbb{F})$ and $\mathrm{GL}_3(\mathbb{F})$ occurring in the Kuznetsov trace formula, where n is any positive integer and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

谨以此文献给我最亲爱的父母！

This is dedicated to my dearest parents!

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Vita

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Notations

- Denote $\mathbb{N} = \{0, 1, 2...\}$ and $\mathbb{N}_+ = \{1, 2, 3, ...\}$.

- The group $\mathbb{Z}/2\mathbb{Z}$ is usually identified with the set $\{0, 1\}$.

- Denote $\mathbb{R}_+ = (0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$.

- Denote by $\mathbb{U} \cong \mathbb{R}_+ \times \mathbb{R}$ the universal cover of $\mathbb{C} \smallsetminus \{0\}$. Each element $z \in \mathbb{U}$ is denoted by $z = xe^{i\phi}$, with $(x, \phi) \in \mathbb{R}_+ \times \mathbb{R}$.

- For $m \in \mathbb{Z}$ define $\delta(m) \in \mathbb{Z}/2\mathbb{Z}$ by $\delta(m) = m(\mathrm{mod}\ 2)$.

- For $s \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, let $[s]_\alpha = \prod_{\kappa=0}^{\alpha-1}(s - \alpha)$ and $(s)_\alpha = \prod_{\kappa=0}^{\alpha-1}(s + \alpha)$ if $\alpha \geqslant 1$, and let $[s]_0 = (s)_0 = 1$.

- For $s \in \mathbb{C}$ let $e(s) = e^{2\pi i s}$.

- For a finite closed interval $[a, b] \subset \mathbb{R}$ define the closed vertical strip $\mathbb{S}[a, b] = \{s \in \mathbb{C} : \mathfrak{Re}\, s \in [a, b]\}$. The open vertical strip $\mathbb{S}(a, b)$ for a finite open interval (a, b) is similarly defined.

- For $\lambda \in \mathbb{C}$ and $r > 0$, define $\mathbb{B}_r(\lambda) = \{s \in \mathbb{C} : |s - \lambda| < r\}$ to be the disc of radius r centered at $s = \lambda$.

- For $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ denote $|\boldsymbol{\lambda}| = \sum_{\ell=1}^n \lambda_\ell$ (this notation works for subsets of $\mathbb{C}^n$, for instance, $(\mathbb{Z}/2\mathbb{Z})^n = \{0, 1\}^n$ and $\mathbb{Z}^n$).

- Define the hyperplane $\mathbb{L}^{n-1} = \{\boldsymbol{\lambda} \in \mathbb{C}^n : |\boldsymbol{\lambda}| = \sum_{\ell=1}^{n} \lambda_\ell = 0\}$.

- Denote by $\boldsymbol{e}^n$ the n-tuple $(1, ..., 1)$.

- For $\boldsymbol{m} = (m_1, ..., m_n) \in \mathbb{Z}^n$ define $\|\boldsymbol{m}\| = (|m_1|, ..., |m_n|)$.

Preface

Bessel functions have been extensively studied since the early 19th century and are presented in various branches of mathematics as well as physics. In number theory, Bessel functions appear in *Voronoĭ's summation formula*, *Petersson's and Kuznetsov's trace formula* for $\mathrm{GL}_2(\mathbb{R})$ (in their simplest versions, for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$). Therefore, understanding the analytic properties of Bessel functions is necessary for understanding arithmetic objects associated to $\mathrm{GL}_2(\mathbb{R})$.

Applying his trace formula, Kuznetsov made the first progress in the direction of the Linnik-Selberg conjecture on averages of Kloosterman sums in [Kuz]. Moreover, both formulae have been heavily used to the subconvexity problems for Hecke L-functions of cuspidal modular forms (see the series of papers by Duke, Friedlander and Iwaniec [DFI1, DFI2, DFI3, DFI4]) and Rankin-Selberg L-functions of two cuspidal forms (see [KMV, Mic, HM]).

In the last decade, several number theorists have worked to generalize the Voronoĭ summation formula to high rank as well as to arbitrary number fields (see, for example, [MS3, MS4, GL1, GL2, IT]), where certain integral transforms, called *Hankel transforms* (*of high rank*), naturally arise. Furthermore, the Kuznetsov trace formula for $\mathrm{SL}_2(\mathbb{C})$ was established in [BM, LG], where the Bessel function associated to a principal series representation of $\mathrm{SL}_2(\mathbb{C})$ was also discovered.

The author of this thesis has dedicated the last two years to studying the analytic theory of Hankel transforms and, more importantly, *fundamental Bessel functions*, or *kernels*, that occur in the Voronoĭ summation formula for $\mathrm{GL}_n(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The work has resulted in two articles, [Qi1] and [Qi2], which constitute Chapter 1, Chapter 2 of this thesis. The motivation of [Qi1] is the collaboration of the author with his advisor Roman Holowinsky and Ritabrata Munshi on hybrid subconvexity bounds for certain Rankin-Selberg L-functions on $\mathrm{GL}_3 \times \mathrm{GL}_2$ (see [HMQ]), during which the author was investigating the analytic properties of Hankel transforms for $\mathrm{GL}_3(\mathbb{R})$ and discovered that their integral kernels can be represented by certain *formal* integrals. The inspiration of [Qi2] is the work [MS1], which provided the author with ideas on establishing the foundation for Hankel transforms for $\mathrm{GL}_n(\mathbb{C})$ from the perspective of harmonic analysis over $\mathbb{C}^\times$.

In general, the Kuznetsov trace formula is much deeper than the Voronoĭ summation formula, and no instances of the corresponding *Bessel functions* are known except for $\mathrm{GL}_2(\mathbb{F})$. To distinguish, the adjective *fundamental* is added to the Bessel functions, or kernels, in the Voronoĭ summation formula as in the last paragraph. According to [IT], it is the Rankin-Selberg $\mathrm{GL}_n \times \mathrm{GL}_1$ local functional equations that provide the underlying structure for fundamental Bessel kernels. The author believes that Rankin-Selberg $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ local functional equations should yield relations between Bessel functions for GL_n and GL_{n-1}, with the occurrence of the corresponding Rankin-Selberg fundamental Bessel kernel of rank $n(n-1)$, and hence fundamental Bessel kernels should be the building blocks of Bessel functions of any rank from an inductive perspective. As a special example, in the case of $\mathrm{GL}_2(\mathbb{F})$, this explains why the Bessel functions occurring in the Kuznetsov trace formula also arise in the Voronoĭ summation formula; see Section 3.2. Furthermore, in attempting to justify his philosophy, the author wrote some notes on formulating Bessel functions for $\mathrm{GL}_3(\mathbb{F})$ in terms of fundamental Bessel kernels, derived from Rankin-Selberg $\mathrm{GL}_3 \times \mathrm{GL}_2$ local functional equations; these are included in Section 3.3.

Chapter 1

Fundamental Bessel Functions

1.1. Introduction

1.1.1. Background

Hankel transforms (of high rank) are introduced as an important constituent of the *Voronoĭ summation formula* by Miller and Schmid in [MS1, MS3, MS4]. This summation formula is a fundamental analytic tool in number theory and has its roots in representation theory.

In this chapter, we shall develop the analytic theory of *fundamental Bessel functions*[I]. These Bessel functions constitute the integral kernels of Hankel transforms. Thus, to motivate our study, we shall start with introducing Hankel transforms and their number theoretic and representation theoretic background.

[I]The Bessel functions studied here are called *fundamental* in order to be distinguished from the Bessel functions for $\mathrm{GL}_n(\mathbb{R})$. Throughout this chapter, we shall drop the adjective *fundamental* for brevity. Moreover, the usual Bessel functions will be referred to as classical Bessel functions.

Some evidences show that fundamental Bessel functions are actually the building blocks of the Bessel functions for $\mathrm{GL}_n(\mathbb{R})$.

Two expressions of a Hankel transform

Let n be a positive integer, and let $(\boldsymbol{\lambda}, \boldsymbol{\delta}) = (\lambda_1, ..., \lambda_n, \delta_1, ..., \delta_n) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$.

The first expression of the Hankel transform of rank n associated with $(\boldsymbol{\lambda}, \boldsymbol{\delta})$ is based on signed Mellin transforms as follows.

Let $\mathscr{S}(\mathbb{R})$ denote the space of Schwartz functions on $\mathbb{R}$. For $\lambda \in \mathbb{C}$, $j \in \mathbb{N}$ and $\eta \in \mathbb{Z}/2\mathbb{Z}$, let υ be a smooth function on $\mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$ such that $\operatorname{sgn}(x)^\eta (\log|x|)^{-j} |x|^{-\lambda}\upsilon(x) \in \mathscr{S}(\mathbb{R})$. For $\delta \in \mathbb{Z}/2\mathbb{Z}$, the *signed Mellin transform* $\mathcal{M}_\delta\upsilon$ *with order* δ of υ is defined by

$$\mathcal{M}_\delta\upsilon(s) = \int_{\mathbb{R}^\times} \upsilon(x)\operatorname{sgn}(x)^\delta|x|^s d^\times x. \tag{1.1.1}$$

Here $d^\times x = |x|^{-1}dx$ is the standard multiplicative Haar measure on $\mathbb{R}^\times$. The Mellin inversion formula is

$$\upsilon(x) = \sum_{\delta\in\mathbb{Z}/2\mathbb{Z}} \frac{\operatorname{sgn}(x)^\delta}{4\pi i}\int_{(\sigma)} \mathcal{M}_\delta\upsilon(s)|x|^{-s}ds, \quad \sigma > -\Re e\,\lambda,$$

where the contour of integration (σ) is the vertical line from $\sigma - i\infty$ to $\sigma + i\infty$.

Let $\mathscr{S}(\mathbb{R}^\times)$ denote the space of smooth functions on $\mathbb{R}^\times$ whose derivatives are rapidly decreasing at both zero and infinity. We associate with $\upsilon \in \mathscr{S}(\mathbb{R}^\times)$ a function Υ on $\mathbb{R}^\times$ satisfying the following two identities

$$\mathcal{M}_\delta\Upsilon(s) = \left(\prod_{\ell=1}^{n} G_{\delta_\ell+\delta}(s-\lambda_\ell)\right)\mathcal{M}_\delta\upsilon(1-s), \quad \delta\in\mathbb{Z}/2\mathbb{Z}, \tag{1.1.2}$$

where $G_\delta(s)$ denotes the gamma factor

$$G_\delta(s) = i^\delta\pi^{\frac12 - s}\frac{\Gamma\left(\frac12(s+\delta)\right)}{\Gamma\left(\frac12(1-s+\delta)\right)} = \begin{cases} 2(2\pi)^{-s}\Gamma(s)\cos\left(\dfrac{\pi s}{2}\right), & \text{if } \delta = 0, \\ 2i(2\pi)^{-s}\Gamma(s)\sin\left(\dfrac{\pi s}{2}\right), & \text{if } \delta = 1, \end{cases} \tag{1.1.3}$$

where for the second equality we apply the duplication formula and Euler's reflection formula of the Gamma function,

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

Υ is called the *Hankel transform of index* $(\boldsymbol{\lambda}, \boldsymbol{\delta})$ of υ[II]. According to [MS3, §6], Υ is smooth on $\mathbb{R}^\times$ and decays rapidly at infinity, along with all its derivatives. At the origin, Υ has singularities of some very particular type. Indeed, $\Upsilon(x) \in \sum_{\ell=1}^{n} \operatorname{sgn}(x)^{\delta_\ell}|x|^{-\lambda_\ell}\mathscr{S}(\mathbb{R})$ when no two components of $\boldsymbol{\lambda}$ differ by an integer, and in the nongeneric case powers of $\log|x|$ will be included.

By the Mellin inversion,

$$\Upsilon(x) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \frac{\operatorname{sgn}(x)^{\delta}}{4\pi i} \int_{(\sigma)} \left(\prod_{\ell=1}^{n} G_{\delta_\ell+\delta}(s-\lambda_\ell) \right) \mathcal{M}_\delta \upsilon(1-s)|x|^{-s} ds, \tag{1.1.4}$$

for $\sigma > \max\{\Re\mathrm{e}\,\lambda_\ell\}$.

In [MS4] there is an alternative description of Υ defined by the *Fourier type transform*, in symbolic notion, as follows

$$\Upsilon(x) = \frac{1}{|x|} \int_{\mathbb{R}^{\times n}} \upsilon\left(\frac{x_1 \ldots x_n}{x}\right) \left(\prod_{\ell=1}^{n} \left(\operatorname{sgn}(x_\ell)^{\delta_\ell} |x_\ell|^{-\lambda_\ell} e(x_\ell) \right) \right) dx_n dx_{n-1} \ldots dx_1, \tag{1.1.5}$$

with $e(x) = e^{2\pi i x}$. The integral in (1.1.5) converges when performed as iterated integral in the order $dx_n dx_{n-1} \ldots dx_1$, starting from x_n, then x_{n-1}, ..., and finally x_1, provided $\Re\mathrm{e}\,\lambda_1 > \ldots > \Re\mathrm{e}\,\lambda_{n-1} > \Re\mathrm{e}\,\lambda_n$, and it has meaning for arbitrary values of $\boldsymbol{\lambda} \in \mathbb{C}^n$ by analytic continuation.

According to [MS4], though *less suggestive than* (1.1.5), the expression (1.1.4) of Hankel transforms is *more useful in applications*. Indeed, all the applications of the Voronoĭ

[II]Note that if υ is the f in [MS4] then $|x|\Upsilon((-)^n x)$ is their $F(x)$.

summation formula so far are based on (1.1.4) with exclusive use of Stirling's asymptotic formula of the Gamma function. On the other hand, there is no occurrence of the Fourier type integral transform (1.1.5) in the literatures other than Miller and Schmid's foundational work.

Assumption. *Subsequently, we shall always assume that the index $\boldsymbol{\lambda}$ satisfies $\sum_{\ell=1}^{n} \lambda_\ell = 0$*[III]*. Accordingly, we define the complex hyperplane $\mathbb{L}^{n-1} = \{\boldsymbol{\lambda} \in \mathbb{C}^n : \sum_{\ell=1}^{n} \lambda_\ell = 0\}$.*

Background of Hankel transforms in number theory and representation theory

For $n = 1$, the number theoretic background lies on the local theory in Tate's thesis at the real place. Actually, in view of (1.1.5), the Hankel transform of rank one and index $(\boldsymbol{\lambda}, \boldsymbol{\delta}) = (0, \delta)$ is essentially the (inverse) Fourier transform,

$$\Upsilon(x) = \int_{\mathbb{R}} \upsilon(y) \operatorname{sgn}(xy)^{\delta} e(xy) dy. \tag{1.1.6}$$

The Voronoĭ summation formula of rank one is the summation formula of Poisson. Recall that Riemann's proof of the functional equation of his ζ-function relies on the Poisson summation formula, whereas Tate's thesis reinterprets this using the Poisson summation formula for the adele ring.

For $n = 2$, the Hankel transform associated with a GL_2-automorphic form has been present in the literatures as part of the Voronoĭ summation formula for GL_2 for decades. See, for instance, [HM, Proposition 1] and the references there. According to [HM, Proposition 1] (see also Remark 1.2.8), we have

$$\Upsilon(x) = \int_{\mathbb{R}^\times} \upsilon(y) J_F(xy) dy, \quad x \in \mathbb{R}^\times, \tag{1.1.7}$$

[III] This condition is just a matter of normalization. Equivalently, the corresponding representations of $\mathrm{GL}_n(\mathbb{R})$ are trivial on the positive component of the center. With this condition on $\boldsymbol{\lambda}$, the associated Bessel functions can be expressed in a simpler way.

where, if F is a Maaß form of eigenvalue $\frac{1}{4}+t^2$ and weight k,

$$
\begin{aligned}
J_F(x) &= -\frac{\pi}{\cosh(\pi t)}\left(Y_{2it}(4\pi\sqrt{x}) + Y_{-2it}(4\pi\sqrt{x})\right)\\
&= \frac{\pi i}{\sinh(\pi t)}\left(J_{2it}(4\pi\sqrt{x}) - J_{-2it}(4\pi\sqrt{x})\right)\\
&= \pi i\left(e^{-\pi t}H^{(1)}_{2it}(4\pi\sqrt{x}) - e^{\pi t}H^{(2)}_{2it}(4\pi\sqrt{x})\right),\\
J_F(-x) &= 4\cosh(\pi t)K_{2it}(4\pi\sqrt{x})\\
&= \frac{\pi i}{\sinh(\pi t)}\left(I_{2it}(4\pi\sqrt{x}) - I_{-2it}(4\pi\sqrt{x})\right), \quad x>0,
\end{aligned}
\tag{1.1.8}
$$

for k even,

$$
\begin{aligned}
J_F(x) &= -\frac{\pi}{\sinh(\pi t)}\left(Y_{2it}(4\pi\sqrt{x}) - Y_{-2it}(4\pi\sqrt{x})\right)\\
&= \frac{\pi i}{\cosh(\pi t)}\left(J_{2it}(4\pi\sqrt{x}) + J_{-2it}(4\pi\sqrt{x})\right)\\
&= \pi i\left(e^{-\pi t}H^{(1)}_{2it}(4\pi\sqrt{x}) + e^{\pi t}H^{(2)}_{2it}(4\pi\sqrt{x})\right)\\
J_F(-x) &= 4\sinh(\pi t)K_{2it}(4\pi\sqrt{x})\\
&= \frac{\pi i}{\cosh(\pi t)}\left(I_{2it}(4\pi\sqrt{x}) - I_{-2it}(4\pi\sqrt{x})\right), \quad x>0,
\end{aligned}
\tag{1.1.9}
$$

for k odd [IV], and if F is a holomorphic cusp form of weight k,

$$
J_F(x) = 2\pi i^k J_{k-1}(4\pi\sqrt{x}), \quad J_F(-x) = 0, \quad x>0. \tag{1.1.10}
$$

Thus the integral kernel J_F has an expression in terms of Bessel functions, where, in standard notation, J_ν, Y_ν, $H^{(1)}_\nu$, $H^{(2)}_\nu$, I_ν and K_ν are the various Bessel functions (see for instance [Wat]). Here, the following connection formulae ([Wat, 3.61 (3, 4, 5, 6), 3.7 (6)]) have been applied in (1.1.8) and (1.1.9),

$$
Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}, \quad Y_{-\nu}(x) = \frac{J_\nu(x) - J_{-\nu}(x)\cos(\pi\nu)}{\sin(\pi\nu)}, \tag{1.1.11}
$$

[IV] For this case there are two insignificant typos in [HM, Proposition 1].

$$H^{(1)}_\nu(x) = \frac{J_{-\nu}(x) - e^{-\pi i \nu} J_\nu(x)}{i \sin(\pi \nu)}, \quad H^{(2)}_\nu(x) = \frac{e^{\pi i \nu} J_\nu(x) - J_{-\nu}(x)}{i \sin(\pi \nu)}, \tag{1.1.12}$$

$$K_\nu(x) = \frac{\pi \left(I_{-\nu}(x) - I_\nu(x)\right)}{2 \sin(\pi \nu)}. \tag{1.1.13}$$

The theory of Bessel functions has been extensively studied since the early 19th century, and we refer the reader to Watson's beautiful book [Wat] for an encyclopedic treatment.

For $n \geqslant 3$, Hankel transforms are formulated in Miller and Schmid [MS3, MS4], given that $(\boldsymbol{\lambda}, \boldsymbol{\delta})$ is a certain parameter of a cuspidal $\mathrm{GL}_n(\mathbb{Z})$-automorphic representation of $\mathrm{GL}_n(\mathbb{R})$. It is the archimedean ingredient that relates the weight functions on two sides of the identity in the Voronoĭ summation formula for $\mathrm{GL}_n(\mathbb{Z})$. For $n = 1, 2$ the Poisson and the Voronoĭ summation formula are also interpreted from their perspective in [MS2].

Using the global theory of $\mathrm{GL}_n \times \mathrm{GL}_1$-Rankin-Selberg L-functions, Inchino and Templier [IT] extend Miller and Schmid's work and prove the Voronoĭ summation formula for any irreducible cuspidal automorphic representation of GL_n over an arbitrary number field for $n \geqslant 2$. According to [IT], the two defining identities (1.1.2) of the associated Hankel transform follows from renormalizing the corresponding local functional equations of the $\mathrm{GL}_n \times \mathrm{GL}_1$-Rankin-Selberg zeta integrals over $\mathbb{R}$.

Bessel kernels

In the case $n \geqslant 3$, when applying the Voronoĭ summation formula, it might have been realized by many authors that, similar to (1.1.6, 1.1.7), Hankel transforms of rank n should also admit integral kernels, that is,

$$\Upsilon(x) = \int_{\mathbb{R}^\times} \upsilon(y) J_{(\lambda, \delta)}(xy) dy.$$

We shall call $J_{(\lambda,\delta)}$ the (*fundamental*) *Bessel kernel of index* $(\boldsymbol{\lambda}, \boldsymbol{\delta})$.

Actually, it will be seen in §1.2.1 that an expression of $J_{(\lambda,\delta)}(\pm x)$, $x \in \mathbb{R}_+ = (0, \infty)$, in terms of certain Mellin-Barnes type integrals involving the Gamma function (see (1.2.7, 1.2.8)) may be easily derived from the first expression (1.1.4) of the Hankel transform of index $(\boldsymbol{\lambda}, \boldsymbol{\delta})$. Moreover, the analytic continuation of $J_{(\lambda,\delta)}(\pm x)$ from $\mathbb{R}_+$ onto the Riemann surface $\mathbb{U}$, the universal cover of $\mathbb{C} \smallsetminus \{0\}$, can be realized as a Barnes type integral via modifying the integral contour of a Mellin-Barnes type integral (see Remark 1.7.10). In the literatures, we have seen applications of the asymptotic expansion of $J_{(\lambda,\delta)}(\pm x)$ obtained from applying Stirling's asymptotic formula of the Gamma function to the Mellin-Barnes type integral (see Appendix 1.A). There are however two limitations of this method. Firstly, it is *only* applicable when $\boldsymbol{\lambda}$ is regarded as fixed constant and hence the dependence on $\boldsymbol{\lambda}$ of the error term can not be clarified. Secondly, it is *not* applicable to a Barnes type integral and therefore the domain of the asymptotic expansion can not be extended from $\mathbb{R}_+$. In this direction from (1.1.4), it seems that we can not proceed any further.

In this chapter, we shall take an approach to Bessel kernels starting from the second expression (1.1.5) of Hankel transforms. This approach is more accessible, at least in symbolic notions, in view of the simpler form of (1.1.5) compared to (1.1.4). Once we can make sense of the symbolic notions in (1.1.5), some well-developed methods from analysis and differential equations may be exploited so that we are able to understand Bessel kernels to a much greater extent.

1.1.2. Outline of this chapter

Bessel functions and their formal integral representations

First of all, in §1.2.1, we introduce the *Bessel function* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *of indices* $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ *and* $\boldsymbol{\varsigma} \in \{+, -\}^n$. It turns out that the Bessel kernel $J_{(\lambda,\delta)}(\pm x)$ can be formulated as a signed sum of $J\big(2\pi x^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\lambda}\big)$, $x \in \mathbb{R}_+$. Our task is therefore understanding each Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

In §1.2.2, with some manipulations on the Fourier type expression (1.1.5) of the Hankel transform of index $(\boldsymbol{\lambda}, \boldsymbol{\delta})$ in a symbolic manner, we obtain a *formal* integral representation of the Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$. If we define $\boldsymbol{\nu} = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ by $\nu_\ell = \lambda_\ell - \lambda_n$, with $\ell = 1, ..., n-1$, then the formal integral is given by

$$J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}) = \int_{\mathbb{R}_+^{n-1}} \left(\prod_{\ell=1}^{n-1} t_\ell^{\nu_\ell - 1} \right) e^{ix\left(\varsigma_n t_1 ... t_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell t_\ell^{-1}\right)} dt_{n-1} ... dt_1. \tag{1.1.14}$$

Justification of this formal integral representation is the main subject of §1.3 and §1.4. For this, we partition the formal integral $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ according to some partition of unity on $\mathbb{R}_+^{n-1}$, and then repeatedly apply *two* kinds of partial integration operators on each resulting integral. In this way, $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ can be transformed into a finite sum of absolutely convergent multiple integrals. This sum of integrals is regarded as the rigorous definition of $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$. However, the simplicity of the expression (1.1.14) is sacrificed after these technical procedures. Furthermore, it is shown that

$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}). \tag{1.1.15}$$

Asymptotics via stationary phase

In §1.5, we either adapt techniques or directly apply results from the method of stationary phase to study the asymptotic behaviour of $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ for large argument.

When all the components of $\boldsymbol{\varsigma}$ are identically $\pm$, we denote $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$, respectively $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$, by $H^\pm(x; \boldsymbol{\lambda})$, respectively $H^\pm_{\boldsymbol{\nu}}(x)$, and call it an *H-Bessel function*[V]. This pair of H-Bessel functions will be of paramount significance in our treatment.

[V] If a statement or a formula includes $\pm$ or $\mp$, then it should be read with $\pm$ and $\mp$ simultaneously replaced by either $+$ and $-$ or $-$ and $+$.

It is shown that $H^{\pm}(x;\boldsymbol{\lambda}) = H^{\pm}_{\nu}(x)$ admits an analytic continuation from $\mathbb{R}_+$ onto the half-plane $\mathbb{H}^{\pm} = \{z \in \mathbb{C} \smallsetminus \{0\} : 0 \leqslant \pm \arg z \leqslant \pi\}$. We have the asymptotic expansion

$$\begin{aligned} H^{\pm}(z;\boldsymbol{\lambda}) = {} & n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} e^{\pm inz} z^{-\frac{n-1}{2}} \\ & \left(\sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\boldsymbol{\lambda}) z^{-m} + O_{\mathfrak{R},M,n}\left(\mathfrak{C}^{2M} |z|^{-M+\frac{n-1}{2}} \right) \right), \end{aligned} \tag{1.1.16}$$

for all $z \in \mathbb{H}^{\pm}$ such that $|z| \geqslant \mathfrak{C}$, where $\mathfrak{C} = \max\{|\lambda_\ell|\} + 1$, $\mathfrak{R} = \max\{|\mathfrak{Re}\,\lambda_\ell|\}$, $M \geqslant 0$, $B_m(\boldsymbol{\lambda})$ is a certain symmetric polynomial in $\boldsymbol{\lambda}$ of degree $2m$, with $B_0(\boldsymbol{\lambda}) = 1$. In particular, these two H-Bessel functions oscillate and decay proportionally to $x^{-\frac{n-1}{2}}$ on $\mathbb{R}_+$.

All the other Bessel functions are called *K-Bessel functions* and are shown to be Schwartz functions at infinity.

Bessel equations

The differential equation, namely *Bessel equation*, satisfied by the Bessel function $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is discovered in §1.6.

Given $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$, there are exactly two Bessel equations

$$\sum_{j=1}^{n} V_{n,j}(\boldsymbol{\lambda}) x^j w^{(j)} + \left(V_{n,0}(\boldsymbol{\lambda}) - \varsigma (in)^n x^n\right) w = 0, \quad \varsigma \in \{+,-\}, \tag{1.1.17}$$

where $V_{n,j}(\boldsymbol{\lambda})$ is some explicitly given symmetric polynomial in $\boldsymbol{\lambda}$ of degree $n-j$. We call ς the *sign* of the Bessel equation (1.1.17). $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ satisfies the Bessel equation of sign $S_n(\boldsymbol{\varsigma}) = \prod_{\ell=1}^{n} \varsigma_\ell$.

The entire §1.7 is devoted to the study of Bessel equations. Let $\mathbb{U}$ denote the Riemann surface associated with $\log z$, that is, the universal cover of $\mathbb{C} \smallsetminus \{0\}$. Replacing x by z to stand for complex variable in the Bessel equation (1.1.17), the domain is extended from $\mathbb{R}_+$ to $\mathbb{U}$. According to the theory of linear ordinary differential equations with analytic coefficients, $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ admits an analytic continuation onto $\mathbb{U}$.

Firstly, since zero is a regular singularity, the Frobenius method may be exploited to find a solution $J_\ell(z;\varsigma,\lambda)$ of (1.1.17), for each $\ell = 1, ..., n$, defined by the following series,

$$J_\ell(z;\varsigma,\lambda) = \sum_{m=0}^{\infty} \frac{(\varsigma i^n)^m z^{n(-\lambda_\ell+m)}}{\prod_{k=1}^{n} \Gamma(\lambda_k - \lambda_\ell + m + 1)}.$$

$J_\ell(z;\varsigma,\lambda)$ are called *Bessel functions of the first kind*, since they generalize the Bessel functions $J_\nu(z)$ and the modified Bessel functions $I_\nu(z)$ of the first kind.

It turns out that each $J(z;\boldsymbol{\varsigma},\lambda)$ may be expressed in terms of $J_\ell(z; S_n(\boldsymbol{\varsigma}),\lambda)$. This leads to the following connection formula

$$J(z;\boldsymbol{\varsigma},\lambda) = e\left(\pm\frac{\sum_{\ell\in L_\mp(\boldsymbol{\varsigma})}\lambda_\ell}{2}\right) H^\pm\left(e^{\pm\pi i\frac{n_\mp(\boldsymbol{\varsigma})}{n}} z;\lambda\right), \tag{1.1.18}$$

where $L_\pm(\boldsymbol{\varsigma}) = \{\ell : \varsigma_\ell = \pm\}$ and $n_\pm(\boldsymbol{\varsigma}) = |L_\pm(\boldsymbol{\varsigma})|$. Thus the Bessel function $J(z;\boldsymbol{\varsigma},\lambda)$ is determined up to a constant by the pair of integers $(n_+(\boldsymbol{\varsigma}), n_-(\boldsymbol{\varsigma}))$, called the *signature* of $J(z;\boldsymbol{\varsigma},\lambda)$.

Secondly, ∞ is an irregular singularity of rank one. The formal solutions at infinity serve as the asymptotic expansions of some actual solutions of Bessel equations.

Let ξ be an n-th root of $\varsigma 1$. There exists a unique formal solution $\widehat{J}(z;\lambda;\xi)$ of the Bessel equation of sign ς in the following form

$$\widehat{J}(z;\lambda;\xi) = e^{in\xi z} z^{-\frac{n-1}{2}} \sum_{m=0}^{\infty} B_m(\lambda;\xi) z^{-m},$$

where $B_m(\lambda;\xi)$ is a symmetric polynomial in λ of degree $2m$, with $B_0(\lambda;\xi) = 1$. The coefficients of $B_m(\lambda;\xi)$ depend only on m, ξ and n. There exists a *unique* solution $J(z;\lambda;\xi)$ of the Bessel equation of sign ς which possesses $\widehat{J}(z;\lambda;\xi)$ as its asymptotic expansion on the sector

$$\mathbb{S}_\xi = \left\{ z \in \mathbb{U} : \left| \arg z - \arg(i\bar{\xi}) \right| < \frac{\pi}{n} \right\},$$

or any of its open subsector.

The study of the theory of asymptotic expansions for ordinary differential equations can be traced back to Poincaré. There are abundant references on this topic, for instance, [CL, Chapter 5], [Was, Chapter III-V] and [Olv, Chapter 7]. However, the author is not aware of any error analysis in the index aspect in the literatures except for differential equations of second order in [Olv]. Nevertheless, with some effort, a very satisfactory error bound is attainable.

For $0 < \vartheta < \frac{1}{2}\pi$ define the sector

$$\mathbb{S}'_{\xi}(\vartheta) = \left\{ z \in \mathbb{U} : \left| \arg z - \arg(i\bar{\xi}) \right| < \pi + \frac{\pi}{n} - \vartheta \right\}.$$

The following asymptotic expansion is established in §1.7.4,

$$J(z; \boldsymbol{\lambda}; \xi) = e^{in\xi z} z^{-\frac{n-1}{2}} \left(\sum_{m=0}^{M-1} B_m(\boldsymbol{\lambda}; \xi) z^{-m} + O_{M,n}\left(\mathfrak{C}^{2M} |z|^{-M} \right) \right) \tag{1.1.19}$$

for all $z \in \mathbb{S}'_{\xi}(\vartheta)$ with $|z| \gg_{M,\vartheta,n} \mathfrak{C}^2$.

For a $2n$-th root of unity ξ, $J(z; \boldsymbol{\lambda}; \xi)$ is called a *Bessel function of the second kind*. We have the following formula that relates all the the Bessel functions of the second kind to either $J(z; \boldsymbol{\lambda}; 1)$ or $J(z; \boldsymbol{\lambda}; -1)$ upon rotating the argument by a $2n$-th root of unity,

$$J(z; \boldsymbol{\lambda}; \xi) = (\pm\xi)^{\frac{n-1}{2}} J(\pm\xi z; \boldsymbol{\lambda}; \pm 1). \tag{1.1.20}$$

Connections between $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ and $J(z; \boldsymbol{\lambda}; \xi)$

Comparing the asymptotic expansions of $H^{\pm}(z; \boldsymbol{\lambda})$ and $J(z; \boldsymbol{\lambda}; \pm 1)$ in (1.1.16) and (1.1.19), we obtain the identity

$$H^{\pm}(z; \boldsymbol{\lambda}) = n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} J(z; \boldsymbol{\lambda}; \pm 1). \tag{1.1.21}$$

It follows from (1.1.18) and (1.1.20) that

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{(\mp 2\pi i)^{\frac{n-1}{2}}}{\sqrt{n}} e\left(\pm \frac{(n-1) n_{\pm}(\boldsymbol{\varsigma})}{4n} \mp \frac{\sum_{\ell \in L_{\pm}(\boldsymbol{\varsigma})} \lambda_\ell}{2} \right) J\left(z; \boldsymbol{\lambda}; \mp e^{\mp \pi i \frac{n_{\pm}(\boldsymbol{\varsigma})}{n}} \right).$$

Thus (1.1.19) may be applied to improve the error estimate in the asymptotic expansion (1.1.16) of the H-Bessel function $H^{\pm}(z;\boldsymbol{\lambda})$ when $|z| \gg_{M,n} \mathfrak{C}^2$ and also to show the exponential decay of K-Bessel functions on $\mathbb{R}_+$.

Connections between $J_\ell(z;\varsigma,\boldsymbol{\lambda})$ and $J(z;\boldsymbol{\lambda};\xi)$

The identity (1.1.21) also yields connection formulae between the two kinds of Bessel functions, in terms of a certain Vandermonde matrix and its inverse.

1.2. Preliminaries on Bessel functions

In §1.2.1 and 1.2.2, we shall introduce the Bessel function $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$, with $\boldsymbol{\varsigma} \in \{+,-\}^n$ and $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$. Two expressions of $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ arise from the two formulae (1.1.4) and (1.1.5) of the Hankel transform of index $(\boldsymbol{\lambda},\boldsymbol{\delta})$. The first is a Mellin-Barnes type contour integral and the second is a formal multiple integral. In §1.2.3 and 1.2.4, some examples of $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ are provided for the purpose of illustration.

Let $\upsilon \in \mathscr{S}(\mathbb{R}^\times)$ be a Schwartz function on $\mathbb{R}^\times$. Without loss of generality, we assume $\upsilon(-y) = (-)^\eta \upsilon(y)$, with $\eta \in \mathbb{Z}/2\mathbb{Z}$.

1.2.1. The definition of the Bessel function $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$

We start with reformulating (1.1.3) as

$$G_\delta(s) = (2\pi)^{-s}\Gamma(s)\left(e\left(\frac{s}{4}\right) + (-)^\delta e\left(-\frac{s}{4}\right)\right).$$

Inserting this formula of G_δ into (1.1.4), $\Upsilon(x)$ then splits as follows

$$\Upsilon(x) = \operatorname{sgn}(x)^\eta \sum_{\boldsymbol{\varsigma}\in\{+,-\}^n} \left(\prod_{\ell=1}^n \varsigma_\ell^{\delta_\ell+\eta}\right) \Upsilon(|x|;\boldsymbol{\varsigma}), \tag{1.2.1}$$

with $\boldsymbol{\varsigma} = (\varsigma_1, ..., \varsigma_n)$, where

$$\Upsilon(x; \boldsymbol{\varsigma}) = \frac{1}{2\pi i} \int_{(\sigma)} \int_0^\infty \upsilon(y) y^{-s} dy \cdot G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) ((2\pi)^n x)^{-s} ds, \quad x \in \mathbb{R}_+, \tag{1.2.2}$$

and

$$G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \prod_{\ell=1}^{n} \Gamma(s - \lambda_\ell) e\left(\frac{\varsigma_\ell (s - \lambda_\ell)}{4}\right). \tag{1.2.3}$$

Since all the derivatives of υ rapidly decay at both zero and infinity, repeating partial integrations yields the bound

$$\int_0^\infty \upsilon(y) y^{-s} dy \ll_{\Re\mathrm{e}\, s, M, \upsilon} (|\Im\mathrm{m}\, s| + 1)^{-M},$$

for any nonnegative integer M. Hence the iterated double integral in (1.2.2) is convergent due to Stirling's formula.

Choose $\rho < \frac{1}{2} - \frac{1}{n}$ so that $\sum_{\ell=1}^n \left(\rho - \Re\mathrm{e}\, \lambda_\ell - \frac{1}{2}\right) < -1$. Without passing through any pole of $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$, we shift the vertical line (σ) to a contour $\mathcal{C}$ that starts from $\rho - i\infty$, ends at $\rho + i\infty$, and remains vertical at infinity. After this contour shift, the double integral in (1.2.2) becomes absolutely convergent by Stirling's formula. Changing the order of integration is therefore legitimate and yields

$$\Upsilon(x; \boldsymbol{\varsigma}) = \int_0^\infty \upsilon(y) J\left(2\pi (xy)^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\lambda}\right) dy, \tag{1.2.4}$$

with

$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{1}{2\pi i} \int_{\mathcal{C}} G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) x^{-ns} ds. \tag{1.2.5}$$

For $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ and $\boldsymbol{\varsigma} \in \{+, -\}^n$, the function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ defined by (1.2.5) is called a *Bessel function* and the integral in (1.2.5) a *Mellin-Barnes type integral*. We view $J\left(x^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\lambda}\right)$ as the inverse Mellin transform of $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

Suitably choosing the integral contour $\mathcal{C}$, it may be verified that $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is a smooth function of x and is analytic with respect to $\boldsymbol{\lambda}$.

Remark 1.2.1. *The contour of integration* (σ) *does not need modification if the components of* $\boldsymbol{\varsigma}$ *are not identical. For further discussions of the integral in the definition* (1.2.5) *of* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *see Remark* 1.7.10.

Remark 1.2.2. *We have*

$$\Upsilon(x) = \int_{\mathbb{R}^\times} \upsilon(y) J_{(\lambda,\delta)}(xy) dy, \quad x \in \mathbb{R}^\times, \tag{1.2.6}$$

for any $\upsilon \in \mathscr{S}(\mathbb{R}^\times)$, *where the Bessel kernel* $J_{(\lambda,\delta)}$ *is given by*

$$J_{(\lambda,\delta)}(\pm x) = \frac{1}{2} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} (\pm)^\delta \sum_{\boldsymbol{\varsigma} \in \{+,-\}^n} \left(\prod_{\ell=1}^n \varsigma_\ell^{\delta_\ell + \delta} \right) J\big(2\pi x^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\lambda}\big), \quad x \in \mathbb{R}_+. \tag{1.2.7}$$

Moreover,

$$J_{(\lambda,\delta)}(x) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \frac{\operatorname{sgn}(x)^\delta}{4\pi i} \int_{\mathcal{C}} \left(\prod_{\ell=1}^n G_{\delta_\ell + \delta}(s - \lambda_\ell) \right) |x|^{-s} ds. \tag{1.2.8}$$

1.2.2. The formal integral representation of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$

In this section, we assume $n \geqslant 2$. Since we shall manipulate the Fourier type integral transform (1.1.5) only in a symbolic manner, the restrictions on the index $\boldsymbol{\lambda}$ that guarantee the convergence of the iterated integral in (1.1.5) will not be imposed here.

With the parity condition on the weight function υ, (1.1.5) may be written as

$$\begin{aligned} \Upsilon(x) = \frac{\operatorname{sgn}(x)^\eta}{|x|} \sum_{\boldsymbol{\varsigma} \in \{+,-\}^n} & \left(\prod_{\ell=1}^n \varsigma_\ell^{\delta_\ell + \eta} \right) \\ & \int_{\mathbb{R}_+^n} \upsilon\left(\frac{x_1 \ldots x_n}{|x|} \right) \left(\prod_{\ell=1}^n x_\ell^{-\lambda_\ell} e(\varsigma_\ell x_\ell) \right) dx_n dx_{n-1} \ldots dx_1. \end{aligned} \tag{1.2.9}$$

Comparing (1.2.9) with (1.2.1)[VI], we arrive at

$$\Upsilon(x; \boldsymbol{\varsigma}) = \frac{1}{|x|} \int_{\mathbb{R}_+^n} \upsilon\left(\frac{x_1 \ldots x_n}{|x|} \right) \left(\prod_{\ell=1}^n x_\ell^{-\lambda_\ell} e(\varsigma_\ell x_\ell) \right) dx_n dx_{n-1} \ldots dx_1.$$

[VI]To justify our comparison, we use the fact that the associated $2^n \times 2^n$ matrix is equal to the n-th tensor power of $\begin{pmatrix} 1 & (-1)^\eta \\ 1 & (-1)^{1+\eta} \end{pmatrix}$ and hence is invertible.

The change of variables $x_n = |x|y(x_1...x_{n-1})^{-1}$, $x_\ell = y_\ell^{-1}$, $\ell = 1, ..., n-1$, turns this further into

$$\Upsilon(x;\boldsymbol{\varsigma}) = \int_{\mathbb{R}_+^n} \upsilon(y)\,(xy)^{-\lambda_n} \left(\prod_{\ell=1}^{n-1} y_\ell^{\lambda_\ell-\lambda_n-1}\right) e\left(\varsigma_n xyy_1...y_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell y_\ell^{-1}\right) dydy_{n-1}...dy_1. \tag{1.2.10}$$

Comparing now (1.2.10) with (1.2.4), if one *formally* changes the order of the integrations, which is *not* permissible since the integral is *not* absolutely convergent, then $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ can be expressed as a symbolic integral as below,

$$J(2\pi x;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = x^{-n\lambda_n} \int_{\mathbb{R}_+^{n-1}} \left(\prod_{\ell=1}^{n-1} y_\ell^{\lambda_\ell-\lambda_n-1}\right) e\left(\varsigma_n x^n y_1...y_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell y_\ell^{-1}\right) dy_{n-1}...dy_1.$$

Another change of variables $y_\ell = t_\ell x^{-1}$, along with the assumption $\sum_{\ell=1}^n \lambda_\ell = 0$, yields

$$J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = \int_{\mathbb{R}_+^{n-1}} \left(\prod_{\ell=1}^{n-1} t_\ell^{\lambda_\ell-\lambda_n-1}\right) e^{ix\left(\varsigma_n t_1...t_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell t_\ell^{-1}\right)} dt_{n-1}...dt_1. \tag{1.2.11}$$

The above integral is *not* absolutely convergent and will be referred to as the *formal integral representation of* $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$.

Remark 1.2.3. *Before realizing its connection with the Fourier type transform* (1.1.5)*, the formal integral representation of* $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ *was derived by the author from* (1.1.4) *based on a symbolic application of the product-convolution principle of the Mellin transform together with the following formula* ([GR, **3.764**])

$$\Gamma(s)e\left(\pm\frac{s}{4}\right) = \int_0^\infty e^{\pm ix} x^s d^\times x, \quad 0 < \mathfrak{Re}\, s < 1. \tag{1.2.12}$$

Though not specified, this principle is implicitly suggested in Miller and Schmid's work, especially, [MS1, Theorem 4.12, Lemma 6.19] *and* [MS3, (5.22, 5.26)].

1.2.3. The classical cases

The case $n = 1$

Proposition 1.2.4. *Suppose* $n = 1$*. Choose the contour* $\mathcal{C}$ *as in* §1.2.1*;* $\mathcal{C}$ *starts from* $\rho - i\infty$ *and ends at* $\rho + i\infty$*, with* $\rho < -\frac{1}{2}$*, and all the nonpositive integers lie on the left side of* $\mathcal{C}$*. We have*

$$e^{\pm ix} = \frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(s) e\left(\pm\frac{s}{4}\right) x^{-s} ds. \tag{1.2.13}$$

Therefore

$$J(x; \pm, 0) = e^{\pm ix}.$$

Proof. Let $\Re\mathfrak{e}\, z > 0$. For $\Re\mathfrak{e}\, s > 0$, we have the formula

$$\Gamma(s) z^{-s} = \int_0^\infty e^{-zx} x^s d^\times x,$$

where the integral is absolutely convergent. The Mellin inversion formula yields

$$e^{-zx} = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma(s) z^{-s} x^{-s} ds, \quad \sigma > 0.$$

Shifting the contour of integration from (σ) to $\mathcal{C}$, one sees that

$$e^{-zx} = \frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(s) z^{-s} x^{-s} ds.$$

Choose $z = e^{\mp\left(\frac{1}{2}\pi - \epsilon\right)i}$, $\pi > \epsilon > 0$. In view of Stirling's formula, the convergence of the integral above is uniform in ϵ. Therefore, we obtain (1.2.13) by letting $\epsilon \to 0$. Q.E.D.

Remark 1.2.5. *Observe that the integral in* (1.2.12) *is only conditionally convergent, the Mellin inversion formula does not apply in the rigorous sense. Nevertheless,* (1.2.13) *should be view as the Mellin inversion of* (1.2.12).

Remark 1.2.6. *It follows from the proof of Proposition* 1.2.4 *that the formula*

$$e^{-e(a)x} = \frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(s) e(-as)\, x^{-s} ds \tag{1.2.14}$$

is valid for any $a \in \left[-\frac{1}{4}, \frac{1}{4}\right]$.

The case $n = 2$

Proposition 1.2.7. *Let* $\lambda \in \mathbb{C}$. *Then*

$$J(x; \pm, \pm, \lambda, -\lambda) = \pm\pi i e^{\pm\pi i\lambda} H^{(1,2)}_{2\lambda}(2x),$$

$$J(x; \pm, \mp, \lambda, -\lambda) = 2e^{\mp\pi i\lambda} K_{2\lambda}(2x).$$

Here $H^{(1)}_\nu$ *and* $H^{(2)}_\nu$ *are Bessel functions of the third kind, also known as Hankel functions, whereas* K_ν *is the modified Bessel function of the second kind, occasionally called the K-Bessel function.*

Proof. The following formulae are derived from [GR, **6.561** 14-16] along with Euler's reflection formula of the Gamma function.

$$\pi\int_0^\infty J_\nu(2\sqrt{x})x^{s-1}dx = \Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\sin\left(\pi\left(s-\frac{\nu}{2}\right)\right)$$

for $-\frac{1}{2}\Re\mathrm{e}\,\nu < \Re\mathrm{e}\, s < \frac{1}{4}$,

$$-\pi\int_0^\infty Y_\nu(2\sqrt{x})x^{s-1}dx = \Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\cos\left(\pi\left(s-\frac{\nu}{2}\right)\right)$$

for $\frac{1}{2}|\Re\mathrm{e}\,\nu| < \Re\mathrm{e}\, s < \frac{1}{4}$, and

$$2\int_0^\infty K_\nu(2\sqrt{x})x^{s-1}dx = \Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)$$

for $\Re\mathrm{e}\, s > \frac{1}{2}|\Re\mathrm{e}\,\nu|$. For $\Re\mathrm{e}\, s$ in the given ranges, these integrals are absolutely convergent. It follows immediately from the Mellin inversion formula that

$$J(x; \pm, \pm, \lambda, -\lambda) = \pm\pi i e^{\pm\pi i\lambda}(J_{2\lambda}(2x) \pm iY_{2\lambda}(2x)), \quad |\Re\mathrm{e}\,\lambda| < \frac{1}{4},$$

$$J(x; \pm, \mp, \lambda, -\lambda) = 2e^{\mp \pi i \lambda} K_{2\lambda}(2x).$$

In view of the analyticity in λ, the first formula remains valid even if $|\mathfrak{Re}\,\lambda| \geqslant \frac{1}{4}$ by the theory of analytic continuation. Finally, we conclude the proof by recollecting the formula $H_\nu^{(1,2)}(x) = J_\nu(x) \pm i Y_\nu(x)$. Q.E.D.

Remark 1.2.8. *Let* $\lambda = it$ *if* F *is a Maaß form of eigenvalue* $\frac{1}{4} + t^2$ *and weight* k*, and let* $\lambda = \frac{1}{2}(k-1)$ *if* F *is a holomorphic cusp form of weight* k*. Then* F *is parametrized by* $(\boldsymbol{\lambda}, \boldsymbol{\delta}) = (\lambda, -\lambda, k (\mathrm{mod}\ 2), 0)$ *and* $J_F = J_{(\boldsymbol{\lambda}, \boldsymbol{\delta})}$*. From the formula* (1.2.7) *of the Bessel kernel, we have*

$$J_{(\boldsymbol{\lambda}, \boldsymbol{\delta})}(x) = J(2\pi\sqrt{x}; +, +, \lambda, -\lambda) + (-)^k J(2\pi\sqrt{x}; -, -, \lambda, -\lambda),$$

$$J_{(\boldsymbol{\lambda}, \boldsymbol{\delta})}(-x) = J(2\pi\sqrt{x}; +, -, \lambda, -\lambda) + (-)^k J(2\pi\sqrt{x}; -, +, \lambda, -\lambda).$$

Thus, Proposition 1.2.7 *implies* (1.1.8, 1.1.9, 1.1.10).

When $x > 0$ and $|\mathfrak{Re}\,\nu| < 1$, we have the following integral representations of Bessel functions ([Wat, 6.21 (10, 11), 6.22 (13)])

$$H_\nu^{(1,2)}(x) = \pm \frac{2e^{\mp \frac{1}{2}\pi i \nu}}{\pi i} \int_0^\infty e^{\pm i x \cosh r} \cosh(\nu r) dr,$$

$$K_\nu(x) = \frac{1}{\cos\left(\frac{1}{2}\pi\nu\right)} \int_0^\infty \cos(x \sinh r) \cosh(\nu r) dr.$$

The change of variables $t = e^r$ yields

$$\pm \pi i e^{\pm \frac{1}{2}\pi i \nu} H_\nu^{(1,2)}(2x) = \int_0^\infty t^{\nu-1} e^{\pm i x (t + t^{-1})} dt,$$

$$2e^{\pm \frac{1}{2}\pi i \nu} K_\nu(2x) = \int_0^\infty t^{\nu-1} e^{\pm i x (t - t^{-1})} dt.$$

The integrals in these formulae are exactly the formal integrals in (1.2.11) in the case $n = 2$. They *conditionally* converge if $|\mathfrak{Re}\,\nu| < 1$, but diverge if otherwise.

1.2.4. A prototypical example

According to [Wat, 3.4 (3, 6), 3.71 (13)],

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x, \quad J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x.$$

The connection formulae in (1.1.12) ([Wat, 3.61 (5, 6)]) then imply that

$$H^{(1)}_{\frac{1}{2}}(x) = -i \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{ix}, \quad H^{(2)}_{\frac{1}{2}}(x) = i \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-ix}.$$

Moreover, [Wat, 3.71 (13)] reads

$$K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}.$$

Therefore, from the formulae in Proposition 1.2.7 we have

$$J\left(x; \pm, \pm, \tfrac{1}{4}, -\tfrac{1}{4}\right) = \left(\frac{\pi}{x}\right)^{\frac{1}{2}} e^{\pm 2ix \pm \frac{1}{4}\pi i}, \quad J\left(x; \pm, \mp, \tfrac{1}{4}, -\tfrac{1}{4}\right) = \left(\frac{\pi}{x}\right)^{\frac{1}{2}} e^{-2x \mp \frac{1}{4}\pi i}.$$

These formulae admit generalizations to arbitrary rank.

Proposition 1.2.9. *For $\boldsymbol{\varsigma} \in \{+, -\}^n$ we define $L_\pm(\boldsymbol{\varsigma}) = \{\ell : \varsigma_\ell = \pm\}$ and $n_\pm(\boldsymbol{\varsigma}) = |L_\pm(\boldsymbol{\varsigma})|$. Put $\xi(\boldsymbol{\varsigma}) = ie^{\pi i \frac{n_-(\boldsymbol{\varsigma}) - n_+(\boldsymbol{\varsigma})}{2n}} = \mp e^{\mp \pi i \frac{n_\pm(\boldsymbol{\varsigma})}{n}}$. Suppose $\boldsymbol{\lambda} = \frac{1}{n}\left(\frac{n-1}{2}, ..., -\frac{n-1}{2}\right)$. Then*

$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{c(\boldsymbol{\varsigma})}{\sqrt{n}} \left(\frac{2\pi}{x}\right)^{\frac{n-1}{2}} e^{in\xi(\boldsymbol{\varsigma})x}, \tag{1.2.15}$$

with $c(\boldsymbol{\varsigma}) = e\left(\mp \frac{n-1}{8} \mp \frac{n_\pm(\boldsymbol{\varsigma})}{2n} \pm \frac{1}{2n} \sum_{\ell \in L_\pm(\boldsymbol{\varsigma})} \ell\right)$.

Proof. Using the multiplication formula of the Gamma function

$$\prod_{k=0}^{n-1} \Gamma\left(s + \frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - ns} \Gamma(ns), \tag{1.2.16}$$

straightforward calculations yield

$$G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = c_1(\boldsymbol{\varsigma}) (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - n\left(s - \frac{n-1}{2n}\right)} \Gamma\left(n\left(s - \frac{n-1}{2n}\right)\right) e\left(\frac{n_+(\boldsymbol{\varsigma}) - n_-(\boldsymbol{\varsigma})}{4} \cdot s\right),$$

with $c_1(\boldsymbol{\varsigma}) = e\left(\mp \frac{(n+1)n_\pm(\boldsymbol{\varsigma})}{4n} \pm \frac{1}{2n} \sum_{\ell \in L_\pm(\boldsymbol{\varsigma})} \ell\right)$. Inserting this into the contour integral in (1.2.5) and making the change of variables from s to $\frac{1}{n}\left(s + \frac{n-1}{2}\right)$, one arrives at

$$J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = \frac{c_1(\boldsymbol{\varsigma})c_2(\boldsymbol{\varsigma})}{\sqrt{n}} \left(\frac{2\pi}{x}\right)^{\frac{n-1}{2}} \frac{1}{2\pi i} \int_{n\mathcal{C}-\frac{n-1}{2}} \Gamma(s) e\left(\frac{n_+(\boldsymbol{\varsigma}) - n_-(\boldsymbol{\varsigma})}{4n} \cdot s\right) (nx)^{-s} ds,$$

with $c_2(\boldsymbol{\varsigma}) = e\left(\mp \frac{n-1}{8} \pm \frac{(n-1)n_\pm(\boldsymbol{\varsigma})}{4n}\right)$. (1.2.15) now follows from (1.2.14) if the contour $\mathcal{C}$ is suitably chosen. Q.E.D.

1.3. The rigorous interpretation of formal integral representations

We first introduce some new notations. Let $d = n-1$, $\boldsymbol{t} = (t_1, ..., t_d) \in \mathbb{R}_+^d$, $\boldsymbol{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{C}^d$ and $\boldsymbol{\varsigma} = (\varsigma_1, ..., \varsigma_d, \varsigma_{d+1}) \in \{+,-\}^{d+1}$. For $a > 0$ define $\mathbb{S}_a^d = \left\{\boldsymbol{\nu} \in \mathbb{C}^d : |\mathfrak{Re}\, \nu_\ell| < a \text{ for all } \ell = 1, ..., d\right\}$. For $\nu \in \mathbb{C}$ define

$$[\nu]_\alpha = \textstyle\prod_{k=0}^{\alpha-1} (\nu - k),\ (\nu)_\alpha = \prod_{k=0}^{\alpha-1} (\nu + k) \ \text{ if } \alpha \geqslant 1, \quad [\nu]_0 = (\nu)_0 = 1.$$

Denote by $p_{\boldsymbol{\nu}}$ the power function

$$p_{\boldsymbol{\nu}}(\boldsymbol{t}) = \prod_{\ell=1}^d t_\ell^{\nu_\ell - 1},$$

let

$$\theta(\boldsymbol{t};\boldsymbol{\varsigma}) = \varsigma_{d+1} t_1 ... t_d + \sum_{\ell=1}^d \varsigma_\ell t_\ell^{-1},$$

and define the formal integral

$$J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}) = \int_{\mathbb{R}_+^d} p_{\boldsymbol{\nu}}(\boldsymbol{t}) e^{ix\theta(\boldsymbol{t};\boldsymbol{\varsigma})} d\boldsymbol{t}. \tag{1.3.1}$$

One sees that the formal integral representation of $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ given in (1.2.11) is equal to $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ if $\nu_\ell = \lambda_\ell - \lambda_{d+1}$, $\ell = 1, ..., d$.

For $d = 1$, it is seen in §1.2.3 that $J_\nu(x;\boldsymbol{\varsigma})$ is conditionally convergent if and only if $|\mathfrak{Re}\,\nu| < 1$ but fails to be absolutely convergent. When $d \geqslant 2$, we are in a worse scenario. The notion of convergence for multiple integrals is always in the absolute sense. Thus, the d-dimensional multiple integral in (1.3.1) alone does not make any sense, since it is clearly not absolutely convergent.

In the following, we shall address this fundamental convergence issue of the formal integral $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$, relying on its structural simplicity, so that it will be provided with mathematically rigorous meanings[VII]. Moreover, it will be shown that our rigorous interpretation of $J_\nu(x;\boldsymbol{\varsigma})$ is a smooth function of x on $\mathbb{R}_+$ as well as an analytic function of $\boldsymbol{\nu}$ on $\mathbb{C}^d$.

1.3.1. Formal partial integration operators

The most crucial observation is that there are *two* kinds of formal partial integrations. The first kind arises from

$$\partial\left(e^{\varsigma_\ell i x t_\ell^{-1}}\right) = -\varsigma_\ell i x t_\ell^{-2} e^{\varsigma_\ell i x t_\ell^{-1}} \partial t_\ell,$$

and the second kind from

$$\partial\left(e^{\varsigma_{d+1} i x t_1 \ldots t_d}\right) = \varsigma_{d+1} i x t_1 \ldots \widehat{t_\ell} \ldots t_d e^{\varsigma_{d+1} i x t_1 \ldots t_d} \partial t_\ell,$$

where $\widehat{t_\ell}$ means that t_ℓ is omitted from the product.

Definition 1.3.1. *Let*

$$\mathscr{T}(\mathbb{R}_+) = \left\{h \in C^\infty(\mathbb{R}_+) : t^\alpha h^{(\alpha)}(t) \ll_\alpha 1 \textit{ for all } \alpha \in \mathbb{N}\right\}.$$

[VII] It turns out that our rigorous interpretation actually coincides with the *Hadamard partie finie* of the formal integral.

$$\begin{tikzcd}[row sep=small] & \boldsymbol{\nu} + \boldsymbol{e}^d + \boldsymbol{e}_\ell \\ \mathcal{P}_{+,\ell} : \boldsymbol{\nu} \arrow[ur] \arrow[dr] \arrow[dr, bend left] & \\ & \boldsymbol{\nu} + \boldsymbol{e}_\ell \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & \boldsymbol{\nu} - \boldsymbol{e}^d \\ \mathcal{P}_{-,\ell} : \boldsymbol{\nu} \arrow[ur] \arrow[ur, bend right] \arrow[dr] & \\ & \boldsymbol{\nu} - \boldsymbol{e}^d - \boldsymbol{e}_\ell \end{tikzcd}$$

Index shifts

For $h(\boldsymbol{t}) \in \bigotimes^d \mathscr{T}(\mathbb{R}_+)$, in the sense that $h(\boldsymbol{t})$ is a linear combination of functions of the form $\prod_{\ell=1}^d h_\ell(t_\ell)$, define the integral

$$J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};h) = \int_{\mathbb{R}_+^d} h(\boldsymbol{t}) p_{\boldsymbol{\nu}}(\boldsymbol{t}) e^{ix\theta(\boldsymbol{t};\boldsymbol{\varsigma})} d\boldsymbol{t}.$$

We call $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};h)$ a J-integral of index $\boldsymbol{\nu}$. Let us introduce an auxiliary space

$$\mathscr{J}_{\boldsymbol{\nu}}(\boldsymbol{\varsigma}) = \mathrm{Span}_{\mathbb{C}[x^{-1}]} \left\{ J_{\boldsymbol{\nu}'}(x;\boldsymbol{\varsigma};h) : \boldsymbol{\nu}' \in \boldsymbol{\nu} + \mathbb{Z}^d, h \in \bigotimes^d \mathscr{T}(\mathbb{R}_+) \right\}.$$

Here $\mathbb{C}[x^{-1}]$ is the ring of polynomials of variable x^{-1} and complex coefficients. Finally, we define $\mathcal{P}_{+,\ell}$ and $\mathcal{P}_{-,\ell}$ to be the two $\mathbb{C}[x^{-1}]$-linear operators on the space $\mathscr{J}_{\boldsymbol{\nu}}(\boldsymbol{\varsigma})$, in symbolic notion, as follows,

$$\begin{aligned} \mathcal{P}_{+,\ell}(J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};h)) = {} & \varsigma_\ell \varsigma_{d+1} J_{\boldsymbol{\nu}+\boldsymbol{e}^d+\boldsymbol{e}_\ell}(x;\boldsymbol{\varsigma};h) \\ & - \varsigma_\ell i(\nu_\ell + 1) x^{-1} J_{\boldsymbol{\nu}+\boldsymbol{e}_\ell}(x;\boldsymbol{\varsigma};h) - \varsigma_\ell i x^{-1} J_{\boldsymbol{\nu}+\boldsymbol{e}_\ell}(x;\boldsymbol{\varsigma};t_\ell \partial_\ell h), \end{aligned}$$

$$\begin{aligned} \mathcal{P}_{-,\ell}(J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};h)) = {} & \varsigma_\ell \varsigma_{d+1} J_{\boldsymbol{\nu}-\boldsymbol{e}^d-\boldsymbol{e}_\ell}(x;\boldsymbol{\varsigma};h) \\ & + \varsigma_{d+1} i(\nu_\ell - 1) x^{-1} J_{\boldsymbol{\nu}-\boldsymbol{e}^d}(x;\boldsymbol{\varsigma};h) + \varsigma_{d+1} i x^{-1} J_{\boldsymbol{\nu}-\boldsymbol{e}^d}(x;\boldsymbol{\varsigma};t_\ell \partial_\ell h), \end{aligned}$$

where $\boldsymbol{e}_\ell = (\underbrace{0, ..., 0, 1}_{\ell}, 0 ..., 0)$ and $\boldsymbol{e}^d = (1, ..., 1)$, and $\partial_\ell h$ is the abbreviated $\partial h / \partial t_\ell$.

The formulations of $\mathcal{P}_{+,\ell}$ and $\mathcal{P}_{-,\ell}$ are quite involved at a first glance. However, the most essential feature of these operators is simply *index shift*!

Observation. *After the operation of $\mathcal{P}_{+,\ell}$ on a J-integral, all the indices of the three resulting J-integrals are nondecreasing and the increment of the ℓ-th index is one greater than the others. The operator $\mathcal{P}_{-,\ell}$ has the effect of decreasing all indices by one except possibly two for the ℓ-th index.*

Lemma 1.3.2. *Let notations be as above.*

(1). *Let $h(\boldsymbol{t}) = \prod_{\ell=1}^{d} h_\ell(t_\ell)$. Suppose that the set $\{1, 2, ..., d\}$ splits into two subsets L_+ and L_- such that*

- *h_ℓ vanishes at infinity if $\ell \in L_-$, and*

- *h_ℓ vanishes in a neighbourhood of zero if $\ell \in L_+$.*

If $\Re\mathrm{e}\, \nu_\ell > 0$ for all $\ell \in L_-$ and $\Re\mathrm{e}\, \nu_\ell < 0$ for all $\ell \in L_+$, then the J-integral $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h)$ absolutely converges.

(2). *Assume the same conditions in* (1). *Moreover, suppose that $\Re\mathrm{e}\, \nu_\ell > 1$ for all $\ell \in L_-$ and $\Re\mathrm{e}\, \nu_\ell < -1$ for all $\ell \in L_+$. Then, for $\ell \in L_-$, all the three J-integrals in the definition of $\mathcal{P}_{+,\ell}(J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h))$ are absolutely convergent and the operation of $\mathcal{P}_{+,\ell}$ on $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h)$ is the actual partial integration of the first kind on the integral over dt_ℓ. Similarly, for $\ell \in L_+$, the operation of $\mathcal{P}_{-,\ell}$ preserves absolute convergence and is the actual partial integration of the second kind on the integral over dt_ℓ.*

(3). *$\mathcal{P}_{+,\ell}$ and $\mathcal{P}_{-,\ell}$ commute with $\mathcal{P}_{+,k}$ and $\mathcal{P}_{-,k}$ if $\ell \neq k$.*

(3). *$\mathcal{P}_{+,l}$ and $\mathcal{P}_{-,l}$ commute with $\mathcal{P}_{+,k}$ and $\mathcal{P}_{-,k}$ if $l \neq k$.*

(4). *Let $\alpha \in \mathbb{N}$. $\mathcal{P}^{\alpha}_{+,l}(J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h))$ is a linear combination of*

$$[\nu_l - 1]_{\alpha_3} x^{-\alpha+\alpha_1} J_{\boldsymbol{\nu}+\alpha_1\boldsymbol{e}^d+\alpha\boldsymbol{e}_l}(x; \boldsymbol{\varsigma}; t_l^{\alpha_2}\partial_l^{\alpha_2} h),$$

and $\mathcal{P}^{\alpha}_{-,l}(J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h))$ is a linear combination of

$$[\nu_l - 1]_{\alpha_3} x^{-\alpha+\alpha_1} J_{\boldsymbol{\nu}-\alpha\boldsymbol{e}^d-\alpha_1\boldsymbol{e}_l}(x; \boldsymbol{\varsigma}; t_l^{\alpha_2}\partial_l^{\alpha_2} h),$$

for $\alpha_1 + \alpha_2 + \alpha_3 \leqslant \alpha$. The coefficients of these linear combinations may be uniformly bounded by a constant depending only on α.

Proof. (1-3) are obvious. The two statements in (4) follow from calculating

$$x^{-\alpha} t_l^{\alpha+\alpha_0} \partial_l^{\alpha_0} \left(h(\boldsymbol{t}) p_{\boldsymbol{\nu}}(\boldsymbol{t}) e^{\varsigma_{d+1} i x t_1 \dots t_d} \right) e^{ix \sum_{k=1}^d \varsigma_k t_k^{-1}}, \quad \alpha_0 \leqslant \alpha,$$

and

$$x^{-\alpha} \partial_l^{\alpha} \left(h(\boldsymbol{t}) p_{\boldsymbol{\nu} - \alpha \boldsymbol{e}^d + \alpha \boldsymbol{e}_l}(\boldsymbol{t}) e^{ix \sum_{k=1}^d \varsigma_k t_k^{-1}} \right) e^{\varsigma_{d+1} i x t_1 \dots t_d}.$$

For the latter, one applies the following formula

$$\frac{d^\alpha \left(e^{a t^{-1}} \right)}{dt^\alpha} = (-)^\alpha \sum_{\beta=1}^{\alpha} \frac{\alpha!(\alpha-1)!}{(\alpha-\beta)!\beta!(\beta-1)!} a^\beta t^{-\alpha-\beta} e^{a t^{-1}}, \quad \alpha \in \mathbb{N}_+, \ a \in \mathbb{C}.$$

Q.E.D.

1.3.2. Partitioning the integral $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$

Let I be a finite set that includes $\{+, -\}$ and let

$$\sum_{\varrho \in I} h_\varrho(t) \equiv 1, \quad t \in \mathbb{R}_+,$$

be a partition of unity on $\mathbb{R}_+$ such that each h_ϱ is a function in $\mathscr{T}(\mathbb{R}_+)$, $h_-(t) \equiv 1$ on a neighbourhood of zero and $h_+(t) \equiv 1$ for large t. Put $h_{\boldsymbol{\varrho}}(\boldsymbol{t}) = \prod_{\ell=1}^d h_{\varrho_\ell}(t_\ell)$ for $\boldsymbol{\varrho} = (\varrho_1, ..., \varrho_d) \in I^d$. We partition the integral $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ into a finite sum of J-integrals

$$J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}) = \sum_{\boldsymbol{\varrho} \in I^d} J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \boldsymbol{\varrho}),$$

with

$$J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \boldsymbol{\varrho}) = J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; h_{\boldsymbol{\varrho}}) = \int_{\mathbb{R}_+^d} h_{\boldsymbol{\varrho}}(\boldsymbol{t}) p_{\boldsymbol{\nu}}(\boldsymbol{t}) e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} d\boldsymbol{t}.$$

1.3.3. The definition of $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$

Let $a > 0$ and assume $\boldsymbol{\nu} \in \mathbb{S}_a$. Let $A \geqslant a+2$ be an integer. For $\boldsymbol{\varrho} \in I^d$ denote $L_\pm(\boldsymbol{\varrho}) = \{\ell : \varrho_\ell = \pm\}$.

We first treat $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ in the case when both $L_+(\boldsymbol{\varrho})$ and $L_-(\boldsymbol{\varrho})$ are nonempty. Define $\mathcal{P}_{+,\boldsymbol{\varrho}} = \prod_{\ell \in L_-(\boldsymbol{\varrho})} \mathcal{P}_{+,\ell}$. This is well-defined due to commutativity (Lemma 1.3.2 (3)). By Lemma 1.3.2 (4) we find that $\mathcal{P}_{+,\boldsymbol{\varrho}}^{2A}(J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}))$ is a linear combination of

$$\left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} [\nu_\ell - 1]_{\alpha_{3,\ell}}\right) x^{-2A|L_-(\boldsymbol{\varrho})| + \sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell}} \cdot J_{\boldsymbol{\nu} + (\sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell}) \boldsymbol{e}^d + 2A \sum_{\ell \in L_-(\boldsymbol{\varrho})} \boldsymbol{e}_\ell} \left(x;\boldsymbol{\varsigma};\left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} t_\ell^{\alpha_{2,\ell}} \partial_\ell^{\alpha_{2,\ell}}\right) h_{\boldsymbol{\varrho}}\right), \tag{1.3.2}$$

with $\alpha_{1,\ell} + \alpha_{2,\ell} + \alpha_{3,\ell} \leqslant 2A$ for each $\ell \in L_-(\boldsymbol{\varrho})$. After this, we choose $\ell_+ \in L_+(\boldsymbol{\varrho})$ and apply $\mathcal{P}_{-,\ell_+}^{A + \sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell}}$ on the J-integral in (1.3.2). By Lemma 1.3.2 (4) we obtain a linear combination of

$$[\nu_{\ell_+} - 1]_{\alpha_3} \left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} [\nu_\ell - 1]_{\alpha_{3,\ell}}\right) x^{-A(2|L_-(\boldsymbol{\varrho})|+1) + \alpha_1} \cdot J_{\boldsymbol{\nu} - A\boldsymbol{e}^d + 2A \sum_{\ell \in L_-(\boldsymbol{\varrho})} \boldsymbol{e}_\ell - \alpha_1 \boldsymbol{e}_{\ell_+}} \left(x;\boldsymbol{\varsigma};\left(t_{\ell_+}^{\alpha_2} \partial_{\ell_+}^{\alpha_2} \prod_{\ell \in L_-(\boldsymbol{\varrho})} t_\ell^{\alpha_{2,\ell}} \partial_\ell^{\alpha_{2,\ell}}\right) h_{\boldsymbol{\varrho}}\right), \tag{1.3.3}$$

with $\alpha_1 + \alpha_2 + \alpha_3 \leqslant \sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell} + A$. It is easy to verify that the real part of the ℓ-th index of the J-integral in (1.3.3) is positive if $\ell \in L_-(\boldsymbol{\varrho})$ and negative if $\ell \in L_+(\boldsymbol{\varrho})$. Therefore, the J-integral in (1.3.3) is absolutely convergent according to Lemma 1.3.2 (1). We define $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ to be the total linear combination of all the J-integrals obtained after these two steps of operations.

When $L_-(\boldsymbol{\varrho}) \neq \emptyset$ but $L_+(\boldsymbol{\varrho}) = \emptyset$, we define $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}) = \mathcal{P}_{+,\boldsymbol{\varrho}}^{A}(J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}))$. It is a linear combination of

$$\left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} [\nu_\ell - 1]_{\alpha_{3,\ell}}\right) x^{-A|L_-(\boldsymbol{\varrho})| + \sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell}} \cdot J_{\boldsymbol{\nu} + (\sum_{\ell \in L_-(\boldsymbol{\varrho})} \alpha_{1,\ell}) \boldsymbol{e}^d + A \sum_{\ell \in L_-(\boldsymbol{\varrho})} \boldsymbol{e}_\ell} \left(x;\boldsymbol{\varsigma};\left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} t_\ell^{\alpha_{2,\ell}} \partial_\ell^{\alpha_{2,\ell}}\right) h_{\boldsymbol{\varrho}}\right), \tag{1.3.4}$$

with $\alpha_{1,\ell} + \alpha_{2,\ell} + \alpha_{3,\ell} \leqslant A$. The J-integral in (1.3.4) is absolutely convergent.

When $L_+(\varrho) \neq \emptyset$ but $L_-(\varrho) = \emptyset$, we choose $\ell_+ \in L_+(\varrho)$ and define $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho) = \mathcal{P}^A_{-,\ell_+}(J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho))$. This is a linear combination of

$$[\nu_{\ell_+} - 1]_{\alpha_3} x^{-A+\alpha_1} J_{\boldsymbol{\nu} - A\boldsymbol{e}^d - \alpha_1 \boldsymbol{e}_{\ell_+}} \left(x; \boldsymbol{\varsigma}; t_{\ell_+}^{\alpha_2} \partial_{\ell_+}^{\alpha_2} h_\varrho \right), \tag{1.3.5}$$

with $\alpha_1 + \alpha_2 + \alpha_3 \leqslant A$. The J-integral in (1.3.5) is again absolutely convergent.

Finally, when both $L_-(\varrho)$ and $L_+(\varrho)$ are empty, we put $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho) = J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$.

Lemma 1.3.3. *The definition of $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ is independent on A and the choice of $\ell_+ \in L_+(\varrho)$.*

Proof. We shall treat the case when both $L_+(\varrho)$ and $L_-(\varrho)$ are nonempty. The other cases are similar and simpler.

Starting from the $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ defined with A, we conduct the following operations in succession for all $\ell \in L_-(\varrho)$: $\mathcal{P}_{+,\ell}$ twice and then $\mathcal{P}_{-,\ell_+}$ once, twice or three times on each resulting J-integral so that the increment of the ℓ-th index is exactly one. In this way, one arrives at the $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ defined with $A + 1$. In view of the assumption $A \geqslant a + 2$, absolute convergence is maintained at each step due to Lemma 1.3.2 (1). Moreover, in our settings, the operations $\mathcal{P}_{+,\ell}$ and $\mathcal{P}_{-,\ell_+}$ are actual partial integrations (Lemma 1.3.2 (2)), so the value is preserved in the process. In conclusion, $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ is independent on A.

Suppose $\ell_+, k_+ \in L_+(\varrho)$. Repeating the process described in the last paragraph A times, but with ℓ_+ replaced by k_+, the $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ defined with ℓ_+ turns into a sum of integrals of an expression symmetric about ℓ_+ and k_+. Interchanging ℓ_+ and k_+ throughout the arguments above, the $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ defined with k_+ is transformed into the same sum of integrals. Thus we conclude that $\mathbb{J}_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho)$ is independent on the choice of ℓ_+. Q.E.D.

Putting these together, we define

$$\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}) = \sum_{\boldsymbol{\varrho}\in I^d} \mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}),$$

and call $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ the *rigorous interpretation of* $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$. The definition of $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ is clearly independent on the partition of unity $\{h_\varrho\}_{\varrho\in I}$ on $\mathbb{R}_+$.

Uniform convergence of the J-integrals in (1.3.3, 1.3.4, 1.3.5) with respect to $\boldsymbol{\nu}$ implies that $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ is an analytic function of $\boldsymbol{\nu}$ on $\mathbb{S}_a^d$ and hence on the whole $\mathbb{C}^d$ since a was arbitrary. Moreover, for any nonnegative integer j, if one chooses $A \geqslant a + j + 2$, differentiating j times under the integral sign for the J-integrals in (1.3.3, 1.3.4, 1.3.5) is legitimate. Therefore, $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ is a smooth function of x.

Henceforth, with some ambiguity, we shall write $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ and $\mathbb{J}_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ as $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ and $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ respectively.

1.4. Equality between $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ and $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$

The goal of this section is to prove that the Bessel function $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is indeed equal to the rigorous interpretation of its formal integral representation $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$.

Proposition 1.4.1. *Suppose that $\boldsymbol{\lambda} \in \mathbb{L}^d$ and $\boldsymbol{\nu} \in \mathbb{C}^d$ satisfy $\nu_\ell = \lambda_\ell - \lambda_{d+1}$, $\ell = 1, ..., d$. Then*

$$J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}).$$

To prove this proposition, one first needs to know how the iterated integral $\Upsilon(x;\boldsymbol{\varsigma})$ given in (1.2.10) is interpreted (compare [MS1, §6] and [MS3, §5]).

Suppose that $\Re\mathrm{e}\,\lambda_1 > ... > \Re\mathrm{e}\,\lambda_d > \Re\mathrm{e}\,\lambda_{d+1}$. Let $\upsilon \in \mathscr{S}(\mathbb{R}_+)$ be a Schwartz function on $\mathbb{R}_+$. Define

$$\Upsilon_{d+1}(x;\boldsymbol{\varsigma}) = \int_{\mathbb{R}_+} \upsilon(y) y^{-\lambda_{d+1}} e(\varsigma_{d+1} xy) dy, \quad x \in \mathbb{R}_+, \tag{1.4.1}$$

and for each $\ell = 1, ..., d$ recursively define

$$\Upsilon_\ell (x; \boldsymbol{\varsigma}) = \int_{\mathbb{R}_+} \Upsilon_{\ell+1} (y; \boldsymbol{\varsigma}) \, y^{\lambda_\ell - \lambda_{\ell+1} - 1} e \left(\varsigma_\ell x y^{-1} \right) dy, \quad x \in \mathbb{R}_+. \tag{1.4.2}$$

Lemma 1.4.2. *Suppose that* $\Re\mathrm{e}\, \lambda_1 > ... > \Re\mathrm{e}\, \lambda_d > \Re\mathrm{e}\, \lambda_{d+1}$. *Recall the definition of* $\mathscr{T}(\mathbb{R}_+)$ *given in Definition* 1.3.1, *and define the space* $\mathscr{T}_\infty(\mathbb{R}_+)$ *of all functions in* $\mathscr{T}(\mathbb{R}_+)$ *that decay rapidly at infinity, along with all their derivatives. Then* $\Upsilon_\ell (x; \boldsymbol{\varsigma}) \in \mathscr{T}_\infty(\mathbb{R}_+)$ *for each* $\ell = 1, ..., d + 1$.

Proof. In the case $\ell = d + 1$, $\Upsilon_{d+1}(x; \boldsymbol{\varsigma})$ is the Fourier transform of a Schwartz function on $\mathbb{R}$ (supported in $\mathbb{R}_+$) and hence is actually a Schwartz function on $\mathbb{R}$. In particular, $\Upsilon_{d+1}(x; \boldsymbol{\varsigma}) \in \mathscr{T}_\infty(\mathbb{R}_+)$. One may also prove this directly via performing partial integration and differentiation under the integral sign on the integral in (1.4.1).

Suppose that $\Upsilon_{\ell+1} (x; \boldsymbol{\varsigma}) \in \mathscr{T}_\infty(\mathbb{R}_+)$. The condition $\Re\mathrm{e}\, \lambda_\ell > \Re\mathrm{e}\, \lambda_{\ell+1}$ secures the convergence of the integral in (1.4.2). Partial integration has the effect of dividing $\varsigma_\ell 2\pi i x$ and results in an integral of the same type but with *the power of* y *raised by one*, so repeating this yields the rapid decay of $\Upsilon_\ell (x; \boldsymbol{\varsigma})$. Moreover, differentiation under the integral sign *decreases the power of* y *by one*, so multiple differentiating $\Upsilon_\ell (x; \boldsymbol{\varsigma})$ is legitimate after repeated partial integrations. From these, it is straightforward to prove that $\Upsilon_\ell(x; \boldsymbol{\varsigma}) \in \mathscr{T}(\mathbb{R}_+)$. Finally, keeping repeating partial integrations yields the rapid decay of all the derivatives of $\Upsilon_\ell (x; \boldsymbol{\varsigma})$. Q.E.D.

The change of variables from y to xy in (1.4.2) yields

$$\Upsilon_\ell (x; \boldsymbol{\varsigma}) = \int_{\mathbb{R}_+} \Upsilon_{\ell+1} (xy; \boldsymbol{\varsigma}) \, x^{\lambda_\ell - \lambda_{\ell+1}} y^{\lambda_\ell - \lambda_{\ell+1} - 1} e \left(\varsigma_\ell y^{-1} \right) dy.$$

Some calculations then show that $\Upsilon_1 (x; \boldsymbol{\varsigma})$ is equal to the iterated integral

$$x^{\nu_1} \int_{\mathbb{R}_+^{d+1}} \upsilon (y) \, y^{-\lambda_{d+1}} \left(\prod_{\ell=1}^{d} y_\ell^{\nu_\ell - 1} \right) e \left(\varsigma_{d+1} x y y_1 ... y_d + \sum_{\ell=1}^{d} \varsigma_\ell y_\ell^{-1} \right) dy dy_d ... dy_1. \tag{1.4.3}$$

Comparing (1.4.3) with (1.2.10), one sees that $\Upsilon(x;\boldsymbol{\varsigma}) = x^{-\lambda_1}\Upsilon_1(x;\boldsymbol{\varsigma})$.

The (actual) partial integration $\mathcal{P}_\ell$ on the integral over dy_ℓ is in correspondence with $\mathcal{P}_{+,\ell}$, whereas the partial integration $\mathcal{P}_{d+1}$ on the integral over dy has the similar effect as $\mathcal{P}_{-,\ell_+}$ of decreasing the powers of all the y_ℓ by one. These observations are crucial to our proof of Proposition 1.4.1 as follows.

Proof of Proposition 1.4.1. Suppose that $\mathfrak{Re}\,\lambda_1 > ... > \mathfrak{Re}\,\lambda_d > \mathfrak{Re}\,\lambda_{d+1}$. We first partition the integral over dy_ℓ in (1.4.3), for each $\ell = 1, ..., d$, into a sum of integrals according to a partition of unity $\{h^{\mathrm{o}}_\varrho\}_{\varrho\in I}$ of $\mathbb{R}_+$. These partitions result in a partition of the integral (1.4.3) into the sum

$$\Upsilon_1\,(x;\boldsymbol{\varsigma}) = \sum_{\boldsymbol{\varrho}\in I^d} \Upsilon_1(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}),$$

with

$$\begin{aligned}\Upsilon_1(x;\boldsymbol{\varsigma};\boldsymbol{\varrho}) = x^{\nu_1}\int_{\mathbb{R}_+^{d+1}} \upsilon\,(y)\,y^{-\lambda_{d+1}}\left(\prod_{\ell=1}^d h^{\mathrm{o}}_{\varrho_\ell}(y_\ell)y_\ell^{\nu_\ell-1}\right)&\\ e\left(\varsigma_{d+1}xyy_1...y_d + \sum_{\ell=1}^d \varsigma_\ell y_\ell^{-1}\right)dy dy_d...dy_1&.\end{aligned}\tag{1.4.4}$$

We now conduct the operations in §1.3.3 with $\mathcal{P}_{+,\ell}$ replaced by $\mathcal{P}_\ell$ and $\mathcal{P}_{-,\ell_+}$ by $\mathcal{P}_{d+1}$ to each integral $\Upsilon_1(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ defined in (1.4.4). While preserving the value, these partial integrations turn the iterated integral $\Upsilon_1(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ into an absolutely convergent multiple integral. We are then able to move the innermost integral over dy to the outermost place. The integral over $dy_d...dy_1$ now becomes the inner integral. Making the change of variables $y_\ell = t_\ell(xy)^{-\frac{1}{d+1}}$ to the inner integral over $dy_d...dy_1$, each partial integration $\mathcal{P}_\ell$ that we did turns into $\mathcal{P}_{+,\ell}$. By the same arguments in the proof of Lemma 1.3.3 showing that $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ is independent on the choice of $\ell_+ \in L_+(\boldsymbol{\varrho})$, the operations of $\mathcal{P}_{d+1}$ that we conducted at the beginning may be reversed and substituted by those of $\mathcal{P}_{-,\ell_+}$. It follows that the inner integral over

$dy_d...dy_1$ is equal to $x^{\lambda_1}\upsilon(y)J_{\boldsymbol{\nu}}\left(2\pi(xy)^{\frac{1}{d+1}};\boldsymbol{\varsigma};\boldsymbol{\varrho}\right)$, with $h_\varrho(t) = h^\circ_\varrho\left(t(xy)^{-\frac{1}{d+1}}\right)$. Summing over $\boldsymbol{\varrho} \in I^d$, we conclude that

$$\Upsilon(x;\boldsymbol{\varsigma}) = x^{-\lambda_1}\Upsilon_1(x;\boldsymbol{\varsigma}) = \int_{\mathbb{R}_+} \upsilon(y)J_{\boldsymbol{\nu}}\left(2\pi(xy)^{\frac{1}{d+1}};\boldsymbol{\varsigma}\right) dy.$$

Therefore, in view of (1.2.4), we have $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$. This equality holds true universally due to the principle of analytic continuation. Q.E.D.

In view of Proposition 1.4.1, we shall subsequently assume that $\boldsymbol{\lambda} \in \mathbb{L}^d$ and $\boldsymbol{\nu} \in \mathbb{C}^d$ always satisfy the relations $\nu_\ell = \lambda_\ell - \lambda_{d+1}$, $\ell = 1, ..., d$.

1.5. H-Bessel functions and K-Bessel functions

According to Proposition 1.2.7, $J_{2\lambda}(x;\pm,\pm) = J(x;\pm,\pm,\lambda,-\lambda)$ is a Hankel function, and $J_{2\lambda}(x;\pm,\mp) = J(x;\pm,\mp,\lambda,-\lambda)$ is a K-Bessel function. There is a remarkable difference between the behaviours of Hankel functions and the K-Bessel function for large argument. The Hankel functions oscillate and decay proportionally to $\frac{1}{\sqrt{x}}$, whereas the K-Bessel function exponentially decays. On the other hand, this phenomena also arises in higher rank for the prototypical example shown in Proposition 1.2.9.

In the following, we shall show that such a categorization stands in general for the Bessel functions $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ of an arbitrary index $\boldsymbol{\nu}$. For this, we shall analyze each integral $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma};\boldsymbol{\varrho})$ in the rigorous interpretation of $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ using *the method of stationary phase*.

First of all, the asymptotic behaviour of $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ for large argument should rely on the existence of a stationary point of the phase function $\theta(\boldsymbol{t};\boldsymbol{\varsigma})$ on $\mathbb{R}_+^d$. We have

$$\theta'(\boldsymbol{t};\boldsymbol{\varsigma}) = \left(\varsigma_{d+1}t_1...\widehat{t_\ell}...t_d - \varsigma_\ell t_\ell^{-2}\right)_{\ell=1}^d.$$

A stationary point of $\theta(\boldsymbol{t};\boldsymbol{\varsigma})$ exists in $\mathbb{R}_+^d$ if and only if $\varsigma_1 = ... = \varsigma_d = \varsigma_{d+1}$, in which case it is equal to $\boldsymbol{t}_0 = (1, ..., 1)$.

Terminology 1.5.1. *We write* $H^{\pm}_{\boldsymbol{\nu}}(x) = J_{\boldsymbol{\nu}}(x; \pm, ..., \pm)$*,* $H^{\pm}(x; \boldsymbol{\lambda}) = J(x; \pm, ... \pm, \boldsymbol{\lambda})$ *and call them H-Bessel functions. If two of the signs* $\varsigma_1, ..., \varsigma_d, \varsigma_{d+1}$ *are different, then* $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$*, or* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$*, is called a K-Bessel function.*

Preparations

We shall retain the notations in §1.3. Moreover, for our purpose we choose a partition of unity $\{h_\varrho\}_{\varrho \in \{-,0,+\}}$ on $\mathbb{R}_+$ such that h_-, h_0 and h_+ are functions in $\mathscr{T}(\mathbb{R}_+)$ supported on $K_- = \left(0, \frac{1}{2}\right]$, $K_0 = \left[\frac{1}{4}, 4\right]$ and $K_+ = [2, \infty)$ respectively. Put $K_{\boldsymbol{\varrho}} = \prod_{\ell=1}^{d} K_{\varrho_\ell}$ and $h_{\boldsymbol{\varrho}}(\boldsymbol{t}) = \prod_{\ell=1}^{d} h_{\varrho_\ell}(t_\ell)$ for $\boldsymbol{\varrho} \in \{-, 0, +\}^d$. Note that $\boldsymbol{t}_0$ is enclosed in the central hypercube $K_{\boldsymbol{0}}$. According to this partition of unity, $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ is partitioned into the sum of 3^d integrals $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \boldsymbol{\varrho})$. In view of (1.3.3, 1.3.4, 1.3.5), $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \boldsymbol{\varrho})$ is a $\mathbb{C}[x^{-1}]$-linear combination of absolutely convergent J-integrals of the form

$$J_{\boldsymbol{\nu}'}(x; \boldsymbol{\varsigma}; h) = \int_{\mathbb{R}_+^d} h(\boldsymbol{t}) p_{\boldsymbol{\nu}'}(\boldsymbol{t}) e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} d\boldsymbol{t}. \tag{1.5.1}$$

Here $h \in \bigotimes^d \mathscr{T}(\mathbb{R}_+)$ is supported in $K_{\boldsymbol{\varrho}}$, and $\boldsymbol{\nu}' \in \boldsymbol{\nu} + \mathbb{Z}^d$ satisfies

$$\Re\mathrm{e}\, \nu'_\ell - \Re\mathrm{e}\, \nu_\ell \geqslant A \text{ if } \ell \in L_-(\boldsymbol{\varrho}), \text{ and } \Re\mathrm{e}\, \nu'_\ell - \Re\mathrm{e}\, \nu_\ell \leqslant -A \text{ if } \ell \in L_+(\boldsymbol{\varrho}), \tag{1.5.2}$$

with $A > \max\{|\Re\mathrm{e}\, \nu_\ell|\} + 2$.

1.5.1. Estimates for $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \boldsymbol{\varrho})$ with $\boldsymbol{\varrho} \neq \boldsymbol{0}$

Let

$$\Theta(\boldsymbol{t}; \boldsymbol{\varsigma}) = \sum_{\ell=1}^{d} \left(t_\ell \partial_\ell \theta(\boldsymbol{t}; \boldsymbol{\varsigma})\right)^2 = \sum_{\ell=1}^{d} \left(\varsigma_{d+1} t_1 ... t_d - \varsigma_\ell t_\ell^{-1}\right)^2. \tag{1.5.3}$$

Lemma 1.5.2. *Let* $\boldsymbol{\varrho} \neq \boldsymbol{0}$*. We have for all* $\boldsymbol{t} \in K_{\boldsymbol{\varrho}}$

$$\Theta(\boldsymbol{t}; \boldsymbol{\varsigma}) \geqslant \frac{1}{16}.$$

Proof. Instead, we shall prove

$$\max \left\{ \left| \varsigma_{d+1} t_1 ... t_d - \varsigma_\ell t_\ell^{-1} \right| : \boldsymbol{t} \in \mathbb{R}_+^d \smallsetminus K_{\mathbf{0}} \text{ and } \ell = 1, ..., d \right\} \geqslant \frac{1}{4}.$$

Firstly, if $t_1 ... t_d < \frac{3}{4}$, then there exists $t_\ell < 1$ and hence $\left| \varsigma_{d+1} t_1 ... t_d - \varsigma_\ell t_\ell^{-1} \right| > 1 - \frac{3}{4} = \frac{1}{4}$. Similarly, if $t_1 ... t_d > \frac{7}{4}$, then there exists $t_\ell > 1$ and hence $\left| \varsigma_{d+1} t_1 ... t_d - \varsigma_\ell t_\ell^{-1} \right| > \frac{7}{4} - 1 > \frac{1}{4}$. Finally, suppose that $\frac{3}{4} \leqslant t_1 ... t_d \leqslant \frac{7}{4}$, then for our choice of $\boldsymbol{t}$ there exists ℓ such that $t_\ell \notin \left(\frac{1}{2}, 2 \right)$, and therefore we still have $\left| \varsigma_{d+1} t_1 ... t_d - \varsigma_\ell t_\ell^{-1} \right| \geqslant \frac{1}{4}$. Q.E.D.

Using (1.5.3), we rewrite the J-integral $J_{\boldsymbol{\nu}'}(x; \boldsymbol{\varsigma}; h)$ in (1.5.1) as below,

$$\sum_{\ell=1}^{d} \int_{\mathbb{R}_+^d} h(\boldsymbol{t}) \left(\varsigma_{d+1} p_{\boldsymbol{\nu}' + \boldsymbol{e}^d + \boldsymbol{e}_\ell}(\boldsymbol{t}) - \varsigma_\ell p_{\boldsymbol{\nu}'}(\boldsymbol{t}) \right) \Theta(\boldsymbol{t}; \boldsymbol{\varsigma})^{-1} \cdot \partial_\ell \theta(\boldsymbol{t}; \boldsymbol{\varsigma}) e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} d\boldsymbol{t}. \tag{1.5.4}$$

We now make use of the *third* kind of partial integrations arising from

$$\partial \left(e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} \right) = ix \cdot \partial_\ell \theta(\boldsymbol{t}; \boldsymbol{\varsigma}) e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} \partial t_\ell.$$

For the ℓ-th integral in (1.5.4), we apply the corresponding partial integration of the third kind. In this way, (1.5.4) turns into

$$\begin{aligned}
& - (ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}_+^d} t_\ell \partial_\ell h \left(\varsigma_{d+1} p_{\boldsymbol{\nu}' + \boldsymbol{e}^d} - \varsigma_\ell p_{\boldsymbol{\nu}' - \boldsymbol{e}_\ell} \right) \Theta^{-1} e^{ix\theta} d\boldsymbol{t} \\
& - (ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}_+^d} h \left(\varsigma_{d+1} (\nu'_\ell + 1) p_{\boldsymbol{\nu}' + \boldsymbol{e}^d} - \varsigma_\ell (\nu'_\ell - 1) p_{\boldsymbol{\nu}' - \boldsymbol{e}_\ell} \right) \Theta^{-1} e^{ix\theta} d\boldsymbol{t} \\
& + \varsigma_{d+1} 2 d^2 (ix)^{-1} \int_{\mathbb{R}_+^d} h p_{\boldsymbol{\nu}' + 3\boldsymbol{e}^d} \Theta^{-2} e^{ix\theta} d\boldsymbol{t} \\
& + 2(ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}_+^d} h \big(\varsigma_\ell (1 - 2d) p_{\boldsymbol{\nu}' + 2\boldsymbol{e}^d - \boldsymbol{e}_\ell} - \varsigma_{d+1} p_{\boldsymbol{\nu}' + \boldsymbol{e}^d - 2\boldsymbol{e}_\ell} + \varsigma_\ell p_{\boldsymbol{\nu}' - 3\boldsymbol{e}_\ell} \big) \Theta^{-2} e^{ix\theta} d\boldsymbol{t} \\
& + 4(ix)^{-1} \sum_{1 \leqslant \ell < k \leqslant d} \varsigma_{d+1} \varsigma_\ell \varsigma_k \int_{\mathbb{R}_+^d} h p_{\boldsymbol{\nu}' + \boldsymbol{e}^d - \boldsymbol{e}_\ell - \boldsymbol{e}_k} \Theta^{-2} e^{ix\theta} d\boldsymbol{t},
\end{aligned}$$

where Θ and θ are the shorthand notations for $\Theta(\boldsymbol{t}; \boldsymbol{\varsigma})$ and $\theta(\boldsymbol{t}; \boldsymbol{\varsigma})$. Since the shifts of indices do not exceed 3, it follows from the condition (1.5.2), combined with Lemma 1.5.2, that all the integrals above absolutely converge provided $A > \mathfrak{r} + 3$.

Repeating the above manipulations, we obtain the following lemma by a straightforward inductive argument.

Lemma 1.5.3. *Let B be a nonnegative integer, and choose $A = \lfloor \mathfrak{r} \rfloor + 3B + 3$. Then $J_{\boldsymbol{\nu}'}(x; \boldsymbol{\varsigma}; h)$ is equal to a linear combination of $\left(\frac{1}{2}(d^2 - d) + 7d + 1\right)^B$ many absolutely convergent integrals of the following form*

$$(ix)^{-B} P(\boldsymbol{\nu}') \int_{\mathbb{R}_+^d} \boldsymbol{t}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} h(\boldsymbol{t}) p_{\boldsymbol{\nu}''}(\boldsymbol{t}) \Theta(\boldsymbol{t}; \boldsymbol{\varsigma})^{-B-B_2} e^{ix\theta(\boldsymbol{t}; \boldsymbol{\varsigma})} d\boldsymbol{t},$$

where $|\boldsymbol{\alpha}| + B_1 + B_2 = B$ ($\boldsymbol{\alpha} \in \mathbb{N}^d$), P is a polynomial of degree B_1 and integer coefficients of size $O_{B,d}(1)$, and $\boldsymbol{\nu}'' \in \boldsymbol{\nu}' + \mathbb{Z}^d$ satisfies $|\nu''_\ell - \nu'_\ell| \leqslant B + 2B_2$ for all $\ell = 1, ..., d$. Recall that in the multi-index notation $|\boldsymbol{\alpha}| = \sum_{\ell=1}^d \alpha_\ell$, $\boldsymbol{t}^{\boldsymbol{\alpha}} = \prod_{\ell=1}^d t_\ell^{\alpha_\ell}$ and $\partial^{\boldsymbol{\alpha}} = \prod_{\ell=1}^d \partial_\ell^{\alpha_\ell}$.

Define $\mathfrak{c} = \max\{|\nu_\ell|\} + 1$ and $\mathfrak{r} = \max\{|\mathfrak{Re}\, \nu_\ell|\}$. Suppose that $x \geqslant \mathfrak{c}$. Applying Lemma 1.5.3 and 1.5.2 to the J-integrals in (1.3.3, 1.3.4, 1.3.5), one obtains the estimate

$$J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}; \varrho) \ll_{\mathfrak{r}, M, d} \left(\frac{\mathfrak{c}}{x}\right)^M,$$

for any given nonnegative integer M. Slight modifications of the above arguments yield a similar estimate for the derivative

$$J_{\boldsymbol{\nu}}^{(j)}(x; \boldsymbol{\varsigma}; \varrho) \ll_{\mathfrak{r}, M, j, d} \left(\frac{\mathfrak{c}}{x}\right)^M. \tag{1.5.5}$$

Remark 1.5.4. *Our proof of* (1.5.5) *is similar to that of* [Hör, Theorem 7.7.1]. *Indeed, $\Theta(\boldsymbol{t}; \boldsymbol{\varsigma})$ plays the same role as $|f'|^2 + \mathfrak{Im}\, f$ in the proof of* [Hör, Theorem 7.7.1], *where f is the phase function there. The non-compactness of K_ϱ however prohibits the application of* [Hör, Theorem 7.7.1] *to the J-integral in* (1.5.1) *in our case.*

1.5.2. Rapid decay of K-Bessel functions

Suppose that there exists $k \in \{1, ..., d\}$ such that $\varsigma_k \neq \varsigma_{d+1}$. Then for any $\boldsymbol{t} \in K_{\mathbf{0}}$

$$\left|\varsigma_{d+1} t_1 ... t_d - \varsigma_k t_k^{-1}\right| > t_k^{-1} \geqslant \frac{1}{4}.$$

Similar to the arguments in §1.5.1, repeating the k-th partial integration of the third kind yields the same bound (1.5.5) in the case $\boldsymbol{\varrho} = \mathbf{0}$.

Remark 1.5.5. *For this, we may also directly apply* [Hör, Theorem 7.7.1].

Theorem 1.5.6. *Let* $\mathfrak{c} = \max\{|\nu_\ell|\} + 1$ *and* $\mathfrak{r} = \max\{|\Re\mathfrak{e}\, \nu_\ell|\}$. *Let j and M be nonnegative integers. Suppose that one of the signs* $\varsigma_1, ..., \varsigma_d$ *is different from* ς_{d+1}. *Then*

$$J_{\boldsymbol{\nu}}^{(j)}(x; \boldsymbol{\varsigma}) \ll_{\mathfrak{r}, M, j, d} \left(\frac{\mathfrak{c}}{x}\right)^M$$

for any $x \geqslant \mathfrak{c}$. *In particular,* $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ *is a Schwartz function at infinity, namely, all derivatives* $J_{\boldsymbol{\nu}}^{(j)}(x; \boldsymbol{\varsigma})$ *rapidly decay at infinity.*

1.5.3. Asymptotic expansions of H-Bessel functions

In the following, we shall adopt the convention $(\pm i)^a = e^{\pm \frac{1}{2} i \pi a}$, $a \in \mathbb{C}$.

We first introduce the function $W_{\boldsymbol{\nu}}(x; \pm)$, which is closely related to the Whittaker function of imaginary argument if $d = 1$ (see [WW, §17.5, 17.6]), defined by

$$W_{\boldsymbol{\nu}}(x; \pm) = (d+1)^{\frac{1}{2}} (\pm 2\pi i)^{-\frac{d}{2}} e^{\mp i (d+1) x} H_{\boldsymbol{\nu}}^{\pm}(x).$$

Write $H_{\boldsymbol{\nu}}^{\pm}(x; \boldsymbol{\varrho}) = J_{\boldsymbol{\nu}}(x; \pm, ..., \pm; \boldsymbol{\varrho})$ and define

$$W_{\boldsymbol{\nu}}(x; \pm; \boldsymbol{\varrho}) = (d+1)^{\frac{1}{2}} (\pm 2\pi i)^{-\frac{d}{2}} e^{\mp i (d+1) x} H_{\boldsymbol{\nu}}^{\pm}(x; \boldsymbol{\varrho}).$$

For $\boldsymbol{\varrho} \neq \mathbf{0}$, the bound (1.5.5) for $H_{\boldsymbol{\nu}}^{\pm}(x; \boldsymbol{\varrho})$ is also valid for $W_{\boldsymbol{\nu}}(x; \pm; \boldsymbol{\varrho})$. Therefore, we are left with analyzing $W_{\boldsymbol{\nu}}(x; \pm; \mathbf{0})$. We have

$$\begin{aligned} W_{\boldsymbol{\nu}}^{(j)}(x; \pm; \mathbf{0}) = {} & (d+1)^{\frac{1}{2}} (\pm 2\pi i)^{-\frac{d}{2}} (\pm i)^j \\ & \int_{K_{\mathbf{0}}} (\theta(\boldsymbol{t}) - d - 1)^j h_{\mathbf{0}}(\boldsymbol{t}) p_{\boldsymbol{\nu}}(\boldsymbol{t}) e^{\pm i x (\theta(\boldsymbol{t}) - d - 1)} d\boldsymbol{t}, \end{aligned} \tag{1.5.6}$$

with

$$\theta(\boldsymbol{t}) = \theta(\boldsymbol{t}; +, ..., +) = t_1 ... t_d + \sum_{\ell=1}^{d} t_\ell^{-1}. \tag{1.5.7}$$

Proposition 1.5.7. [Hör, Theorem 7.7.5]. *Let $K \subset \mathbb{R}^d$ be a compact set, X an open neighbourhood of K and M a nonnegative integer. If $u(\boldsymbol{t}) \in C_0^{2M}(K)$, $f(\boldsymbol{t}) \in C^{3M+1}(X)$ and $\Im\mathfrak{m}\, f \geqslant 0$ in X, $\Im\mathfrak{m}\, f(\boldsymbol{t}_0) = 0$, $f'(\boldsymbol{t}_0) = 0$, $\det f''(\boldsymbol{t}_0) \neq 0$ and $f' \neq 0$ in $K \setminus \{\boldsymbol{t}_0\}$, then for $x > 0$*

$$\left| \int_K u(\boldsymbol{t}) e^{ixf(\boldsymbol{t})} d\boldsymbol{t} - e^{ixf(\boldsymbol{t}_0)} \left((2\pi i)^{-d} \det f''(\boldsymbol{t}_0) \right)^{-\frac{1}{2}} \sum_{m=0}^{M-1} x^{-m-\frac{d}{2}} \mathcal{L}_m u \right| \ll x^{-M} \sum_{|\alpha| \leqslant 2M} \sup |D^\alpha u| .$$

Here the implied constant depends only on M, f, K and d. With

$$g(\boldsymbol{t}) = f(\boldsymbol{t}) - f(\boldsymbol{t}_0) - \frac{1}{2} \langle f''(\boldsymbol{t}_0)(\boldsymbol{t} - \boldsymbol{t}_0), \boldsymbol{t} - \boldsymbol{t}_0 \rangle$$

which vanishes of third order at $\boldsymbol{t}_0$, we have

$$\mathcal{L}_m u = i^{-m} \sum_{r=0}^{2m} \frac{1}{2^{m+r}(m+r)!r!} \left\langle f''(\boldsymbol{t}_0)^{-1} D, D \right\rangle^{m+r} (g^r u)(\boldsymbol{t}_0).^{\text{VIII}}$$

This is a differential operator of order $2m$ acting on u at $\boldsymbol{t}_0$. The coefficients are rational homogeneous functions of degree $-m$ in $f''(\boldsymbol{t}_0)$, ..., $f^{(2m+2)}(\boldsymbol{t}_0)$ with denominator $(\det f''(\boldsymbol{t}_0))^{3m}$. In every term the total number of derivatives of u and of f'' is at most $2m$.

We now apply Proposition 1.5.7 to the integral in (1.5.6). For this, we let

$$K = K_{\boldsymbol{0}} = \left[\tfrac{1}{4}, 4\right]^d, \quad X = \left(\tfrac{1}{5}, 5\right)^d,$$

$$f(\boldsymbol{t}) = \pm\left(\theta(\boldsymbol{t}) - d - 1\right), \quad f'(\boldsymbol{t}) = \pm\left(t_1 ... \widehat{t_\ell} ... t_d - t_\ell^{-2}\right)_{\ell=1}^d, \quad \boldsymbol{t}_0 = (1, ..., 1),$$

$$f''(\boldsymbol{t}_0) = \pm \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}, \quad \det f''(\boldsymbol{t}_0) = (\pm)^d (d+1), \quad g(\boldsymbol{t}) = \pm G(\boldsymbol{t}),$$

$$f''(\boldsymbol{t}_0)^{-1} = \pm \frac{1}{d+1} \begin{pmatrix} d & -1 & \cdots & -1 \\ -1 & d & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & d \end{pmatrix},$$

[VIII] According to Hörmander, $D = -i(\partial_1, ..., \partial_d)$.

$$u(\boldsymbol{t}) = (d+1)^{\frac{1}{2}} (\pm 2\pi i)^{-\frac{d}{2}} (\pm i)^j \, (\theta(\boldsymbol{t}) - d - 1)^j \, p_{\boldsymbol{\nu}}(\boldsymbol{t}) h_{\boldsymbol{0}}(\boldsymbol{t}),$$

with

$$(1.5.8) \qquad G(\boldsymbol{t}) = t_1 ... t_d + \sum_{\ell=1}^{d} \left(-t_\ell^2 + (d+1) t_\ell + t_\ell^{-1} \right) - \sum_{1 \leqslant \ell < k \leqslant d} t_\ell t_k - \frac{(d+1)(d+2)}{2}.$$

Proposition 1.5.7 yields the following asymptotic expansion of $W_{\boldsymbol{\nu}}^{(j)}(x; \pm; \boldsymbol{0})$,

$$W_{\boldsymbol{\nu}}^{(j)}(x; \pm; \boldsymbol{0}) = \sum_{m=0}^{M-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\nu}) x^{-m-\frac{d}{2}} + O_{\mathfrak{r}, M, j, d} \left(\mathfrak{c}^{2M} x^{-M} \right), \quad x \in \mathbb{R}_+,$$

with

$$(1.5.9) \qquad B_{m,j}(\boldsymbol{\nu}) = \sum_{r=0}^{2m} \frac{(-)^{m+r} \mathcal{L}^{m+r} \left(G^r (\theta - d - 1)^j p_{\boldsymbol{\nu}} \right) (\boldsymbol{t}_0)}{(2(d+1))^{m+r} (m+r)! r!},$$

where $\mathcal{L}$ is the second-order differential operator given by

$$(1.5.10) \qquad \mathcal{L} = d \sum_{\ell=1}^{d} \partial_\ell^2 - 2 \sum_{1 \leqslant \ell < k \leqslant d} \partial_\ell \partial_k.$$

Lemma 1.5.8. *We have $B_{m,j}(\boldsymbol{\nu}) = 0$ if $m < j$. Otherwise, $B_{m,j}(\boldsymbol{\nu}) \in \mathbb{Q}[\boldsymbol{\nu}]$ is a symmetric polynomial of degree $2m - 2j$. In particular, $B_{m,j}(\boldsymbol{\nu}) \ll_{m,j,d} \mathfrak{c}^{2m-2j}$ for $m \geqslant j$.*

Proof. The symmetry of $B_{m,j}(\boldsymbol{\nu})$ is clear from definition. Since $\theta - d - 1$ vanishes of second order at $\boldsymbol{t}_0$, $2j$ many differentiations are required to remove the zero of $(\theta - d - 1)^j$ at $\boldsymbol{t}_0$. From this, along with the descriptions of the differential operator $\mathcal{L}_m$ in Proposition 1.5.7, one proves the lemma. Q.E.D.

Furthermore, in view of (1.5.5), the total contribution to $W_{\boldsymbol{\nu}}^{(j)}(x; \pm)$ from all those $W_{\boldsymbol{\nu}}^{(j)}(x; \pm; \boldsymbol{\varrho})$ with $\boldsymbol{\varrho} \neq \boldsymbol{0}$ is of size $O_{\mathfrak{r}, M, j, d} \left(\mathfrak{c}^M x^{-M} \right)$ and hence may be absorbed into the error term in the asymptotic expansion of $W_{\boldsymbol{\nu}}^{(j)}(x; \pm; \boldsymbol{0})$.

In conclusion, the following proposition is established.

Proposition 1.5.9. *Let M, j be nonnegative integers such that $M \geqslant j$. Then for $x \geqslant \mathfrak{c}$*

$$W_{\boldsymbol{\nu}}^{(j)}(x; \pm) = \sum_{m=j}^{M-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\nu}) x^{-m-\frac{d}{2}} + O_{\mathfrak{r},M,j,d}\left(\mathfrak{c}^{2M} x^{-M}\right).$$

Corollary 1.5.10. *Let N, j be nonnegative integers such that $N \geqslant j$, and let $\epsilon > 0$.*

(1). *We have $W_{\boldsymbol{\nu}}^{(j)}(x; \pm) \ll_{\mathfrak{r},j,d} \mathfrak{c}^{2j} x^{-j}$ for $x \geqslant \mathfrak{c}$.*

(2). *If $x \geqslant \mathfrak{c}^{2+\epsilon}$, then*

$$W_{\boldsymbol{\nu}}^{(j)}(x; \pm) = \sum_{m=j}^{N-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\nu}) x^{-m-\frac{d}{2}} + O_{\mathfrak{r},N,j,\epsilon,d}\left(\mathfrak{c}^{2N} x^{-N-\frac{d}{2}}\right).$$

Proof. On letting $M = j$, Proposition 1.5.9 implies (1). On choosing M sufficiently large so that $(2+\epsilon)\left(M - N + \frac{d}{2}\right) \geqslant 2(M-N)$, Proposition 1.5.9 and Lemma 1.5.8 yield

$$\begin{aligned} & W_{\boldsymbol{\nu}}^{(j)}(x; \pm) - \sum_{m=j}^{N-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\nu}) x^{-m-\frac{d}{2}} \\ = & \sum_{m=N}^{M-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\nu}) x^{-m-\frac{d}{2}} + O_{\mathfrak{r},j,M,d}\left(\mathfrak{c}^{2M} x^{-M}\right) = O_{\mathfrak{r},j,N,\epsilon,d}\left(\mathfrak{c}^{2N} x^{-N-\frac{d}{2}}\right). \end{aligned}$$

Q.E.D.

Finally, the asymptotic expansion of $H^{\pm}(x; \boldsymbol{\lambda}) (= H_{\boldsymbol{\nu}}^{\pm}(x))$ is formulated as below.

Theorem 1.5.11. *Let $\mathfrak{C} = \max\{|\lambda_\ell|\} + 1$ and $\mathfrak{R} = \max\{|\mathfrak{Re}\, \lambda_\ell|\}$. Let M be a nonnegative integer.*

(1). *Define $W(x; \pm, \boldsymbol{\lambda}) = \sqrt{n}(\pm 2\pi i)^{-\frac{n-1}{2}} e^{\mp inx} H^{\pm}(x; \boldsymbol{\lambda})$. Let $M \geqslant j \geqslant 0$. Then*

$$W^{(j)}(x; \pm, \boldsymbol{\lambda}) = \sum_{m=j}^{M-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\lambda}) x^{-m-\frac{n-1}{2}} + O_{\mathfrak{R},M,j,n}\left(\mathfrak{C}^{2M} x^{-M}\right)$$

for all $x \geqslant \mathfrak{C}$. Here $B_{m,j}(\boldsymbol{\lambda}) \in \mathbb{Q}[\boldsymbol{\lambda}]$ is a symmetric polynomial in $\boldsymbol{\lambda}$ of degree $2m$, with $B_{0,0}(\boldsymbol{\lambda}) = 1$. The coefficients of $B_{m,j}(\boldsymbol{\lambda})$ depends only on m, j and d.

(2). *Let $B_m(\boldsymbol{\lambda}) = B_{m,0}(\boldsymbol{\lambda})$. Then for $x \geqslant \mathfrak{C}$*

$$H^{\pm}(x; \boldsymbol{\lambda}) = n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} e^{\pm inx} x^{-\frac{n-1}{2}} \left(\sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\boldsymbol{\lambda}) x^{-m} + O_{\mathfrak{R},M,d}\left(\mathfrak{C}^{2M} x^{-M+\frac{n-1}{2}}\right) \right).$$

Proof. This theorem is a direct consequence of Proposition 1.5.9 and Lemma 1.5.8. It is only left to verify the symmetry of $B_{m,j}(\boldsymbol{\lambda}) = B_{m,j}(\boldsymbol{\nu})$ with respect to $\boldsymbol{\lambda}$. Indeed, in view of (1.2.3, 1.2.5), $H^{\pm}(x; \boldsymbol{\lambda})$ is symmetric with respect to $\boldsymbol{\lambda}$, so $B_{m,j}(\boldsymbol{\lambda})$ must be represented by a symmetric polynomial in $\boldsymbol{\lambda}$ modulo $\sum_{\ell=1}^{d+1} \lambda_\ell$. Q.E.D.

Corollary 1.5.12. *Let M be a nonnegative integer, and let $\epsilon > 0$. Then for $x \geqslant \mathfrak{C}^{2+\epsilon}$*

$$H^{\pm}(x; \boldsymbol{\lambda}) = n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} e^{\pm i n x} x^{-\frac{n-1}{2}} \left(\sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\boldsymbol{\lambda}) x^{-m} + O_{\mathfrak{R}, M, \epsilon, n}\left(\mathfrak{C}^{2M} x^{-M} \right) \right).$$

1.5.4. Concluding remarks

On the analytic continuations of H-Bessel functions

Our observation is that the phase function θ defined by (1.5.7) is always positive on $\mathbb{R}_+^d$. It follows that if one replaces x by $z = x e^{i\omega}$, with $x > 0$ and $0 \leqslant \pm\omega \leqslant \pi$, then the various J-integrals in the rigorous interpretation of $H_{\boldsymbol{\nu}}^{\pm}(z)$ remain absolutely convergent, uniformly with respect to z, since $\left| e^{\pm i z \theta(\boldsymbol{t})} \right| = e^{\mp x \sin\omega\, \theta(\boldsymbol{t})} \leqslant 1$. Therefore, the resulting integral $H_{\boldsymbol{\nu}}^{\pm}(z)$ gives rise to an analytic continuation of $H_{\boldsymbol{\nu}}^{\pm}(x)$ onto the half-plane $\mathbb{H}^{\pm} = \{ z \in \mathbb{C} \smallsetminus \{0\} : 0 \leqslant \pm \arg z \leqslant \pi \}$. In view of Proposition 1.4.1, one may define $H^{\pm}(z; \boldsymbol{\lambda}) = H_{\boldsymbol{\nu}}^{\pm}(z)$ and regard it as the analytic continuation of $H^{\pm}(x; \boldsymbol{\lambda})$ from $\mathbb{R}_+$ onto $\mathbb{H}^{\pm}$. Furthermore, with slight modifications of the arguments above, where the phase function f is now chosen to be $\pm e^{i\omega}(\theta - d - 1)$ in the application of Proposition 1.5.7, the domain of validity for the asymptotic expansions in Theorem 1.5.11 may be extended from $\mathbb{R}_+$ onto $\mathbb{H}^{\pm}$. For example, we have

$$\begin{aligned} H^{\pm}(z; \boldsymbol{\lambda}) &= n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} e^{\pm i n z} z^{-\frac{n-1}{2}} \\ &\quad \left(\sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\boldsymbol{\lambda}) z^{-m} + O_{\mathfrak{R}, M, n}\left(\mathfrak{C}^{2M} |z|^{-M + \frac{n-1}{2}} \right) \right), \end{aligned} \tag{1.5.11}$$

for all $z \in \mathbb{H}^{\pm}$ such that $|z| \geqslant \mathfrak{C}$.

Obviously, the above method of obtaining the analytic continuation of $H^{\pm}_{\nu}$ does not apply to K-Bessel functions.

On the asymptotic of the Bessel kernel $J_{(\lambda,\delta)}$

As in (1.2.7), $J_{(\lambda,\delta)}(\pm x)$ is a combination of $J\big(2\pi x^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\lambda}\big)$, and hence its asymptotic follows immediately from Theorem 1.5.6 and 1.5.11.

Theorem 1.5.13. *Let* $(\boldsymbol{\lambda}, \boldsymbol{\delta}) \in \mathbb{L}^{n-1} \times (\mathbb{Z}/2\mathbb{Z})^n$. *Let* $M \geqslant 0$. *Then, for* $x > 0$, *we may write*

$$J_{(\lambda,\delta)}\left(x^n\right) = \sum_{\pm} \frac{(\pm)^{\sum \delta_l} e\left(\pm\left(nx + \frac{n-1}{8}\right)\right)}{n^{\frac{1}{2}} x^{\frac{n-1}{2}}} W^{\pm}_{\lambda}(x) + E^{+}_{(\lambda,\delta)}(x),$$
$$J_{(\lambda,\delta)}\left(-x^n\right) = E^{-}_{(\lambda,\delta)}(x),$$

if n is even, and

$$J_{(\lambda,\delta)}\left(\pm x^n\right) = \frac{(\pm)^{\sum \delta_l} e\left(\pm\left(nx + \frac{n-1}{8}\right)\right)}{n^{\frac{1}{2}} x^{\frac{n-1}{2}}} W^{\pm}_{\lambda}(x) + E^{\pm}_{(\lambda,\delta)}(x),$$

if n is odd, such that

$$W^{\pm}_{\lambda}(x) = \sum_{m=0}^{M-1} B^{\pm}_m(\boldsymbol{\lambda}) x^{-m} + O_{\Re, M, n}\left(\mathfrak{C}^{2M} x^{-M + \frac{n-1}{2}}\right),$$

and

$$E^{\pm}_{(\lambda,\delta)}(x) = O_{\Re, M, n}\left(\mathfrak{C}^{M} x^{-M}\right),$$

for $x \geqslant \mathfrak{C}$. *With the notations in Theorem* 1.5.11, *we have* $W^{\pm}_{\lambda}(x) = (2\pi x)^{\frac{n-1}{2}} W^{\pm}(2\pi x; \boldsymbol{\lambda})$ *and* $B^{\pm}_m(\boldsymbol{\lambda}) = (\pm 2\pi i)^{-m} B_m(\boldsymbol{\lambda})$.

On the implied constants of estimates

All the implied constants that occur in this section are of exponential dependence on the real parts of the indices. If one considers the d-th symmetric lift of a holomorphic Hecke cusp form of weight k, the estimates are particularly awful in the k aspect.

In §1.6 and §1.7, we shall further explore the theory of Bessel functions from the perspective of differential equations. Consequently, if the argument is sufficiently large, then all the estimates in this section can be improved so that the dependence on the index can be completely eliminated.

On the coefficients in the asymptotics

One feature of the method of stationary phase is the explicit formula of the coefficients in the asymptotic expansion in terms of certain partial differential operators. In the present case of $H^{\pm}(x; \boldsymbol{\lambda}) = H^{\pm}_{\boldsymbol{\nu}}(x)$, (1.5.9) provides an explicit formula of $B_m(\boldsymbol{\lambda}) = B_{m,0}(\boldsymbol{\nu})$. To compute $\mathcal{L}^{m+r}\left(G^r p_{\boldsymbol{\nu}}\right)(\boldsymbol{t}_0)$ appearing in (1.5.9), we observe that the function G defined in (1.5.8) does not only vanish of third order at $\boldsymbol{t}_0$. Actually, $\partial^{\boldsymbol{\alpha}} G(\boldsymbol{t}_0)$ vanishes except for $\boldsymbol{\alpha} = (0, ..., 0, \alpha, 0..., 0)$, with $\alpha \geqslant 3$. In the exceptional case we have $\partial^{\boldsymbol{\alpha}} G(\boldsymbol{t}_0) = (-)^{\alpha} \alpha!$. However, the resulting expression is considerably complicated. To illustrate, we consider the case $d = 1$.

When $d = 1$, we have $\mathcal{L} = (d/dt)^2$. For $2m \geqslant r \geqslant 1$,

$$\begin{aligned}
&(d/dt)^{2m+2r}\left(G^r p_\nu\right)(1) \\
&= (2m+2r)! \sum_{\alpha=0}^{2m-r} \left| \left\{ (\alpha_1, ..., \alpha_r) : \sum_{q=1}^{r} \alpha_q = 2m + 2r - \alpha, \alpha_q \geqslant 3 \right\} \right| \frac{(-)^{\alpha}[\nu - 1]_{\alpha}}{\alpha!} \\
&= (2m+2r)! \sum_{\alpha=0}^{2m-r} \binom{2m-\alpha-1}{r-1} \frac{(1-\nu)_{\alpha}}{\alpha!}.
\end{aligned}$$

Therefore (1.5.9) yields

$$B_{m,0}(\nu) = \left(-\frac{1}{4}\right)^m \left(\frac{(1-\nu)_{2m}}{m!} + \sum_{r=1}^{2m} \frac{(-)^r (2m+2r)!}{4^r (m+r)! r!} \sum_{\alpha=0}^{2m-r} \binom{2m-\alpha-1}{r-1} \frac{(1-\nu)_{\alpha}}{\alpha!} \right).$$

However, this expression of $B_{m,0}(\nu)$ is not in its simplest form. Indeed, we have the asymptotic expansions of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ ([Wat, 7.2 (1, 2)])

$$H_\nu^{(1,2)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{\pm i\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} \left(\sum_{m=0}^{\infty} \frac{(\pm)^m \left(\frac{1}{2}-\nu\right)_m \left(\frac{1}{2}+\nu\right)_m}{m!(2ix)^m}\right),$$

which are deducible from Hankel's integral representations ([Wat, 6.12 (3, 4)]). In view of Proposition 1.2.7 and Theorem 1.5.11, one obtains

$$B_{m,0}(\nu) = \frac{\left(\frac{1}{2}-\nu\right)_m \left(\frac{1}{2}+\nu\right)_m}{4^m m!}.$$

Therefore, we have the following combinatoric identity

$$\begin{aligned} \frac{(-)^m \left(\frac{1}{2}-\nu\right)_m \left(\frac{1}{2}+\nu\right)_m}{m!} &= \frac{(1-\nu)_{2m}}{m!} \\ &\quad + \sum_{r=1}^{2m} \frac{(-)^r (2m+2r)!}{4^r (m+r)! r!} \sum_{\alpha=0}^{2m-r} \binom{2m-\alpha-1}{r-1} \frac{(1-\nu)_\alpha}{\alpha!}. \end{aligned} \tag{1.5.12}$$

It seems however hard to find an elementary proof of this identity.

1.6. Recurrence formulae and differential equations for Bessel functions

Making use of certain recurrence formulae for $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$, we shall derive the differential equation satisfied by $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

1.6.1. The recurrence formulae

Applying the formal partial integrations of either the first or the second kind and the differentiation under the integral sign on the formal integral expression of $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ in (1.3.1), one obtains the recurrence formulae

$$\nu_\ell (ix)^{-1} J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}) = \varsigma_\ell J_{\boldsymbol{\nu} - \boldsymbol{e}_\ell}(x; \boldsymbol{\varsigma}) - \varsigma_{d+1} J_{\boldsymbol{\nu} + \boldsymbol{e}^d}(x; \boldsymbol{\varsigma}) \tag{1.6.1}$$

for $\ell = 1, ..., d$, and

$$J'_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}) = \varsigma_{d+1} i J_{\boldsymbol{\nu}+\boldsymbol{e}^d}(x; \boldsymbol{\varsigma}) + i \sum_{\ell=1}^{d} \varsigma_\ell J_{\boldsymbol{\nu}-\boldsymbol{e}_\ell}(x; \boldsymbol{\varsigma}). \tag{1.6.2}$$

It is easy to verify (1.6.1) and (1.6.2) using the rigorous interpretation of $J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma})$ established in §1.3.3. Moreover, using (1.6.1), one may reformulate (1.6.2) as below,

$$J'_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}) = \varsigma_{d+1} i (d+1) J_{\boldsymbol{\nu}+\boldsymbol{e}^d}(x; \boldsymbol{\varsigma}) + \frac{\sum_{\ell=1}^{d} \nu_\ell}{x} J_{\boldsymbol{\nu}}(x; \boldsymbol{\varsigma}). \tag{1.6.3}$$

1.6.2. The differential equations

Lemma 1.6.1. *Define* $\boldsymbol{e}^\ell = (\underbrace{1, ..., 1}_{\ell}, 0..., 0)$, $\ell = 1, ..., d$, *and denote* $\boldsymbol{e}^0 = \boldsymbol{e}^{d+1} = (0, ..., 0)$ *for convenience. Let* $\nu_{d+1} = 0$.

(1). *For* $\ell = 1, ..., d+1$ *we have*

$$J'_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x; \boldsymbol{\varsigma}) = \varsigma_\ell i (d+1) J_{\boldsymbol{\nu}+\boldsymbol{e}^{\ell-1}}(x; \boldsymbol{\varsigma}) - \frac{\Lambda_{d-\ell+1}(\boldsymbol{\nu}) + d - \ell + 1}{x} J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x; \boldsymbol{\varsigma}), \tag{1.6.4}$$

with

$$\Lambda_m(\boldsymbol{\nu}) = -\sum_{k=1}^{d} \nu_k + (d+1)\nu_{d-m+1}, \quad m = 0, ..., d.$$

(2). *For* $0 \leqslant j \leqslant k \leqslant d+1$ *define*

$$U_{k,j}(\boldsymbol{\nu}) = \begin{cases} 1, & \text{if } j = k, \\ -\left(\Lambda_j(\boldsymbol{\nu}) + k - 1\right) U_{k-1,j}(\boldsymbol{\nu}) + U_{k-1,j-1}(\boldsymbol{\nu}), & \text{if } 0 \leqslant j \leqslant k-1, \end{cases}$$

with the notation $U_{k,-1}(\boldsymbol{\nu}) = 0$, *and*

$$S_0(\boldsymbol{\varsigma}) = +, \quad S_j(\boldsymbol{\varsigma}) = \prod_{m=0}^{j-1} \varsigma_{d-m+1} \text{ for } j = 1, ..., d+1.$$

Then

$$J_{\boldsymbol{\nu}}^{(k)}(x; \boldsymbol{\varsigma}) = \sum_{j=0}^{k} S_j(\boldsymbol{\varsigma}) (i(d+1))^j U_{k,j}(\boldsymbol{\nu}) x^{j-k} J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x; \boldsymbol{\varsigma}). \tag{1.6.5}$$

Proof. By (1.6.3) and (1.6.1),

$$
\begin{aligned}
J'_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x;\boldsymbol{\varsigma}) &= \varsigma_{d+1} i(d+1) J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell+\boldsymbol{e}^d}(x;\boldsymbol{\varsigma}) + \frac{\sum_{k=1}^d \nu_k + \ell}{x} J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x;\boldsymbol{\varsigma}) \\
&= i(d+1)\left(-\frac{\nu_\ell + 1}{ix} J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x;\boldsymbol{\varsigma}) + \varsigma_\ell J_{\boldsymbol{\nu}+\boldsymbol{e}^{\ell-1}}(x;\boldsymbol{\varsigma}) \right) + \frac{\sum_{k=1}^d \nu_k + \ell}{x} J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x;\boldsymbol{\varsigma}) \\
&= \varsigma_\ell i(d+1) J_{\boldsymbol{\nu}+\boldsymbol{e}^{\ell-1}}(x;\boldsymbol{\varsigma}) + \frac{\sum_{k=1}^d \nu_k - (d+1)\nu_\ell + \ell - d - 1}{x} J_{\boldsymbol{\nu}+\boldsymbol{e}^\ell}(x;\boldsymbol{\varsigma}).
\end{aligned}
$$

This proves (1.6.4).

(1.6.5) is trivial when $k = 0$. Suppose that $k \geqslant 1$ and that (1.6.5) is already proven for $k-1$. The inductive hypothesis and (1.6.4) imply

$$
\begin{aligned}
J_{\boldsymbol{\nu}}^{(k)}(x;\boldsymbol{\varsigma}) &= \sum_{j=0}^{k-1} S_j(\boldsymbol{\varsigma})(i(d+1))^j U_{k-1,j}(\boldsymbol{\nu}) x^{j-k+1} \\
&\quad \big((j-k+1) x^{-1} J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x;\boldsymbol{\varsigma}) \\
&\qquad + \varsigma_{d-j+1} i(d+1) J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j}}(x;\boldsymbol{\varsigma}) - (\Lambda_j(\boldsymbol{\nu}) + j) x^{-1} J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x;\boldsymbol{\varsigma}) \big) \\
&= -\sum_{j=0}^{k-1} S_j(\boldsymbol{\varsigma})(i(d+1))^j U_{k-1,j}(\boldsymbol{\nu})(\Lambda_j(\boldsymbol{\nu}) + k - 1) x^{j-k} J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x;\boldsymbol{\varsigma}) \\
&\quad + \sum_{j=1}^{k} S_{j-1}(\boldsymbol{\varsigma}) \varsigma_{d-j+2} (i(d+1))^j U_{k-1,j-1}(\boldsymbol{\nu}) x^{j-k} J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x;\boldsymbol{\varsigma}).
\end{aligned}
$$

Then (1.6.5) follows from the definitions of $U_{k,j}(\boldsymbol{\nu})$ and $S_j(\boldsymbol{\varsigma})$. Q.E.D.

Lemma 1.6.1 (2) may be recapitulated as

$$
X_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}) = D(x)^{-1} U(\boldsymbol{\nu}) D(x) S(\boldsymbol{\varsigma}) Y_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}), \tag{1.6.6}
$$

where $X_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}) = \left(J_{\boldsymbol{\nu}}^{(k)}(x;\boldsymbol{\varsigma}) \right)_{k=0}^{d+1}$ and $Y_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma}) = \left(J_{\boldsymbol{\nu}+\boldsymbol{e}^{d-j+1}}(x;\boldsymbol{\varsigma}) \right)_{j=0}^{d+1}$ are column vectors of functions, $S(\boldsymbol{\varsigma}) = \operatorname{diag}\left(S_j(\boldsymbol{\varsigma})(i(d+1))^j \right)_{j=0}^{d+1}$ and $D(x) = \operatorname{diag}\left(x^j \right)_{j=0}^{d+1}$ are diagonal matrices, and $U(\boldsymbol{\nu})$ is the lower triangular unipotent $(d+2) \times (d+2)$ matrix whose $(k+1, j+1)$-th entry is equal to $U_{k,j}(\boldsymbol{\nu})$. The inverse matrix $U(\boldsymbol{\nu})^{-1}$ is again a lower triangular unipotent matrix. Let $V_{k,j}(\boldsymbol{\nu})$ denote the $(k+1, j+1)$-th entry of $U(\boldsymbol{\nu})^{-1}$. It is evident that $V_{k,j}(\boldsymbol{\nu})$ is a polynomial in $\boldsymbol{\nu}$ of degree $k - j$ and integral coefficients.

Observe that $J_{\boldsymbol{\nu}+\boldsymbol{e}^{d+1}}(x;\boldsymbol{\varsigma}) = J_{\boldsymbol{\nu}+\boldsymbol{e}^{0}}(x;\boldsymbol{\varsigma}) = J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$. Therefore, (1.6.6) implies that $J_{\boldsymbol{\nu}}(x;\boldsymbol{\varsigma})$ satisfies the following linear differential equation of order $d+1$

$$\sum_{j=1}^{d+1} V_{d+1,j}(\boldsymbol{\nu}) x^{j-d-1} w^{(j)} + \left(V_{d+1,0}(\boldsymbol{\nu}) x^{-d-1} - S_{d+1}(\boldsymbol{\varsigma})(i(d+1))^{d+1}\right) w = 0. \tag{1.6.7}$$

1.6.3. Calculations of the coefficients in the differential equations

Definition 1.6.2. *Let $\boldsymbol{\Lambda} = \{\Lambda_m\}_{m=0}^{\infty}$ be a sequence of complex numbers.*

(1). *For $k, j \geqslant -1$ inductively define a double sequence of polynomials $U_{k,j}(\boldsymbol{\Lambda})$ in $\boldsymbol{\Lambda}$ by the initial conditions*

$$U_{-1,-1}(\boldsymbol{\Lambda}) = 1, \quad U_{k,-1}(\boldsymbol{\Lambda}) = U_{-1,j}(\boldsymbol{\Lambda}) = 0 \ \text{ if } k, j \geqslant 0,$$

and the recurrence relation

$$U_{k,j}(\boldsymbol{\Lambda}) = -\left(\Lambda_j + k - 1\right) U_{k-1,j}(\boldsymbol{\Lambda}) + U_{k-1,j-1}(\boldsymbol{\Lambda}), \quad k, j \geqslant 0. \tag{1.6.8}$$

(2). *For $j, m \geqslant -1$ with $(j,m) \neq (-1,-1)$ define a double sequence of integers $A_{j,m}$ by the initial conditions*

$$A_{-1,0} = 1, \quad A_{-1,m} = A_{j,-1} = 0 \ \text{ if } m \geqslant 1, j \geqslant 0,$$

and the recurrence relation

$$A_{j,m} = jA_{j,m-1} + A_{j-1,m}, \quad j, m \geqslant 0. \tag{1.6.9}$$

(3). *For $k, m \geqslant 0$ we define $\sigma_{k,m}(\boldsymbol{\Lambda})$ to be the elementary symmetric polynomial in $\Lambda_0, ..., \Lambda_k$ of degree m, with the convention that $\sigma_{k,m}(\boldsymbol{\Lambda}) = 0$ if $m \geqslant k+2$. Moreover, we denote*

$$\sigma_{-1,0}(\boldsymbol{\Lambda}) = 1, \quad \sigma_{k,-1}(\boldsymbol{\Lambda}) = \sigma_{-1,m}(\boldsymbol{\Lambda}) = 0 \ \text{ if } k \geqslant -1, m \geqslant 1.$$

Observe that, with the above notations as initial conditions, $\sigma_{k,m}(\mathbf{\Lambda})$ may also be inductively defined by the recurrence relation

$$\sigma_{k,m}(\mathbf{\Lambda}) = \Lambda_k \sigma_{k-1,m-1}(\mathbf{\Lambda}) + \sigma_{k-1,m}(\mathbf{\Lambda}), \quad k, m \geqslant 0. \tag{1.6.10}$$

(4). *For $k \geqslant 0$, $j \geqslant -1$ define*

$$V_{k,j}(\mathbf{\Lambda}) = \begin{cases} 0, & \text{if } j > k, \\ \displaystyle\sum_{m=0}^{k-j} A_{j,k-j-m}\sigma_{k-1,m}(\mathbf{\Lambda}), & \text{if } k \geqslant j. \end{cases} \tag{1.6.11}$$

Lemma 1.6.3. *Let notations be as above.*

(1). *$U_{k,j}(\mathbf{\Lambda})$ is a polynomial in $\Lambda_0, ..., \Lambda_j$. $U_{k,j}(\mathbf{\Lambda}) = 0$ if $j > k$, and $U_{k,k}(\mathbf{\Lambda}) = 1$. $U_{k,0}(\mathbf{\Lambda}) = [-\Lambda_0]_k$ for $k \geqslant 0$.*

(2). *$A_{j,0} = 1$, and $A_{j,1} = \frac{1}{2}j(j+1)$.*

(3). *$V_{k,j}(\mathbf{\Lambda})$ is a symmetric polynomial in $\Lambda_0, ..., \Lambda_{k-1}$. $V_{k,k}(\mathbf{\Lambda}) = 1$. $V_{k,-1}(\mathbf{\Lambda}) = 0$ and $V_{k,k-1}(\mathbf{\Lambda}) = \sigma_{k-1,1}(\mathbf{\Lambda}) + \frac{1}{2}k(k-1)$ for $k \geqslant 0$.*

(4). *$V_{k,j}(\mathbf{\Lambda})$ satisfies the following recurrence relation*

$$V_{k,j}(\mathbf{\Lambda}) = (\Lambda_{k-1} + j)V_{k-1,j}(\mathbf{\Lambda}) + V_{k-1,j-1}(\mathbf{\Lambda}), \quad k \geqslant 1, j \geqslant 0. \tag{1.6.12}$$

Proof. (1-3) are evident from the definitions.

(4). (1.6.12) is obvious if $j \geqslant k$. If $k > j$, then the recurrence relations (1.6.10, 1.6.9) for $\sigma_{k,m}(\mathbf{\Lambda})$ and $A_{j,m}$, in conjunction with the definition (1.6.11) of $V_{k,j}(\mathbf{\Lambda})$, yield

$$\begin{aligned} V_{k,j}(\mathbf{\Lambda}) &= \sum_{m=0}^{k-j} A_{j,k-j-m}\sigma_{k-1,m}(\mathbf{\Lambda}) \\ &= \Lambda_{k-1} \sum_{m=1}^{k-j} A_{j,k-j-m}\sigma_{k-2,m-1}(\mathbf{\Lambda}) + \sum_{m=0}^{k-j} A_{j,k-j-m}\sigma_{k-2,m}(\mathbf{\Lambda}) \\ &= \Lambda_{k-1} \sum_{m=0}^{k-j-1} A_{j,k-j-m-1}\sigma_{k-2,m}(\mathbf{\Lambda}) \end{aligned}$$

$$
\begin{aligned}
&+ j \sum_{m=0}^{k-j-1} A_{j,k-j-m-1}\sigma_{k-2,m}(\mathbf{\Lambda}) + \sum_{m=0}^{k-j} A_{j-1,k-j-m}\sigma_{k-2,m}(\mathbf{\Lambda}) \\
= &(\Lambda_{k-1} + j)V_{k-1,j}(\mathbf{\Lambda}) + V_{k-1,j-1}(\mathbf{\Lambda}).
\end{aligned}
$$

Q.E.D.

Lemma 1.6.4. *For $k \geqslant 0$ and $j \geqslant -1$ such that $k \geqslant j$, we have*

$$
\sum_{\ell=j}^{k} U_{k,\ell}(\mathbf{\Lambda})V_{\ell,j}(\mathbf{\Lambda}) = \delta_{k,j}, \tag{1.6.13}
$$

where $\delta_{k,j}$ denotes Kronecker's delta symbol. Consequently,

$$
\sum_{\ell=j}^{k} V_{k,\ell}(\mathbf{\Lambda})U_{\ell,j}(\mathbf{\Lambda}) = \delta_{k,j}. \tag{1.6.14}
$$

Proof. (1.6.13) is obvious if either $k = j$ or $j = -1$. In the proof we may therefore assume that $k - 1 \geqslant j \geqslant 0$ and that (1.6.13) is already proven for smaller values of $k - j$ as well as for smaller values of j and the same $k - j$.

By the recurrence relations (1.6.8, 1.6.12) for $U_{k,j}(\mathbf{\Lambda})$ and $V_{k,j}(\mathbf{\Lambda})$ and the induction hypothesis,

$$
\begin{aligned}
&\sum_{\ell=j}^{k} U_{k,\ell}(\mathbf{\Lambda})V_{\ell,j}(\mathbf{\Lambda}) \\
= &- \sum_{\ell=j}^{k-1}(k - 1 + \Lambda_\ell)U_{k-1,\ell}(\mathbf{\Lambda})V_{\ell,j}(\mathbf{\Lambda}) + \sum_{\ell=j}^{k} U_{k-1,\ell-1}(\mathbf{\Lambda})V_{\ell,j}(\mathbf{\Lambda}) \\
= &- (k-1)\delta_{k-1,j} - \sum_{\ell=j}^{k-1} \Lambda_\ell U_{k-1,\ell}(\mathbf{\Lambda})V_{\ell,j}(\mathbf{\Lambda}) + \sum_{\ell=j+1}^{k} \Lambda_{\ell-1}U_{k-1,\ell-1}(\mathbf{\Lambda})V_{\ell-1,j}(\mathbf{\Lambda}) \\
&+ j \sum_{\ell=j+1}^{k} U_{k-1,\ell-1}(\mathbf{\Lambda})V_{\ell-1,j}(\mathbf{\Lambda}) + \sum_{\ell=j}^{k} U_{k-1,\ell-1}(\mathbf{\Lambda})V_{\ell-1,j-1}(\mathbf{\Lambda}) \\
= &- (k-1)\delta_{k-1,j} + 0 + j\delta_{k-1,j} + \delta_{k-1,j-1} = 0.
\end{aligned}
$$

This completes the proof of (1.6.13). Q.E.D.

Finally, we have the following explicit formulae for $A_{j,m}$.

Lemma 1.6.5. *We have* $A_{0,0} = 1$, $A_{0,m} = 0$ *if* $m \geqslant 1$, *and*

$$A_{j,m} = \sum_{r=1}^{j} \frac{(-)^{j-r} r^{m+j}}{r!(j-r)!} \ \text{if } j \geqslant 1, m \geqslant 0. \tag{1.6.15}$$

Proof. It is easily seen that $A_{0,0} = 1$ and $A_{0,m} = 0$ if $m \geqslant 1$.

It is straightforward to verify that the sequence given by (1.6.15) satisfies the recurrence relation (1.6.9), so it is left to show that (1.6.15) holds true for $m = 0$. Initially, $A_{j,0} = 1$, and hence one must verify

$$\sum_{r=1}^{j} \frac{(-)^{j-r} r^{j}}{r!(j-r)!} = 1.$$

This however follows from considering all the identities obtained by differentiating the following binomial identity up to j times and then evaluating at $x = 1$,

$$(x-1)^j - (-1)^j = j! \sum_{r=1}^{j} \frac{(-1)^{j-r}}{r!(j-r)!} x^r.$$

Q.E.D.

1.6.4. Conclusion

We first observe that, when $0 \leqslant j \leqslant k \leqslant d+1$, both $U_{k,j}(\mathbf{\Lambda})$ and $V_{k,j}(\mathbf{\Lambda})$ are polynomials in $\Lambda_0, ..., \Lambda_d$ according to Lemma 1.6.3 (1, 3). If one puts $\Lambda_m = \Lambda_m(\boldsymbol{\nu})$ for $m = 0, ..., d$, then $U_{k,j}(\boldsymbol{\nu}) = U_{k,j}(\mathbf{\Lambda})$. It follows from Lemma 1.6.4 that $V_{k,j}(\boldsymbol{\nu}) = V_{k,j}(\mathbf{\Lambda})$. Moreover, the relations $\nu_\ell = \lambda_\ell - \lambda_{d+1}$, $\ell = 1, ..., d$, along with the assumption $\sum_{\ell=1}^{d+1} \lambda_\ell = 0$, yields

$$\Lambda_m(\boldsymbol{\nu}) = (d+1)\lambda_{d-m+1}.$$

Now we can reformulate (1.6.7) in the following theorem.

Theorem 1.6.6. *The Bessel function* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *satisfies the following linear differential equation of order* $d+1$

$$\sum_{j=1}^{d+1} V_{d+1,j}(\boldsymbol{\lambda}) x^j w^{(j)} + \left(V_{d+1,0}(\boldsymbol{\lambda}) - S_{d+1}(\boldsymbol{\varsigma})(i(d+1))^{d+1} x^{d+1}\right) w = 0, \tag{1.6.16}$$

where

$$S_{d+1}(\boldsymbol{\varsigma}) = \prod_{\ell=1}^{d+1} \varsigma_\ell, \quad V_{d+1,j}(\boldsymbol{\lambda}) = \sum_{m=0}^{d-j+1} A_{j,d-j-m+1}(d+1)^m \sigma_m(\boldsymbol{\lambda}),$$

$\sigma_m(\boldsymbol{\lambda})$ *denotes the elementary symmetric polynomial in* $\boldsymbol{\lambda}$ *of degree m, with* $\sigma_1(\boldsymbol{\lambda}) = 0$*, and* $A_{j,m}$ *is recurrently defined in Definition* 1.6.2 (3) *and explicitly given in Lemma* 1.6.5. *We shall call the equation* (1.6.16) *a Bessel equation of index* $\boldsymbol{\lambda}$*, or simply a Bessel equation if the index* $\boldsymbol{\lambda}$ *is given.*

For a given index $\boldsymbol{\lambda}$, (1.6.16) only provides two Bessel equations. The sign $S_{d+1}(\boldsymbol{\varsigma})$ determines which one of the two Bessel equations a Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ satisfies.

Definition 1.6.7. *We call* $S_{d+1}(\boldsymbol{\varsigma}) = \prod_{\ell=1}^{d+1} \varsigma_\ell$ *the sign of the Bessel function* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *as well as the Bessel equation satisfied by* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

Finally, we collect some simple facts on $V_{d+1,j}(\boldsymbol{\lambda})$ in the following lemma, which will play important roles later in the study of Bessel equations. See (1.6.14) in Lemma 1.6.4 and Lemma 1.6.3 (3).

Lemma 1.6.8. *We have*

(1). $\sum_{j=0}^{d+1} V_{d+1,j}(\boldsymbol{\lambda}) [-(d+1)\lambda_{d+1}]_j = 0.$

(2). $V_{d+1,d}(\boldsymbol{\lambda}) = \frac{1}{2} d(d+1).$

Remark 1.6.9. *If we define*

$$\boldsymbol{J}(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = J\left((d+1)^{-1} x; \boldsymbol{\varsigma}, (d+1)^{-1} \boldsymbol{\lambda}\right),$$

then this normalized Bessel function satisfies a differential equation with coefficients free of powers of $(d+1)$*, that is,*

$$\sum_{j=1}^{d+1} \boldsymbol{V}_{d+1,j}(\boldsymbol{\lambda}) x^j w^{(j)} + \left(\boldsymbol{V}_{d+1,0}(\boldsymbol{\lambda}) - S_{d+1}(\boldsymbol{\varsigma}) i^{d+1} x^{d+1}\right) w = 0,$$

with

$$\boldsymbol{V}_{d+1,j}(\boldsymbol{\lambda}) = \sum_{m=0}^{d-j+1} A_{j,d-j-m+1}\sigma_m(\boldsymbol{\lambda}).$$

In particular, if $d = 1$, $\boldsymbol{\lambda} = (\lambda, -\lambda)$, then the two normalized Bessel equations are

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} + \left(-\lambda^2 \pm x^2\right) w = 0.$$

These are exactly the Bessel equation and the modified Bessel equation of index λ.

1.7. Bessel equations

The theory of linear ordinary differential equations with analytic coefficients[IX] will be employed in this section to study Bessel equations.

Subsequently, we shall use z instead of x to indicate complex variable. For $\varsigma \in \{+, -\}$ and $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$, we introduce the Bessel differential operator

$$\nabla_{\varsigma,\boldsymbol{\lambda}} = \sum_{j=1}^{n} V_{n,j}(\boldsymbol{\lambda})z^j\frac{d^j}{dz^j} + V_{n,0}(\boldsymbol{\lambda}) - \varsigma(in)^n z^n. \tag{1.7.1}$$

The Bessel equation of index $\boldsymbol{\lambda}$ and sign ς may be written as

$$\nabla_{\varsigma,\boldsymbol{\lambda}}(w) = 0. \tag{1.7.2}$$

We shall study Bessel equations on the Riemann surface $\mathbb{U}$ associated with $\log z$, that is, the universal cover of $\mathbb{C} \smallsetminus \{0\}$. Each element in $\mathbb{U}$ is represented by a pair (x, ω) with modulus $x \in \mathbb{R}_+$ and argument $\omega \in \mathbb{R}$, and will be denoted by $z = xe^{i\omega} = e^{\log x + i\omega}$ with some ambiguity. Conventionally, define $z^\lambda = e^{\lambda \log z}$ for $z \in \mathbb{U}, \lambda \in \mathbb{C}$, $\bar{z} = e^{-\log z}$, and moreover let $1 = e^0$, $-1 = e^{\pi i}$ and $\pm i = e^{\pm\frac{1}{2}\pi i}$.

First of all, since Bessel equations are nonsingular on $\mathbb{U}$, all the solutions of Bessel equations are analytic on $\mathbb{U}$.

[IX][CL, Chapter 4, 5] and [Was, Chapter II-V] are the main references that we follow, and the reader is referred to these books for terminologies and definitions.

Each Bessel equation has only two singularities at $z = 0$ and $z = \infty$. According to the classification of singularities, 0 is a *regular singularity*, so the Frobenius method gives rise to solutions of Bessel equations developed in series of ascending powers of z, or possibly logarithmic sums of this kind of series, whereas ∞ is an *irregular singularity of rank one*, and therefore one may find certain formal solutions that are the asymptotic expansions of some actual solutions of Bessel equations.

When studying the asymptotic expansions for the Bessel equation (1.7.2), it is more convenient to consider its corresponding system of differential equations,

$$w' = B(z; \varsigma, \boldsymbol{\lambda}) w, \tag{1.7.3}$$

with

$$B(z; \varsigma, \boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -V_{n,0}(\boldsymbol{\lambda}) z^{-n} + \varsigma (in)^n & -V_{n,1}(\boldsymbol{\lambda}) z^{-n+1} & \cdots & \cdots & -V_{n,n-1}(\boldsymbol{\lambda}) z^{-1} \end{pmatrix}.$$

A simple but important observation is as follows.

Lemma 1.7.1. *Let $\varsigma \in \{+, -\}$ and a be an integer. If $\varphi(z)$ is a solution of the Bessel equation of sign ς, then $\varphi\left(e^{\pi i \frac{a}{n}} z\right)$ satisfies the Bessel equation of sign $(-)^a \varsigma$.*

Variants of Lemma 1.7.1, Lemma 1.7.3, 1.7.9 and 1.7.21, will play important roles later in §1.8 when we study the connection formulae for various kinds of Bessel functions.

1.7.1. Bessel functions of the first kind

The indicial equation associated with $\nabla_{\varsigma,\lambda}$ is given as below,

$$\sum_{j=0}^{n} [\rho]_j V_{n,j}(\boldsymbol{\lambda}) = 0.$$

Let $P_\lambda(\rho)$ denote the polynomial on the left of this equation. Lemma 1.6.8 (1) along with the symmetry of $V_{n,j}(\boldsymbol{\lambda})$ yields the following identity,

$$\sum_{j=0}^{n}[-n\lambda_\ell]_j V_{n,j}(\boldsymbol{\lambda}) = 0,$$

for each $\ell = 1, ..., n$. Therefore,

$$P_\lambda(\rho) = \prod_{\ell=1}^{n}(\rho + n\lambda_\ell).$$

Consider the formal series

$$\sum_{m=0}^{\infty} c_m z^{\rho+m},$$

where the index ρ and the coefficients c_m, with $c_0 \neq 0$, are to be determined. It is easy to see that

$$\nabla_{\varsigma,\lambda} \sum_{m=0}^{\infty} c_m z^{\rho+m} = \sum_{m=0}^{\infty} c_m P_\lambda(\rho + m) z^{\rho+m} - \varsigma(in)^n \sum_{m=0}^{\infty} c_m z^{\rho+m+n}.$$

If the following equations are satisfied

$$\begin{aligned} &c_m P_\lambda(\rho + m) = 0, \quad n > m \geqslant 1, \\ &c_m P_\lambda(\rho + m) - \varsigma(in)^n c_{m-n} = 0, \quad m \geqslant n, \end{aligned} \tag{1.7.4}$$

then

$$\nabla_{\varsigma,\lambda} \sum_{m=0}^{\infty} c_m z^{\rho+m} = c_0 P_\lambda(\rho) z^\rho.$$

Given $\ell \in \{1, ..., n\}$. Choose $\rho = -n\lambda_\ell$ and let $c_0 = \prod_{k=1}^{n} \Gamma\left(\lambda_k - \lambda_\ell + 1\right)^{-1}$. Suppose, for the moment, that no two components of $n\boldsymbol{\lambda}$ differ by an integer. Then $P_\lambda(-n\lambda_\ell + m) \neq 0$ for any $m \geqslant 1$ and $c_0 \neq 0$, and hence the system of equations (1.7.4) is uniquely solvable. It follows that

$$\sum_{m=0}^{\infty} \frac{(\varsigma i^n)^m z^{n(-\lambda_\ell+m)}}{\prod_{k=1}^{n} \Gamma\left(\lambda_k - \lambda_\ell + m + 1\right)} \tag{1.7.5}$$

is a formal solution of the differential equation (1.7.2).

Now suppose that $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ is unrestricted. The series in (1.7.5) is absolutely convergent, locally uniformly convergent with respect to $\boldsymbol{\lambda}$, and hence gives rise to an analytic function of z on the Riemann surface $\mathbb{U}$, as well as an analytic function of $\boldsymbol{\lambda}$. We denote by $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ the analytic function given by the series (1.7.5) and call it a *Bessel function of the first kind*. It is evident that $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ is an actual solution of (1.7.2).

Definition 1.7.2. *Let $\mathbb{D}^{n-1}$ denote the set of $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ such that no two components of $\boldsymbol{\lambda}$ differ by an integer. We call an index $\boldsymbol{\lambda}$ generic if $\boldsymbol{\lambda} \in \mathbb{D}^{n-1}$.*

When $\boldsymbol{\lambda} \in \mathbb{D}^{n-1}$, all the $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ constitute a fundamental set of solutions, since the leading term in the expression (1.7.5) of $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ does not vanish. However, this is no longer the case if $\boldsymbol{\lambda} \notin \mathbb{D}^{n-1}$. Indeed, if $\lambda_\ell - \lambda_k$ is an integer, $k \neq \ell$, then $J_\ell(z; \varsigma, \boldsymbol{\lambda}) = (\varsigma i^n)^{\lambda_\ell - \lambda_k} J_k(z; \varsigma, \boldsymbol{\lambda})$. There are other solutions arising as certain logarithmic sums of series of ascending powers of z. Roughly speaking, powers of $\log z$ may occur in some solutions. For more details the reader may consult [CL, §4.8].

Lemma 1.7.3. *Let a be an integer. We have*

$$J_\ell\left(e^{\pi i \frac{a}{n}} z; \varsigma, \boldsymbol{\lambda}\right) = e^{-\pi i a \lambda_\ell} J_\ell(z; (-)^a \varsigma, \boldsymbol{\lambda}).$$

Remark 1.7.4. *If $n = 2$, then we have the following formulae according to* [Wat, 3.1 (8), 3.7 (2)],

$$J_1(z; +, \lambda, -\lambda) = J_{-2\lambda}(2z), \quad J_2(z; +, \lambda, -\lambda) = J_{2\lambda}(2z),$$

$$J_1(z; -, \lambda, -\lambda) = I_{-2\lambda}(2z), \quad J_2(z; -, \lambda, -\lambda) = I_{2\lambda}(2z).$$

1.7.2. The analytic continuation of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$

For any given $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$, since $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ satisfies the Bessel equation of sign $S_n(\boldsymbol{\varsigma})$, it admits a unique analytic continuation $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ onto $\mathbb{U}$. Recall the definition

$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{1}{2\pi i} \int_{\mathcal{C}} G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) x^{-ns} ds, \quad x \in \mathbb{R}_+,$$

where $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \prod_{k=1}^{n} \Gamma(s - \lambda_k) e\left(\frac{1}{4} \varsigma_k (s - \lambda_k)\right)$ and $\mathcal{C}$ is a suitable contour.

Let $\varsigma = S_n(\boldsymbol{\varsigma})$. For the moment, let us assume that $\boldsymbol{\lambda}$ is generic. For $\ell = 1, ..., n$ and $m = 0, 1, 2, ...$, $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ has a simple pole at $\lambda_\ell - m$ with residue

$$(-)^m \frac{1}{m!} e\left(\frac{\sum_{k=1}^{n} \varsigma_k (\lambda_\ell - \lambda_k - m)}{4}\right) \prod_{k \neq \ell} \Gamma(\lambda_\ell - \lambda_k - m) =$$

$$\pi^{n-1} e\left(-\frac{\sum_{k=1}^{n} \varsigma_k \lambda_k}{4}\right) e\left(\frac{\sum_{k=1}^{n} \varsigma_k \lambda_\ell}{4}\right) \left(\prod_{k \neq \ell} \frac{1}{\sin(\pi(\lambda_\ell - \lambda_k))}\right) \frac{(\varsigma i^n)^m}{\prod_{k=1}^{n} \Gamma(\lambda_k - \lambda_\ell + m + 1)}.$$

Here we have used Euler's reflection formula for the Gamma function. Applying Cauchy's residue theorem, $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is developed into an absolutely convergent series on shifting the contour $\mathcal{C}$ far left, and, in view of (1.7.5), we obtain

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \pi^{n-1} E(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \sum_{\ell=1}^{n} E_\ell(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) S_\ell(\boldsymbol{\lambda}) J_\ell(z; \varsigma, \boldsymbol{\lambda}), \quad z \in \mathbb{U}, \tag{1.7.6}$$

with

$$E(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(-\frac{\sum_{k=1}^{n} \varsigma_k \lambda_k}{4}\right), \ E_l(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(\frac{\sum_{k=1}^{n} \varsigma_k \lambda_l}{4}\right), \ S_l(\boldsymbol{\lambda}) = \prod_{k \neq l} \frac{1}{\sin(\pi(\lambda_l - \lambda_k))}.$$

Because of the possible vanishing of $\sin(\pi(\lambda_\ell - \lambda_k))$, the definition of $S_\ell(\boldsymbol{\lambda})$ may fail to make sense if $\boldsymbol{\lambda}$ is not generic. In order to properly interpret (1.7.6) in the non-generic case, one has to pass to the limit, that is,

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \pi^{n-1} E(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \cdot \lim_{\substack{\boldsymbol{\lambda}' \to \boldsymbol{\lambda} \\ \boldsymbol{\lambda}' \in \mathbb{D}^{n-1}}} \sum_{\ell=1}^{n} E_\ell(\boldsymbol{\varsigma}, \boldsymbol{\lambda}') S_\ell(\boldsymbol{\lambda}') J_\ell(z; \varsigma, \boldsymbol{\lambda}'). \tag{1.7.7}$$

We recollect the definitions of $L_\pm(\boldsymbol{\varsigma})$ and $n_\pm(\boldsymbol{\varsigma})$ introduced in Proposition 1.2.9.

Definition 1.7.5. *Let $\boldsymbol{\varsigma} \in \{+,-\}^n$. We define $L_\pm(\boldsymbol{\varsigma}) = \{\ell : \varsigma_\ell = \pm\}$ and $n_\pm(\boldsymbol{\varsigma}) = |L_\pm(\boldsymbol{\varsigma})|$. The pair of integers $(n_+(\boldsymbol{\varsigma}), n_-(\boldsymbol{\varsigma}))$ is called the signature of $\boldsymbol{\varsigma}$, as well as the signature of the Bessel function $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.*

With Definition 1.7.5, we reformulate (1.7.6, 1.7.7) in the following lemma.

Lemma 1.7.6. *We have*

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \pi^{n-1} E(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \sum_{\ell=1}^{n} E_\ell(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) S_\ell(\boldsymbol{\lambda}) J_\ell\big(z; (-)^{n_-(\varsigma)}, \boldsymbol{\lambda}\big),$$

with $E(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(-\frac{1}{4}\sum_{k \in L_+(\varsigma)} + \frac{1}{4}\sum_{k \in L_-(\varsigma)} \lambda_k\right)$, $E_\ell(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(\frac{1}{4}(n_+(\boldsymbol{\varsigma}) - n_-(\boldsymbol{\varsigma}))\lambda_\ell\right)$ and $S_\ell(\boldsymbol{\lambda}) = 1/\prod_{k \neq \ell} \sin(\pi(\lambda_\ell - \lambda_k))$. When $\boldsymbol{\lambda}$ is not generic, the right hand side is to be replaced by its limit.

Remark 1.7.7. *In view of Proposition* 1.2.7 *and Remark* 1.7.4*, Lemma* 1.7.6 *is equivalent to the connection formulae in* (1.1.12, 1.1.13) (*see* [Wat, 3.61(5, 6), 3.7 (6)]).

Remark 1.7.8. *In the case when $\boldsymbol{\lambda} = \frac{1}{n}\left(\frac{n-1}{2}, ..., -\frac{n-1}{2}\right)$, the formula in Lemma* 1.7.6 *amounts to splitting the Taylor series expansion of $e^{in\xi(\varsigma)x}$ in* (1.2.15) *according to the residue class of indices modulo n. To see this, one requires the multiplicative formula of the Gamma function* (1.2.16) *as well as the trigonometric identity*

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Using Lemma 1.7.3 and 1.7.6, one proves the following lemma, which implies that the Bessel function $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is determined by its signature up to a constant multiple.

Lemma 1.7.9. *Define $H^\pm(z; \boldsymbol{\lambda}) = J(z; \pm, ..., \pm, \boldsymbol{\lambda})$. Then*

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(\pm \frac{\sum_{\ell \in L_\mp(\varsigma)} \lambda_\ell}{2}\right) H^\pm\left(e^{\pm\pi i \frac{n_\mp(\varsigma)}{n}} z; \boldsymbol{\lambda}\right).$$

Remark 1.7.10. *We have the following Barnes type integral representation,*

$$J(z;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = \frac{1}{2\pi i}\int_{\mathcal{C}'} G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})z^{-ns}ds, \quad z \in \mathbb{U}, \tag{1.7.8}$$

where $\mathcal{C}'$ *is a contour that starts from and returns to* $-\infty$ *after encircling the poles of the integrand counter-clockwise. Compare* [Wat, §6.5]. *Lemma* 1.7.9 *may also be seen from this integral representation.*

When $-\frac{n_-(\boldsymbol{\varsigma})}{n}\pi < \arg z < \frac{n_+(\boldsymbol{\varsigma})}{n}\pi$, *the contour* $\mathcal{C}'$ *may be opened out to the vertical line* (σ), *with* $\sigma > \max\{\Re\mathrm{e}\,\lambda_\ell\}$. *Thus*

$$J(z;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = \frac{1}{2\pi i}\int_{(\sigma)} G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})z^{-ns}ds, \quad -\frac{n_-(\boldsymbol{\varsigma})}{n}\pi < \arg z < \frac{n_+(\boldsymbol{\varsigma})}{n}\pi. \tag{1.7.9}$$

On the boundary rays $\arg z = \pm\frac{n_\pm(\boldsymbol{\varsigma})}{n}\pi$, *the contour* (σ) *should be shifted to* $\mathcal{C}$ *defined as in* §1.2.1, *in order to secure convergence.*

The contour integrals in (1.7.8, 1.7.9) *absolutely converge, locally uniformly in both* z *and* $\boldsymbol{\lambda}$. *To see these, one uses Stirling's formula to examine the behaviour of the integrand* $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})z^{-ns}$ *on integral contours, where for* (1.7.8) *a transformation of* $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ *by Euler's reflection formula is required.*

1.7.3. Asymptotics for Bessel equations and Bessel functions of the second kind

Subsequently, we proceed to investigate the asymptotics at infinity for Bessel equations.

Definition 1.7.11. *For* $\varsigma \in \{+,-\}$ *and a positive integer* N, *we let* $\mathbb{X}_N(\varsigma)$ *denote the set of* N*-th roots of* $\varsigma 1$.[X]

Before delving into our general study, let us first consider the prototypical example given in Proposition 1.2.9.

[X]Under certain circumstances, it is suitable to view an element ξ of $\mathbb{X}_N(\varsigma)$ as a point in $\mathbb{U}$ instead of $\mathbb{C} \smallsetminus \{0\}$. This however should be clear from the context.

Proposition 1.7.12. *For any $\xi \in \mathbb{X}_{2n}(+)$, the function $z^{-\frac{n-1}{2}} e^{in\xi z}$ is a solution of the Bessel equation of index $\frac{1}{n}\left(\frac{n-1}{2}, ..., -\frac{n-1}{2}\right)$ and sign ξ^n.*

Proof. When $\Im\mathfrak{m}\, \xi \geqslant 0$, this can be seen from Proposition 1.2.9 and Theorem 1.6.6. For arbitrary ξ, one makes use of Lemma 1.7.1. Q.E.D.

Formal solutions of Bessel equations at infinity

Following [CL, Chapter 5], we shall consider the system of differential equations (1.7.3). We have

$$B(\infty; \varsigma, \boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varsigma (in)^n & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

If one let $\mathbb{X}_n(\varsigma) = \{\xi_1, ..., \xi_n\}$, then the eigenvalues of $B(\infty; \varsigma, \boldsymbol{\lambda})$ are $in\xi_1, ..., in\xi_n$. The conjugation by the following matrix diagonalizes $B(\infty; \varsigma, \boldsymbol{\lambda})$,

$$T = \frac{1}{n} \begin{pmatrix} 1 & (in\xi_1)^{-1} & \cdots & (in\xi_1)^{-n+1} \\ 1 & (in\xi_2)^{-1} & \cdots & (in\xi_2)^{-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (in\xi_n)^{-1} & \cdots & (in\xi_n)^{-n+1} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ in\xi_1 & in\xi_2 & \cdots & in\xi_n \\ \cdots & \cdots & \cdots & \cdots \\ (in\xi_1)^{n-1} & (in\xi_2)^{n-1} & \cdots & (in\xi_n)^{n-1} \end{pmatrix}.$$

Thus, the substitution $u = Tw$ turns the system of differential equations (1.7.3) into

$$u' = A(z)u, \tag{1.7.10}$$

where $A(z) = TB(z; \varsigma, \boldsymbol{\lambda})T^{-1}$ is a matrix of polynomials in z^{-1} of degree n,

$$A(z) = \sum_{j=0}^{n} z^{-j} A_j,$$

with

$$(1.7.11)\qquad \begin{aligned} A_0 &= \varDelta = \operatorname{diag}\,(in\xi_\ell)_{\ell=1}^{n}\,, \\ A_j &= -i^{-j+1}n^{-j}V_{n,n-j}(\lambda)\left(\xi_k\xi_\ell^{-j}\right)_{k,\ell=1}^{n},\ j=1,...,n. \end{aligned}$$

It is convenient to put $A_j = 0$ if $j > n$. The dependence on ς, λ and the ordering of the eigenvalues have been suppressed in our notations in the interest of brevity.

Suppose $\widehat{\Phi}$ is a formal solution matrix for (1.7.10) of the form

$$\widehat{\Phi}(z) = P(z)z^R e^{Qz},$$

where P is a formal power series in z^{-1},

$$P(z) = \sum_{m=0}^{\infty} z^{-m}P_m,$$

and R, Q are constant diagonal matrices. Since

$$\widehat{\Phi}' = P'z^R e^{Qz} + z^{-1}PRz^R e^{Qz} + Pz^R Qe^{Qz} = \left(P' + z^{-1}PR + PQ\right) z^R e^{Qz},$$

the differential equation (1.7.10) yields

$$\sum_{m=0}^{\infty} z^{-m-1}P_m(R - mI) + \sum_{m=0}^{\infty} z^{-m}P_m Q = \left(\sum_{j=0}^{\infty} z^{-j}A_j\right)\left(\sum_{m=0}^{\infty} z^{-m}P_m\right),$$

where I denotes the identity matrix. Comparing the coefficients of various powers of z^{-1}, it follows that $\widehat{\Phi}$ is a formal solution matrix for (1.7.10) if and only if R, Q and P_m satisfy the following equations

$$(1.7.12)\qquad \begin{aligned} P_0Q - \varDelta P_0 &= 0 \\ P_{m+1}Q - \varDelta P_{m+1} &= \sum_{j=1}^{m+1} A_jP_{m-j+1} + P_m(mI - R), \quad m \geqslant 0. \end{aligned}$$

A solution of the first equation in (1.7.12) is given by

$$(1.7.13)\qquad Q = \varDelta, \quad P_0 = I.$$

Using (1.7.13), the second equation in (1.7.12) for $m = 0$ becomes

$$P_1\Delta - \Delta P_1 = A_1 - R. \tag{1.7.14}$$

Since Δ is diagonal, the diagonal entries of the left side of (1.7.14) are zero, and hence the diagonal entries of R must be identical with those of A_1. In view of (1.7.11) and Lemma 1.6.8 (2), we have

$$A_1 = -\frac{1}{n}V_{n,n-1}(\boldsymbol{\lambda}) \cdot \left(\xi_k\xi_\ell^{-1}\right)_{k,\ell=1}^n = -\frac{n-1}{2}\left(\xi_k\xi_\ell^{-1}\right)_{k,\ell=1}^n,$$

and therefore

$$R = -\frac{n-1}{2}I. \tag{1.7.15}$$

Let $p_{1,k\ell}$ denote the (k,ℓ)-th entry of P_1. It follows from (1.7.11, 1.7.14) that

$$in(\xi_\ell - \xi_k)p_{1,k\ell} = -\frac{n-1}{2}\xi_k\xi_\ell^{-1}, \quad k \neq \ell. \tag{1.7.16}$$

The off-diagonal entries of P_1 are uniquely determined by (1.7.16). Therefore, a solution of (1.7.14) is

$$P_1 = D_1 + P_1^o, \tag{1.7.17}$$

where D_1 is any diagonal matrix and P_1^o is the matrix with diagonal entries zero and (k,ℓ)-th entry $p_{1,k\ell}$, $k \neq \ell$. To determine D_1, one resorts to the second equation in (1.7.12) for $m = 1$, which, in view of (1.7.13, 1.7.15, 1.7.17), may be written as

$$P_2\Delta - \Delta P_2 - \left(A_1 + \frac{n-1}{2}\right)D_1 - \frac{n+1}{2}P_1^o = A_1P_1^o + A_2 + D_1.$$

The matrix on the left side has zero diagonal entries. It follows that D_1 must be equal to the diagonal part of $-A_1P_1^o - A_2$.

In general, using (1.7.13, 1.7.15), the second equation in (1.7.12) may be written as

$$P_{m+1}\Delta - \Delta P_{m+1} = \sum_{j=1}^{m+1} A_j P_{m-j+1} + \left(m + \frac{n-1}{2}\right) P_m, \quad m \geqslant 0. \tag{1.7.18}$$

Applying (1.7.18), an induction on m implies that

$$P_m = D_m + P_m^o, \quad m \geqslant 1,$$

where D_m and P_m^o are inductively defined as follows. Put $D_0 = I$. Let mD_m be the diagonal part of

$$-\sum_{j=2}^{m+1} A_j D_{m-j+1} - \sum_{j=1}^{m} A_j P_{m-j+1}^o,$$

and let P_{m+1}^o be the matrix with diagonal entries zero such that $P_{m+1}^o\Delta - \Delta P_{m+1}^o$ is the off-diagonal part of

$$\sum_{j=1}^{m+1} A_j D_{m-j+1} + \sum_{j=1}^{m} A_j P_{m-j+1}^o + \left(m + \frac{n-1}{2}\right) P_m^o.$$

In this way, an inductive construction of the formal solution matrix of (1.7.10) is completed for the given initial choices $Q = \Delta$, $P_0 = I$.

With the observations that A_j is of degree j in $\boldsymbol{\lambda}$ for $j \geqslant 2$ and that A_1 is constant, we may show the following lemma using an inductive argument.

Lemma 1.7.13. *The entries of P_m are symmetric polynomial in $\boldsymbol{\lambda}$. If $m \geqslant 1$, then the off-diagonal entries of P_m have degree at most $2m-2$, whereas the degree of each diagonal entry is exactly $2m$.*

The first row of $T^{-1}\widehat{\Phi}$ constitute a fundamental system of formal solutions of the Bessel equation (1.7.2). Some calculations yield the following proposition, where for derivatives of order higher than $n-1$ the differential equation (1.7.2) is applied.

Proposition 1.7.14. *Let $\varsigma \in \{+,-\}$ and $\xi \in \mathbb{X}_n(\varsigma)$. There exists a unique sequence of symmetric polynomials $B_m(\boldsymbol{\lambda};\xi)$ in $\boldsymbol{\lambda}$ of degree $2m$ and coefficients depending only on m, ξ and n, normalized so that $B_0(\boldsymbol{\lambda};\xi) = 1$, such that*

$$e^{in\xi z} z^{-\frac{n-1}{2}} \sum_{m=0}^{\infty} B_m(\boldsymbol{\lambda};\xi) z^{-m} \tag{1.7.19}$$

is a formal solution of the Bessel equation of sign ς (1.7.2)*. We shall denote the formal series in* (1.7.19) *by $\widehat{J}(z;\boldsymbol{\lambda};\xi)$. Moreover, the j-th formal derivative $\widehat{J}^{(j)}(z;\boldsymbol{\lambda};\xi)$ is also of the form as* (1.7.19)*, but with coefficients depending on j as well.*

Remark 1.7.15. *The above arguments are essentially adapted from the proof of* [CL, Chapter 5, Theorem 2.1]. *This construction of the formal solution and Lemma* 1.7.13 *will be required later in* § 1.7.4 *for the error analysis.*

However, This method is not the best for the actual computation of the coefficients $B_m(\boldsymbol{\lambda};\xi)$. We may derive the recurrent relations for $B_m(\boldsymbol{\lambda};\xi)$ by a more direct but less suggestive approach as follows.

The substitution $w = e^{in\xi z} z^{-\frac{n-1}{2}} u$ transforms the Bessel equation (1.7.2) *into*

$$\sum_{j=0}^{n} W_j(z;\boldsymbol{\lambda}) u^{(j)} = 0,$$

where $W_j(z;\boldsymbol{\lambda})$ is a polynomial in z^{-1} of degree $n-j$,

$$W_j(z;\boldsymbol{\lambda}) = \sum_{k=0}^{n-j} W_{j,k}(\boldsymbol{\lambda}) z^{-k},$$

with

$$\begin{aligned} W_{0,0}(\boldsymbol{\lambda}) &= (in\xi)^n - \varsigma(in)^n = 0, \\ W_{j,k}(\boldsymbol{\lambda}) &= \frac{(in\xi)^{n-j-k}}{j!(n-j-k)!} \sum_{r=0}^{k} \frac{(n-r)!}{(k-r)!} \left[-\frac{n-1}{2}\right]_{k-r} V_{n,n-r}(\boldsymbol{\lambda}), \quad (j,k) \neq (0,0). \end{aligned}$$

We have

$$W_{0,1}(\boldsymbol{\lambda}) = (in\xi)^{n-1}\left(n\left(-\frac{n-1}{2}\right)V_{n,n}(\boldsymbol{\lambda}) + V_{n,n-1}(\boldsymbol{\lambda})\right) = 0,$$

but $W_{1,0}(\boldsymbol{\lambda}) = n(in\xi)^{n-1}$ *is nonzero. Some calculations show that* $B_m(\boldsymbol{\lambda};\xi)$ *satisfy the following recurrence relations*

$$(m-1)W_{1,0}(\boldsymbol{\lambda})B_{m-1}(\boldsymbol{\lambda};\xi) = \sum_{k=2}^{\min\{m,n\}} B_{m-k}(\boldsymbol{\lambda};\xi)\sum_{j=0}^{k} W_{j,k-j}(\boldsymbol{\lambda})[k-m]_j, \quad m \geqslant 2.$$

If $n=2$*, for a fourth root of unity* $\xi = \pm 1, \pm i$ *one may calculate in this way to obtain*

$$B_m(\lambda,-\lambda;\xi) = \frac{\left(\frac{1}{2}-2\lambda\right)_m\left(\frac{1}{2}+2\lambda\right)_m}{(4i\xi)^m m!}.$$

Bessel functions of the second kind

Bessel functions of the second kind are solutions of Bessel equations defined according to their asymptotic expansions at infinity. We shall apply several results in the asymptotic theory of ordinary differential equations from [Was, Chapter IV].

Firstly, [Was, Theorem 12.3] implies the following lemma.

Lemma 1.7.16 (Existence of solutions)**.** *Let* $\varsigma \in \{+,-\}$, $\xi \in \mathbb{X}_n(\varsigma)$*, and* $\mathbb{S} \subset \mathbb{U}$ *be an open sector with vertex at the origin and a positive central angle not exceeding* π*. Then there exists a solution of the Bessel equation of sign* ς (1.7.2) *that has the asymptotic expansion* $\widehat{J}(z;\boldsymbol{\lambda};\xi)$ *defined in* (1.7.19) *on* $\mathbb{S}$*. Moreover, each derivative of this solution has the formal derivative of* $\widehat{J}(z;\boldsymbol{\lambda};\xi)$ *of the same order as its asymptotic expansion.*

For two distinct $\xi, \xi' \in \mathbb{X}_n(\varsigma)$, the ray emitted from the origin on which

$$\mathfrak{Re}\,((i\xi - i\xi')z) = -\mathfrak{Im}\,((\xi-\xi')z) = 0$$

is called a *separation ray*.

We first consider the case $n = 2$. It is clear that the separation rays constitute either the real or the imaginary axis and thus separate $\mathbb{C} \smallsetminus \{0\}$ into two half-planes. Accordingly, we define $\mathbb{S}_{\pm 1} = \{z : \pm\mathfrak{Im}\, z > 0\}$ and $\mathbb{S}_{\pm i} = \{z : \pm\mathfrak{Re}\, z > 0\}$.

In the case $n \geqslant 3$, there are $2n$ distinct separation rays in $\mathbb{C} \smallsetminus \{0\}$ given by the equations

$$\arg z = \arg(i\xi'), \quad \xi' \in \mathbb{X}_{2n}(+).$$

These separation rays divide $\mathbb{C} \smallsetminus \{0\}$ into $2n$ many open sectors

$$\mathbb{S}_\xi^\pm = \left\{ z : 0 < \pm \left(\arg z - \arg(i\bar\xi) \right) < \frac{\pi}{n} \right\}, \quad \xi \in \mathbb{X}_n(\varsigma). \tag{1.7.20}$$

In both sectors $\mathbb{S}_\xi^+$ and $\mathbb{S}_\xi^-$ we have

$$\mathfrak{Re}\,(i\xi z) < \mathfrak{Re}\,(i\xi' z) \ \text{ for all } \xi' \in \mathbb{X}_n(\varsigma),\ \xi' \neq \xi. \tag{1.7.21}$$

Let $\mathbb{S}_\xi$ be the sector on which (1.7.21) is satisfied. It is evident that

$$\mathbb{S}_\xi = \left\{ z : \left| \arg z - \arg(i\bar\xi) \right| < \frac{\pi}{n} \right\}. \tag{1.7.22}$$

Lemma 1.7.17. *Let $\varsigma \in \{+, -\}$ and $\xi \in \mathbb{X}_n(\varsigma)$.*

(1. Existence of asymptotics). *If $n \geqslant 3$, on the sector $\mathbb{S}_\xi^\pm$, all the solutions of the Bessel equation of sign ς have asymptotic representation a multiple of $\widehat{J}(z; \boldsymbol{\lambda}; \xi')$ for some $\xi' \in \mathbb{X}_n(\varsigma)$. If $n = 2$, the same assertion is true with $\mathbb{S}_\xi^\pm$ replaced by $\mathbb{S}_\xi$.*

(2. Uniqueness of the solution). *There is a unique solution of the Bessel equation of sign ς that possesses $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ as its asymptotic expansion on $\mathbb{S}_\xi$ or any of its open subsector, and we shall denote this solution by $J(z; \boldsymbol{\lambda}; \xi)$. Moreover, $J^{(j)}(z; \boldsymbol{\lambda}; \xi) \sim \widehat{J}^{(j)}(z; \boldsymbol{\lambda}; \xi)$ on $\mathbb{S}_\xi$ for any $j \geqslant 0$.*

Proof. (1) follows directly from [Was, Theorem 15.1].

For $n = 2$, since (1.7.21) holds for the sector $\mathbb{S}_\xi$, (2) is true according to [Was, Corollary to Theorem 15.3]. Similarly, if $n \geqslant 3$, (2) is true with $\mathbb{S}_\xi$ replaced by $\mathbb{S}_\xi^\pm$. Thus there exists a unique solution of the Bessel equation of sign ς possessing $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ as its asymptotic expansion on $\mathbb{S}_\xi^\pm$ or any of its open subsector. For the moment, we denote this solution by $J^\pm(z; \boldsymbol{\lambda}; \xi)$. On the other hand, because $\mathbb{S}_\xi$ has central angle $\frac{2}{n}\pi < \pi$, there exists a solution $J(z; \boldsymbol{\lambda}; \xi)$ with asymptotic $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ on a given open subsector $\mathbb{S} \subset \mathbb{S}_\xi$ due to Lemma 1.7.16. Observe that at least one of $\mathbb{S} \cap \mathbb{S}_\xi^+$ and $\mathbb{S} \cap \mathbb{S}_\xi^-$ is a nonempty open sector, say $\mathbb{S} \cap \mathbb{S}_\xi^+ \neq \emptyset$, then the uniqueness of $J(z; \boldsymbol{\lambda}; \xi)$ follows from that of $J^+(z; \boldsymbol{\lambda}; \xi)$ along with the principle of analytic continuation. Q.E.D.

Proposition 1.7.18. *Let $\varsigma \in \{+, -\}$, $\xi \in \mathbb{X}_n(\varsigma)$, ϑ be a small positive constant, say $0 < \vartheta < \frac{1}{2}\pi$, and define*

$$\mathbb{S}'_\xi(\vartheta) = \left\{ z : \left| \arg z - \arg(i\bar{\xi}) \right| < \pi + \frac{\pi}{n} - \vartheta \right\}. \tag{1.7.23}$$

Then $J(z; \boldsymbol{\lambda}; \xi)$ is the unique solution of the Bessel equation of sign ς that has the asymptotic expansion $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ on $\mathbb{S}'_\xi(\vartheta)$. Moreover, $J^{(j)}(z; \boldsymbol{\lambda}; \xi) \sim \widehat{J}^{(j)}(z; \boldsymbol{\lambda}; \xi)$ on $\mathbb{S}'_\xi(\vartheta)$ for any $j \geqslant 0$.

Proof. Following from Lemma 1.7.16, there exists a solution of the Bessel equation of sign ς that has the asymptotic expansion $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ on the open sector

$$\mathbb{S}_\xi^\pm(\vartheta) = \left\{ z : \frac{\pi}{n} - \vartheta < \pm \left(\arg z - \arg(i\bar{\xi}) \right) < \pi + \frac{\pi}{n} - \vartheta \right\}.$$

On the nonempty open sector $\mathbb{S}_\xi \cap \mathbb{S}_\xi^\pm(\vartheta)$ this solution must be identical with $J(z; \boldsymbol{\lambda}; \xi)$ by Lemma 1.7.17 (2) and hence is equal to $J(z; \boldsymbol{\lambda}; \xi)$ on $\mathbb{S}_\xi \cup \mathbb{S}_\xi^\pm(\vartheta)$ due to the principle of analytic continuation. Therefore, the region of validity of the asymptotic $J(z; \boldsymbol{\lambda}; \xi) \sim \widehat{J}(z; \boldsymbol{\lambda}; \xi)$ may be widened from $\mathbb{S}_\xi$ onto $\mathbb{S}'_\xi(\vartheta) = \mathbb{S}_\xi \cup \mathbb{S}_\xi^+(\vartheta) \cup \mathbb{S}_\xi^-(\vartheta)$. In the same way, Lemma 1.7.16 and 1.7.17 (2) also imply that $J^{(j)}(z; \boldsymbol{\lambda}; \xi) \sim \widehat{J}^{(j)}(z; \boldsymbol{\lambda}; \xi)$ on $\mathbb{S}'_\xi(\vartheta)$. Q.E.D.

Corollary 1.7.19. *Let $\varsigma \in \{+, -\}$. All the $J(z; \boldsymbol{\lambda}; \xi)$, with $\xi \in \mathbb{X}_n(\varsigma)$, form a fundamental set of solutions of the Bessel equation of sign ς.*

Remark 1.7.20. *If $n = 2$, by* [Wat, 3.7 (8), 3.71 (18), 7.2 (1, 2), 7.23 (1, 2)] *we have the following formula of $J(z; \lambda, -\lambda; \xi)$, with $\xi = \pm 1, \pm i$, and the corresponding sector on which its asymptotic expansion is valid*

$$J(z; \lambda, -\lambda; 1) = \sqrt{\pi i} e^{\pi i \lambda} H^{(1)}_{2\lambda}(2z), \quad \mathbb{S}'_1(\vartheta) = \{z : -\pi + \vartheta < \arg z < 2\pi - \vartheta\};$$
$$J(z; \lambda, -\lambda; -1) = \sqrt{-\pi i} e^{-\pi i \lambda} H^{(2)}_{2\lambda}(2z), \quad \mathbb{S}'_{-1}(\vartheta) = \{z : -2\pi + \vartheta < \arg z < \pi - \vartheta\};$$
$$J(z; \lambda, -\lambda; i) = \frac{2}{\sqrt{\pi}} K_{2\lambda}(2z), \quad \mathbb{S}'_i(\vartheta) = \left\{z : |\arg z| < \frac{3}{2}\pi - \vartheta\right\};$$
$$J(z; \lambda, -\lambda; -i) = 2\sqrt{\pi} I_{2\lambda}(2z) - \frac{2i}{\sqrt{\pi}} e^{2\pi i \lambda} K_{2\lambda}(2z),$$
$$\mathbb{S}'_{-i}(\vartheta) = \left\{z : -\frac{1}{2}\pi + \vartheta < \arg z < \frac{5}{2}\pi - \vartheta\right\}.$$

Lemma 1.7.21. *Let $\xi \in \mathbb{X}_{2n}(+)$. We have*

$$J(z; \boldsymbol{\lambda}; \xi) = (\pm\xi)^{\frac{n-1}{2}} J(\pm\xi z; \boldsymbol{\lambda}; \pm 1),$$

and $B_m(\boldsymbol{\lambda}; \xi) = (\pm\xi)^{-m} B_m(\boldsymbol{\lambda}; \pm 1)$.

Proof. By Lemma 1.7.1, $(\pm\xi)^{\frac{n-1}{2}} J(\pm\xi z; \boldsymbol{\lambda}; \pm 1)$ is a solution of one of the two Bessel equations of index $\boldsymbol{\lambda}$. In view of Proposition 1.7.14 and Lemma 1.7.17 (2), it possesses $\widehat{J}(z; \boldsymbol{\lambda}; \xi)$ as its asymptotic expansion on $\mathbb{S}_\xi$ and hence must be identical with $J(z; \boldsymbol{\lambda}; \xi)$. Q.E.D.

Terminology 1.7.22. *For $\xi \in \mathbb{X}_{2n}(+)$, $J(z; \boldsymbol{\lambda}; \xi)$ is called a Bessel function of the second kind.*

Remark 1.7.23. *The results in this section do not provide any information on the asymptotics near zero of Bessel functions of the second kind, and therefore their connections with Bessel functions of the first kind can not be clarified here. We shall nevertheless find the* *connection formulae between the two kinds of Bessel functions later in* §1.8*, appealing to the asymptotic expansion of the H-Bessel function* $H^{\pm}(z;\boldsymbol{\lambda})$ *on the half-plane* $\mathbb{H}^{\pm}$ *that we showed earlier in* §1.5*.*

1.7.4. Error Bounds for asymptotic expansions

The error bound for the asymptotic expansion of $J(z;\boldsymbol{\lambda};\xi)$ with dependence on $\boldsymbol{\lambda}$ is always desirable for potential applications in analytic number theory. However, the author does not find any general results on the error analysis for differential equations of order higher than two. We shall nevertheless combine and generalize the ideas from [CL, §5.4] and [Olv, §7.2] to obtain an almost optimal error estimate for the asymptotic expansion of the Bessel function $J(z;\boldsymbol{\lambda};\xi)$. Observe that both of their methods have drawbacks for generalizations. [Olv] hardly uses the viewpoint from differential systems as only the second-order case is treated, whereas [CL, §5.4] is restricted to the positive real axis for more clarified expositions.

Preparations

We retain the notations from §1.7.3. For a positive integer M denote by $P_{(M)}$ the polynomial in z^{-1},

$$P_{(M)}(z) = \sum_{m=0}^{M} z^{-m} P_m,$$

and by $\widehat{\Phi}_{(M)}$ the truncation of $\widehat{\Phi}$,

$$\widehat{\Phi}_{(M)}(z) = P_{(M)}(z) z^{-\frac{n-1}{2}} e^{\Lambda z}.$$

By Lemma 1.7.13, we have $|z^{-m}P_m| \ll_{m,n} \mathfrak{C}^{2m}|z|^{-m}$, so $P_{(M)}^{-1}$ exists as an analytic function for $|z| > c_1\mathfrak{C}^2$, where c_1 is some constant depending only on M and n. Moreover,

$$\left|P_{(M)}(z)\right|, \left|P_{(M)}^{-1}(z)\right| = O_{M,n}(1), \quad |z| > c_1\mathfrak{C}^2. \tag{1.7.24}$$

Let $A_{(M)}$ and $E_{(M)}$ be defined by

$$A_{(M)} = \widehat{\Phi}'_{(M)}\widehat{\Phi}^{-1}_{(M)}, \quad E_{(M)} = A - A_{(M)}.$$

$A_{(M)}$ and $E_{(M)}$ are clearly analytic for $|z| > c_1\mathfrak{C}^2$. Since

$$E_{(M)}P_{(M)} = AP_{(M)} - \left(P'_{(M)} - \frac{n-1}{2}z^{-1}P_{(M)} + P_{(M)}\varDelta\right),$$

it follows from the construction of $\widehat{\Phi}$ in §1.7.3 that $E_{(M)}P_{(M)}$ is a polynomial in z^{-1} of the form $\sum_{m=M+1}^{M+n} z^{-m}E_m$ so that

$$\begin{aligned} E_{M+1} &= P^o_{M+1}\varDelta - \varDelta P^o_{M+1}, \\ E_m &= \sum_{j=m-M}^{\min\{m,n\}} A_j P_{m-j}, \quad M+1 < m \leqslant M+n. \end{aligned}$$

Therefore, in view of Lemma 1.7.13, $|E_{M+1}| \lll_{M,n} \mathfrak{C}^{2M}$ and $|E_m| \lll_{m,n} \mathfrak{C}^{m+M}$ for $M+1 < m \leqslant M+n$. It follows that $|E_{(M)}(z)P_{(M)}(z)| \lll_{M,n} \mathfrak{C}^{2M}z^{-M-1}$ for $|z| > c_1\mathfrak{C}^2$, and this, combined with (1.7.24), yields

(1.7.25) $$|E_{(M)}(z)| = O_{M,n}\left(\mathfrak{C}^{2M}|z|^{-M-1}\right).$$

By the definition of $A_{(M)}$, for $|z| > c_1\mathfrak{C}^2$, $\widehat{\Phi}_{(M)}$ is a fundamental matrix of the system

(1.7.26) $$u' = A_{(M)}u.$$

We shall regard the differential system (1.7.10), that is,

(1.7.27) $$u' = Au = A_{(M)}u + E_{(M)}u,$$

as a nonhomogeneous system with (1.7.26) as the corresponding homogeneous system.

Construction of a solution

Given $\ell \in \{1, ..., n\}$, let

$$\widehat{\varphi}_{(M),\ell}(z) = p_{(M),\ell}(z)z^{-\frac{n-1}{2}}e^{in\xi_\ell z}$$

be the ℓ-th column vector of the matrix $\widehat{\Phi}_{(M)}$, where $p_{(M),\ell}$ is the ℓ-th column vector of $P_{(M)}$. Using a version of the variation-of-constants formula and the method of successive approximations, we shall construct a solution $\varphi_{(M),\ell}$ of (1.7.10), for z in some suitable domain, satisfying

$$\left|\varphi_{(M),\ell}(z)\right| = O_{M,n}\left(|z|^{-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)}\right), \tag{1.7.28}$$

and

$$\left|\varphi_{(M),\ell}(z) - \widehat{\varphi}_{(M),\ell}(z)\right| = O_{M,n}\left(\mathfrak{C}^{2M}|z|^{-M-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)}\right), \tag{1.7.29}$$

with the implied constant in (1.7.29) also depending on the chosen domain.

Step 1. *Constructing the domain and the contours for the integral equation.* For $C \geqslant c_1\mathfrak{C}^2$ and $0 < \vartheta < \frac{1}{2}\pi$, define the domain $\mathbb{D}(C;\vartheta) \subset \mathbb{U}$ by

$$\mathbb{D}(C;\vartheta) = \{z : |\arg z| \leqslant \pi, |z| > C\} \cup \left\{z : \pi < |\arg z| < \frac{3}{2}\pi - \vartheta,\ \Re\mathrm{e}\ z < -C\right\}.$$

For $k \neq \ell$ let $\omega(\ell,k) = \arg(i\bar{\xi}_\ell - i\bar{\xi}_k) = \arg(i\bar{\xi}_\ell) + \arg(1 - \xi_k\bar{\xi}_\ell)$, and define

$$\mathbb{D}_{\xi_\ell}(C;\vartheta) = \bigcap_{k\neq\ell} e^{i\omega(\ell,k)} \cdot \mathbb{D}(C;\vartheta).$$

With the observation that

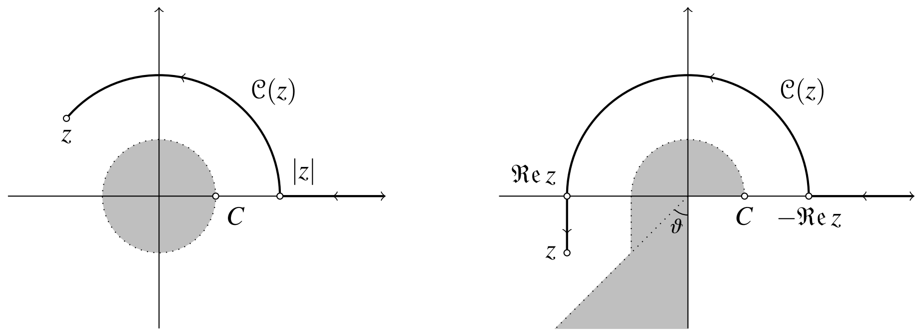
$$\left\{\arg(1 - \xi_k\bar{\xi}_\ell) : k \neq \ell\right\} = \left\{\left(\frac{1}{2} - \frac{a}{n}\right)\pi : a = 1, ..., n-1\right\},$$

it is straightforward to show that $\mathbb{D}_{\xi_\ell}(C;\vartheta) = i\bar{\xi}_\ell\mathbb{D}'(C;\vartheta)$, where $\mathbb{D}'(C;\vartheta)$ is defined to be the union of the sector

$$\left\{z : |\arg z| \leqslant \frac{\pi}{2} + \frac{\pi}{n},\ |z| > C\right\}$$

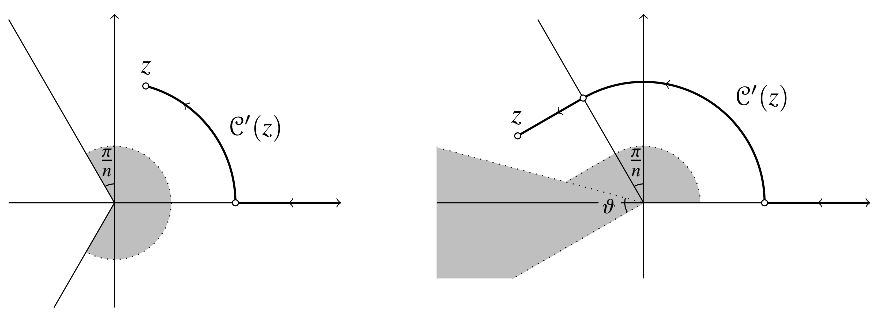
and the following two domains

$$\left\{z : \frac{\pi}{2} + \frac{\pi}{n} < \arg z < \pi + \frac{\pi}{n} - \vartheta,\ \Im\mathrm{m}\left(e^{-\frac{1}{n}\pi i}z\right) > C\right\},$$
$$\left\{z : -\pi - \frac{\pi}{n} + \vartheta < \arg z < -\frac{\pi}{2} - \frac{\pi}{n},\ \Im\mathrm{m}\left(e^{\frac{1}{n}\pi i}z\right) < -C\right\}.$$



$|\arg z| \leqslant \pi$ $\qquad$ $\pi < \arg z < \frac{3}{2}\pi - \vartheta$

Figure 1.1: $\mathcal{C}(z) \subset \mathbb{D}(C;\vartheta)$



$|\arg z| \leqslant \frac{1}{2}\pi + \frac{1}{n}\pi$ $\qquad$ $\frac{1}{2}\pi + \frac{1}{n}\pi < \arg z < \pi + \frac{1}{n}\pi - \vartheta$

Figure 1.2: $\mathcal{C}'(z) \subset \mathbb{D}'(C;\vartheta)$

For $z \in \mathbb{D}(C;\vartheta)$ we define a contour $\mathcal{C}(z) \subset \mathbb{D}(C;\vartheta)$ that starts from ∞ and ends at z; see Figure 1.1. For $z \in \mathbb{D}(C;\vartheta)$ with $|\arg z| \leqslant \pi$, the contour $\mathcal{C}(z)$ consists of the part of the positive axis where the magnitude exceeds $|z|$ and an arc of the circle centered at the origin of radius $|z|$, with angle not exceeding π and endpoint z. For $z \in \mathbb{D}(C;\vartheta)$ with $\pi < |\arg z| < \frac{3}{2}\pi - \vartheta$, the definition of the contour $\mathcal{C}(z)$ is modified so that the circular arc has radius $-\mathfrak{Re}\, z$ instead of $|z|$ and ends at $\mathfrak{Re}\, z$ on the negative real axis, and that $\mathcal{C}(z)$ also consists of a vertical line segment joining $\mathfrak{Re}\, z$ and z. The most crucial property that $\mathcal{C}(z)$ satisfies is the *nonincreasing* of $\mathfrak{Re}\,\zeta$ along $\mathcal{C}(z)$.

We also define a contour $\mathcal{C}'(z)$ for $z \in \mathbb{D}'(C;\vartheta)$ of a similar shape as $\mathcal{C}(z)$ illustrated in Figure 1.2.

Step 2. *Solving the integral equation via successive approximations.* We first split $\widehat{\Phi}_{(M)}^{-1}$ into n parts

$$\widehat{\Phi}_{(M)}^{-1} = \sum_{k=1}^{n} \Psi_{(M)}^{(k)},$$

where the j-th row of $\Psi_{(M)}^{(k)}$ is identical with the k-th row of $\widehat{\Phi}_{(M)}^{-1}$, or identically zero, according as $j = k$ or not.

The integral equation to be considered is the following

$$u(z) = \widehat{\varphi}_{(M),\ell}(z) + \sum_{k\neq\ell} \int_{\infty e^{i\omega(\ell,k)}}^{z} K_k(z,\zeta)u(\zeta)d\zeta + \int_{\infty i\bar{\xi}_\ell}^{z} K_\ell(z,\zeta)u(\zeta)d\zeta, \tag{1.7.30}$$

where

$$K_k(z,\zeta) = \widehat{\Phi}_{(M)}(z)\Psi_{(M)}^{(k)}(\zeta)E_{(M)}(\zeta), \quad z,\zeta \in \mathbb{D}_{\xi_\ell}(C;\vartheta), k = 1,...,n,$$

the integral in the sum is integrated on the contour $e^{i\omega(\ell,k)}\mathcal{C}\left(e^{-i\omega(\ell,k)}z\right)$, whereas the last integral is on the contour $i\bar{\xi}_\ell\mathcal{C}'\left(-i\xi_\ell z\right)$. Clearly, all these contours lie in $\mathbb{D}_{\xi_\ell}(C;\vartheta)$. Most importantly, we note that $\mathfrak{Re}\,((i\xi_\ell - i\xi_k)\zeta)$ is a negative multiple of $\mathfrak{Re}\,\left(e^{-i\omega(\ell,k)}\zeta\right)$ and hence is *nondecreasing* along the contour $e^{i\omega(\ell,k)}\mathcal{C}\left(e^{-i\omega(\ell,k)}z\right)$.

By direct verification, it follows that if $u(z) = \varphi(z)$ satisfies (1.7.30), with the integrals convergent, then φ satisfies (1.7.27).

In order to solve (1.7.30), define the successive approximations

$$\varphi^0(z) \equiv 0,$$

$$\varphi^{\alpha+1}(z) = \widehat{\varphi}_{(M),\ell}(z) + \sum_{k \neq \ell} \int_{\infty e^{i\omega(\ell,k)}}^{z} K_k(z,\zeta)\varphi^{\alpha}(\zeta)d\zeta + \int_{\infty i\bar{\xi}_\ell}^{z} K_\ell(z,\zeta)\varphi^{\alpha}(\zeta)d\zeta. \tag{1.7.31}$$

The (j,r)-th entry of the matrix $\widehat{\Phi}_{(M)}(z)\Psi^{(k)}_{(M)}(\zeta)$ is given by

$$\left(\widehat{\Phi}_{(M)}(z)\Psi^{(k)}_{(M)}(\zeta)\right)_{jr} = \left(P_{(M)}(z)\right)_{jk}\left(P^{-1}_{(M)}(\zeta)\right)_{kr}\left(\frac{z}{\zeta}\right)^{-\frac{n-1}{2}} e^{in\xi_k(z-\zeta)}.$$

It follows from (1.7.24, 1.7.25) that

$$|K_k(z,\zeta)| \leqslant c_2 \mathfrak{C}^{2M}|z|^{-\frac{n-1}{2}}|\zeta|^{-M-1+\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_k(z-\zeta))}, \tag{1.7.32}$$

for some constant c_2 depending only on M and n. Furthermore, we may appropriately choose c_2 such that

$$\int_{\infty i\bar{\xi}_\ell}^{z} |\zeta|^{-M-1}|d\zeta|, \int_{\infty e^{i\omega(\ell,k)}}^{z} |\zeta|^{-M-1}|d\zeta| \leqslant c_2 C^{-M}, \quad k \neq \ell. \tag{1.7.33}$$

According to (1.7.31), $\varphi^1(z) = \widehat{\varphi}_{(M),\ell}(z) = p_{(M),\ell}(z)z^{-\frac{n-1}{2}}e^{in\xi_\ell z}$, so

$$\left|\varphi^1(z) - \varphi^0(z)\right| = \left|\varphi^1(z)\right| \leqslant c_2|z|^{-\frac{n-1}{2}}e^{\Re\mathrm{e}\,(in\xi_\ell z)}, \quad z \in \mathbb{D}_{\xi_\ell}(C;\vartheta).$$

We shall show by induction that for all $z \in \mathbb{D}_{\xi_\ell}(C;\vartheta)$

$$\left|\varphi^{\alpha}(z) - \varphi^{\alpha-1}(z)\right| \leqslant c_2\left(nc_2^2\mathfrak{C}^{2M}C^{-M}\right)^{\alpha-1}|z|^{-\frac{n-1}{2}}e^{\Re\mathrm{e}\,(in\xi_\ell z)}. \tag{1.7.34}$$

Let $z \in \mathbb{D}_{\xi_\ell}(C;\vartheta)$. Assume that (1.7.34) holds. From (1.7.31) we have

$$\left|\varphi^{\alpha+1}(z) - \varphi^{\alpha}(z)\right| \leqslant \sum_{k\neq\ell} R_k + R_\ell,$$

with

$$R_k = \int_{\infty e^{i\omega(\ell,k)}}^{z} |K_k(z,\zeta)| \left|\varphi^{\alpha}(\zeta) - \varphi^{\alpha-1}(\zeta)\right| |d\zeta|,$$
$$R_\ell = \int_{\infty i\bar{\xi}_\ell}^{z} |K_\ell(z,\zeta)| \left|\varphi^{\alpha}(\zeta) - \varphi^{\alpha-1}(\zeta)\right| |d\zeta|.$$

It follows from (1.7.32, 1.7.34) that R_k has bound

$$c_2^2\mathfrak{C}^{2M}\left(nc_2^2\mathfrak{C}^{2M}C^{-M}\right)^{\alpha-1}|z|^{-\frac{n-1}{2}}e^{\mathfrak{Re}\,(in\xi_\ell z)}\int_{\infty e^{i\omega(\ell,k)}}^{z}|\zeta|^{-M-1}e^{\mathfrak{Re}\,(in(\xi_\ell-\xi_k)(\zeta-z))}|d\zeta|.$$

Since $\mathfrak{Re}\,((i\xi_\ell - i\xi_k)\zeta)$ is nondecreasing on the integral contour,

$$R_k \leqslant c_2^2\mathfrak{C}^{2M}\left(nc_2^2\mathfrak{C}^{2M}C^{-M}\right)^{\alpha-1}|z|^{-\frac{n-1}{2}}e^{\mathfrak{Re}\,(in\xi_\ell z)}\int_{\infty e^{i\omega(\ell,k)}}^{z}|\zeta|^{-M-1}|d\zeta|,$$

and (1.7.33) further yields

$$R_k \leqslant c_2 n^{\alpha-1}\left(c_2^2\mathfrak{C}^{2M}C^{-M}\right)^{\alpha}|z|^{-\frac{n-1}{2}}e^{\mathfrak{Re}\,(in\xi_\ell z)}.$$

Similar arguments show that R_ℓ has the same bound as R_k. Thus (1.7.34) is true with α replaced by $\alpha + 1$.

Set the constant $C = c\mathfrak{C}^2$ such that $c^M \geqslant 2nc_2^2$. Then $nc_2^2\mathfrak{C}^{2M}C^{-M} \leqslant \frac{1}{2}$, and therefore the series $\sum_{\alpha=1}^{\infty}(\varphi^{\alpha}(z) - \varphi^{\alpha-1}(z))$ absolutely and locally uniformly converges. The limit function $\varphi_{(M),\ell}(z)$ satisfies (1.7.28) for all $z \in \mathbb{D}_{\xi_\ell}(C;\vartheta)$. More precisely,

$$\left|\varphi_{(M),\ell}(z)\right| \leqslant 2c_2|z|^{-\frac{n-1}{2}}e^{\mathfrak{Re}\,(in\xi_\ell z)}, \quad z \in \mathbb{D}_{\xi_\ell}(C;\vartheta). \tag{1.7.35}$$

Using a standard argument for successive approximations, it follows that $\varphi_{(M),\ell}$ satisfies the integral equation (1.7.30) and hence the differential system (1.7.27).

The proof of the error bound (1.7.29) is similar. Since $\varphi_{(M),\ell}(z)$ is a solution of the integral equation (1.7.30), we have

$$\left|\varphi_{(M),\ell}(z) - \widehat{\varphi}_{(M),\ell}(z)\right| \leqslant \sum_{k\neq\ell} S_k + S_\ell,$$

where

$$S_k = \int_{\infty e^{i\omega(\ell,k)}}^{z} |K_k(z,\zeta)| \left|\varphi_{(M),\ell}(\zeta)\right| |d\zeta|\,, \;\; S_\ell = \int_{\infty i\bar{\xi}_\ell}^{z} |K_\ell(z,\zeta)| \left|\varphi_{(M),\ell}(\zeta)\right| |d\zeta|\,.$$

With the observation that $|\zeta| \geqslant \sin\vartheta \cdot |z|$ for $z \in \mathbb{D}_{\xi_\ell}(C;\vartheta)$ and ζ on the integral contours given above, we may replace (1.7.33) by the following

$$\text{(1.7.36)} \qquad \int_{\infty i\bar{\xi}_\ell}^{z} |\zeta|^{-M-1} |d\zeta|\,, \int_{\infty e^{i\omega(\ell,k)}}^{z} |\zeta|^{-M-1} |d\zeta| \leqslant c_2 |z|^{-M}, \quad k \neq \ell,$$

with c_2 now also depending on ϑ.

The bounds (1.7.32, 1.7.35) of $K_k(z,\zeta)$ and $\varphi_{(M),\ell}(z)$ along with (1.7.36) yield

$$\begin{aligned} S_k &\leqslant 2c_2^2 \mathfrak{C}^{2M} |z|^{-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)} \int_{\infty e^{i\omega(\ell,k)}}^{z} |\zeta|^{-M-1} e^{\Re\mathrm{e}\,(in(\xi_\ell-\xi_k)(\zeta-z))} |d\zeta| \\ &\leqslant 2c_2^2 \mathfrak{C}^{2M} |z|^{-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)} \int_{\infty e^{i\omega(\ell,k)}}^{z} |\zeta|^{-M-1} |d\zeta| \\ &\leqslant 2c_2^3 \mathfrak{C}^{2M} |z|^{-M-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)}. \end{aligned}$$

Again, the second inequality follows from the fact that $\Re\mathrm{e}\,((i\xi_\ell - i\xi_k)\zeta)$ is nondecreasing on the integral contour. Similarly, S_ℓ has the same bound as S_k. Thus (1.7.29) is proven and can be made precise as below

$$\text{(1.7.37)} \qquad \left|\varphi_{(M),\ell}(z) - \widehat{\varphi}_{(M),\ell}(z)\right| \leqslant 2nc_2^3 \mathfrak{C}^{2M} |z|^{-M-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)}, \quad z \in \mathbb{D}_{\xi_\ell}(C;\vartheta).$$

Conclusion

Restricting to the sector $\mathbb{S}^{\pm}_{\xi_\ell} \cap \{z : |z| > C\} \subset \mathbb{D}_{\xi_\ell}(C;\vartheta)$, with $\mathbb{S}^{\pm}_{\xi_\ell}$ replaced by $\mathbb{S}_{\xi_\ell}$ if $n = 2$, each $\varphi_{(M),\ell}$ has an asymptotic representation a multiple of $\widehat{\varphi}_k$ for some k according to Lemma 1.7.17 (1). Since $\Re\mathrm{e}\,(i\xi_\ell z) < \Re\mathrm{e}\,(i\xi_j z)$ for all $j \neq \ell$, the bound (1.7.28) forces $k = \ell$. Therefore, for any positive integer M, $\varphi_{(M),\ell}$ is identical with the unique solution φ_ℓ of the differential system (1.7.10) with asymptotic expansion $\widehat{\varphi}_\ell$ on $\mathbb{S}^{\pm}_{\xi_\ell}$ (see Lemma 1.7.17).

Replacing $\varphi_{(M),\ell}$ by φ_ℓ and absorbing the M-th term of $\widehat{\varphi}_{(M),\ell}$ into the error bound, we may reformulate (1.7.37) as the following error bound for φ_ℓ

$$\left|\varphi_\ell(z) - \widehat{\varphi}_{(M-1),\ell}(z)\right| = O_{M,\vartheta,n}\left(\mathfrak{C}^{2M}|z|^{-M-\frac{n-1}{2}} e^{\Re\mathrm{e}\,(in\xi_\ell z)}\right), \quad z \in \mathbb{D}_{\xi_\ell}(C;\vartheta). \tag{1.7.38}$$

Moreover, in view of the definition of the sector $\mathbb{S}'_{\xi_\ell}(\vartheta)$ given in (1.7.23), we have

$$\mathbb{S}'_{\xi_\ell}(\vartheta) \cap \left\{z : |z| > \frac{C}{\sin\vartheta}\right\} \subset \mathbb{D}_{\xi_\ell}(C;\vartheta). \tag{1.7.39}$$

Thus, the following theorem is finally established by (1.7.38) and (1.7.39).

Theorem 1.7.24. *Let $\varsigma \in \{+,-\}$, $\xi \in \mathbb{X}_n(\varsigma)$, $0 < \vartheta < \frac{1}{2}\pi$, $\mathbb{S}'_\xi(\vartheta)$ be the sector defined as in* (1.7.23)*, and M be a positive integer. Then there exists a constant c, depending only on M, ϑ and n, such that*

$$J(z;\boldsymbol{\lambda};\xi) = e^{in\xi z} z^{-\frac{n-1}{2}}\left(\sum_{m=0}^{M-1} B_m(\boldsymbol{\lambda};\xi) z^{-m} + O_{M,\vartheta,n}\left(\mathfrak{C}^{2M}|z|^{-M}\right)\right) \tag{1.7.40}$$

for all $z \in \mathbb{S}'_\xi(\vartheta)$ such that $|z| > c\mathfrak{C}^2$. Similar asymptotic is valid for all the derivatives of $J(z;\boldsymbol{\lambda};\xi)$, where the constant c and the implied constant of the error estimate are allowed to depend on the order of the derivative.

Finally, we remark that, since $B_m(\boldsymbol{\lambda};\xi)z^{-m}$ is of size $O_{m,n}\left(\mathfrak{C}^{2m}|z|^{-m}\right)$, the error bound in (1.7.40) is optimal, given that ϑ is fixed.

1.8. Connections between various types of Bessel functions

Recall from §1.5.4 that the asymptotic expansion in Theorem 1.5.11 remains valid for the H-Bessel function $H^\pm(z;\boldsymbol{\lambda})$ on the half-plane $\mathbb{H}^\pm = \{z : 0 \leqslant \pm\arg z \leqslant \pi\}$ (see (1.5.11)). With the observations that $H^\pm(z;\boldsymbol{\lambda})$ satisfies the Bessel equation of sign $(\pm)^n$, that the asymptotic expansions of $\sqrt{n}(\pm 2\pi i)^{-\frac{n-1}{2}} H^\pm(z;\boldsymbol{\lambda})$ and $J(z;\boldsymbol{\lambda};\pm 1)$ have exactly the same form and the same leading term due to Theorem 1.5.11 and Proposition 1.7.14, and that $\mathbb{S}_{\pm 1} = \left\{z : \left(\frac{1}{2} - \frac{1}{n}\right)\pi < \pm \arg z < \left(\frac{1}{2} + \frac{1}{n}\right)\pi\right\} \subset \mathbb{H}^{\pm}$, Lemma 1.7.17 (2) implies the following theorem.

Theorem 1.8.1. *We have*

$$H^{\pm}(z; \boldsymbol{\lambda}) = n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} J(z; \boldsymbol{\lambda}; \pm 1),$$

and $B_m(\boldsymbol{\lambda}; \pm 1) = (\pm i)^{-m} B_m(\boldsymbol{\lambda})$.

Remark 1.8.2. $B_m(\boldsymbol{\lambda}; \pm 1)$ *can only be obtained from certain recurrence relations in* §1.7.3 *from the differential equation aspect. On the other hand, using the stationary phase method,* (1.5.9) *in* §1.5.3 *yields an explicit formula of* $B_m(\boldsymbol{\lambda})$. *Thus, Theorem* 1.8.1 *indicates that the recurrence relations for* $B_m(\boldsymbol{\lambda}; \pm 1)$ *are actually solvable!*

As consequences of Theorem 1.8.1, we can establish the connections between various Bessel functions, that is, $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$, $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ and $J(z; \boldsymbol{\lambda}; \xi)$. Recall that $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ has already been expressed in terms of $J_\ell(z; \varsigma, \boldsymbol{\lambda})$ in Lemma 1.7.6.

1.8.1. Relations between $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ and $J(z; \boldsymbol{\lambda}; \xi)$

$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is equal to a multiple of $H^{\pm}\left(e^{\pm \pi i \frac{n_{\mp}(\boldsymbol{\varsigma})}{n}} z; \boldsymbol{\lambda}\right)$ in view of Lemma 1.7.9, whereas $J(z; \boldsymbol{\lambda}; \xi)$ is a multiple of $J(\pm \xi z; \boldsymbol{\lambda}; \pm 1)$ due to Lemma 1.7.21. Furthermore, the equality, up to constant, between $H^{\pm}(z; \boldsymbol{\lambda})$ and $J(z; \boldsymbol{\lambda}; \pm 1)$ has just been established in Theorem 1.8.1. We then arrive at the following corollary.

Corollary 1.8.3. *Let* $L_{\pm}(\boldsymbol{\varsigma}) = \{\ell : \varsigma_\ell = \pm\}$ *and* $n_{\pm}(\boldsymbol{\varsigma}) = |L_{\pm}(\boldsymbol{\varsigma})|$ *be as in Definition* 1.7.5. *Let* $c(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(\mp \frac{n-1}{8} \pm \frac{(n-1) n_{\pm}(\boldsymbol{\varsigma})}{4n} \mp \frac{1}{2} \sum_{\ell \in L_{\pm}(\boldsymbol{\varsigma})} \lambda_\ell\right)$ *and* $\xi(\boldsymbol{\varsigma}) = \mp e^{\mp \pi i \frac{n_{\pm}(\boldsymbol{\varsigma})}{n}}$. *Then*

$$J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{(2\pi)^{\frac{n-1}{2}} c(\boldsymbol{\varsigma}, \boldsymbol{\lambda})}{\sqrt{n}} J(z; \boldsymbol{\lambda}; \xi(\boldsymbol{\varsigma})).$$

Here, it is understood that $\arg \xi(\boldsymbol{\varsigma}) = \frac{n_-(\boldsymbol{\varsigma})}{n}\pi = \pi - \frac{n_+(\boldsymbol{\varsigma})}{n}\pi$.

Corollary 1.8.3 shows that $J(z;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ should really be categorized in the class of Bessel functions of the second kind. Moreover, the asymptotic behaviours of the Bessel functions $J(z;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ are classified by their signatures $(n_+(\boldsymbol{\varsigma}),n_-(\boldsymbol{\varsigma}))$. Therefore, $J(z;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is *uniquely* determined by its signature up to a constant multiple.

1.8.2. Relations connecting the two kinds of Bessel functions

From Lemma 1.7.21 and Theorem 1.8.1, one sees that $J(z;\boldsymbol{\lambda};\xi)$ is a constant multiple of $H^+(\xi z;\boldsymbol{\lambda})$. On the other hand, $H^+(z;\boldsymbol{\lambda})$ can be expressed in terms of Bessel functions of the first kind in view of Lemma 1.7.6. Finally, using Lemma 1.7.3, the following corollary is readily established.

Corollary 1.8.4. *Let $\varsigma\in\{+,-\}$. If $\xi\in\mathbb{X}_n(\varsigma)$, then*

$$J(z;\boldsymbol{\lambda};\xi) = \sqrt{n}\left(-\frac{\pi i\xi}{2}\right)^{\frac{n-1}{2}}\sum_{\ell=1}^{n}\left(i\bar{\xi}\right)^{n\lambda_\ell}S_\ell(\boldsymbol{\lambda})J_\ell(z;\varsigma,\boldsymbol{\lambda}),$$

with $S_\ell(\boldsymbol{\lambda}) = 1/\prod_{k\neq\ell}\sin\left(\pi(\lambda_\ell-\lambda_k)\right)$. According to our convention, we have $(-i\xi)^{\frac{n-1}{2}} = e^{\frac{n-1}{2}\left(-\frac{1}{2}\pi i+i\arg\xi\right)}$ and $\left(i\bar{\xi}\right)^{n\lambda_\ell} = e^{\frac{1}{2}\pi in\lambda_\ell-in\lambda_\ell\arg\xi}$. When $\boldsymbol{\lambda}$ is not generic, the right hand side should be replaced by its limit.

We now fix an integer a and let $\xi_j = e^{\pi i\frac{2j+a-2}{n}}\in\mathbb{X}_n\left((-)^a\right)$, with $j=1,...,n$. It follows from Corollary 1.8.4 that

$$X(z;\boldsymbol{\lambda}) = \sqrt{n}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}}e^{-\frac{1}{4}\pi i(n-1)}\cdot DV(\boldsymbol{\lambda})S(\boldsymbol{\lambda})E(\boldsymbol{\lambda})Y(z;\boldsymbol{\lambda}),$$

with

$$X(z;\boldsymbol{\lambda}) = \left(J\left(z;\boldsymbol{\lambda};\xi_j\right)\right)_{j=1}^n,\quad Y(z;\boldsymbol{\lambda}) = \left(J_\ell\left(z;(-)^a,\boldsymbol{\lambda}\right)\right)_{\ell=1}^n,$$
$$D = \mathrm{diag}\left(\xi_j^{\frac{n-1}{2}}\right)_{j=1}^n,\quad E(\boldsymbol{\lambda}) = \mathrm{diag}\left(e^{\pi i\left(\frac{1}{2}n-a\right)\lambda_\ell}\right)_{\ell=1}^n,\quad S(\boldsymbol{\lambda}) = \mathrm{diag}\left(S_\ell(\boldsymbol{\lambda})\right)_{\ell=1}^n,$$

$$V(\boldsymbol{\lambda}) = \left(e^{-2\pi i(j-1)\lambda_\ell}\right)_{j,\ell=1}^{n}.$$

Observe that $V(\boldsymbol{\lambda})$ is a *Vandermonde matrix*.

Lemma 1.8.5. *For an n-tuple $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{C}^n$ we define the Vandermonde matrix $V = \left(x_\ell^{j-1}\right)_{j,\ell=1}^{n}$. For $d = 0, 1, ..., n-1$ and $m = 1, ..., n$, let $\sigma_{m,d}$ denote the elementary symmetric polynomial in $x_1, ..., \widehat{x_m}, ..., x_n$ of degree d, and let $\tau_m = \prod_{k\neq m}(x_m - x_k)$. If $\boldsymbol{x}$ is generic in the sense that all the components of $\boldsymbol{x}$ are distinct, then V is invertible, and furthermore, the inverse of V is $((-)^{n-j}\sigma_{m,n-j}\tau_m^{-1})_{m,j=1}^{n}$.*

Proof of Lemma 1.8.5. It is a well-known fact that V is invertible whenever $\boldsymbol{x}$ is generic. If one denotes by $w_{m,j}$ the (m, j)-th entry of V^{-1}, then

$$\sum_{j=1}^{n} w_{m,j} x_\ell^{j-1} = \delta_{m,\ell}.$$

The Lagrange interpolation formula implies the following identity of polynomials

$$\sum_{j=1}^{n} w_{m,j} x^{j-1} = \prod_{k\neq m} \frac{x - x_k}{x_m - x_k}.$$

Identifying the coefficient of x^{j-1} on both sides yields the desired formula of $w_{m,j}$. Q.E.D.

Corollary 1.8.6. *Let a be a given integer. For $j = 1, ..., n$ define $\xi_j = e^{\pi i \frac{2j+a-2}{n}}$. For $d = 0, 1, ..., n-1$ and $\ell = 1, ..., n$, let $\sigma_{\ell,d}(\boldsymbol{\lambda})$ denote the elementary symmetric polynomial in $e^{-2\pi i\lambda_1}, ..., \widehat{e^{-2\pi i\lambda_\ell}}, ..., e^{-2\pi i\lambda_n}$ of degree d. Then*

$$J_\ell\left(z; (-)^a, \boldsymbol{\lambda}\right) = \frac{e^{\frac{3}{4}\pi i(n-1)}}{\sqrt{n}(2\pi)^{\frac{n-1}{2}}} e^{\pi i\left(\frac{1}{2}n+a-2\right)\lambda_\ell} \sum_{j=1}^{n} (-)^{n-j} \xi_j^{-\frac{n-1}{2}} \sigma_{\ell,n-j}(\boldsymbol{\lambda}) J\left(z; \boldsymbol{\lambda}; \xi_j\right).$$

Proof. Choosing $x_\ell = e^{-2\pi i\lambda_\ell}$ in Lemma 1.8.5, one sees that if $\boldsymbol{\lambda}$ is generic then the matrix $V(\boldsymbol{\lambda})$ is invertible and its inverse is given by

$$\left((-2i)^{1-n} \cdot (-)^{n-j} \sigma_{\ell,n-j}(\boldsymbol{\lambda}) e^{\pi i(n-2)\lambda_\ell} S_\ell(\boldsymbol{\lambda})\right)_{\ell,j=1}^{n}.$$

Some straightforward calculations then complete the proof. Q.E.D.

Remark 1.8.7. *In view of Proposition* 1.2.7*, Remark* 1.7.4 *and* 1.7.20*, when* $n = 2$*, Corollary* 1.8.4 *corresponds to the connection formulae* ([Wat, 3.61(5, 6), 3.7 (6)]),

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\pi i\nu} J_\nu(z)}{i\sin(\pi\nu)}, \quad H_\nu^{(2)}(z) = \frac{e^{\pi i\nu} J_\nu(z) - J_{-\nu}(z)}{i\sin(\pi\nu)},$$
$$K_\nu(z) = \frac{\pi\left(I_{-\nu}(z) - I_\nu(z)\right)}{2\sin(\pi\nu)}, \qquad \pi I_\nu(z) - ie^{\pi i\nu} K_\nu(z) = \frac{\pi i\left(e^{-\pi i\nu} I_\nu(z) - e^{\pi i\nu} I_{-\nu}(z)\right)}{2\sin(\pi\nu)},$$

whereas Corollary 1.8.6*, with* $a = 0$ *or* 1*, amounts to the formulae* (*see* [Wat, 3.61(1, 2), 3.7 (6)])

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2}, \qquad J_{-\nu}(z) = \frac{e^{\pi i\nu} H_\nu^{(1)}(z) + e^{-\pi i\nu} H_\nu^{(2)}(z)}{2},$$
$$I_\nu(z) = \frac{ie^{\pi i\nu} K_\nu(z) + \left(\pi I_\nu(z) - ie^{\pi i\nu} K_\nu(z)\right)}{\pi}, \quad I_{-\nu}(z) = \frac{ie^{-\pi i\nu} K_\nu(z) + \left(\pi I_\nu(z) - ie^{\pi i\nu} K_\nu(z)\right)}{\pi}.$$

1.9. H-Bessel functions and K-Bessel functions, II

In this concluding section, we apply Theorem 1.7.24 to improve the results in §1.5 on the asymptotics of Bessel functions $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ and the Bessel kernel $J_{(\boldsymbol{\lambda}, \boldsymbol{\delta})}(\pm x)$ for $x \geqslant \mathfrak{C}^2$.

1.9.1. Asymptotic expansions of H-Bessel functions

The following proposition is a direct consequence of Theorem 1.7.24 and 1.8.1.

Proposition 1.9.1. *Let* $0 < \vartheta < \frac{1}{2}\pi$.

(1). *Let* M *be a positive integer. We have*

$$H^{\pm}(z; \boldsymbol{\lambda}) = n^{-\frac{1}{2}} (\pm 2\pi i)^{\frac{n-1}{2}} e^{\pm inz} z^{-\frac{n-1}{2}} \left(\sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\boldsymbol{\lambda}) z^{-m} + O_{M,\vartheta,n}\left(\mathfrak{C}^{2M} |z|^{-M}\right) \right), \tag{1.9.1}$$

for all $z \in \mathbb{S}'_{\pm 1}(\vartheta)$ *such that* $|z| \geqslant_{M,\vartheta,n} \mathfrak{C}^2$.

(2). *Define* $W(z; \pm, \boldsymbol{\lambda}) = \sqrt{n} (\pm 2\pi i)^{-\frac{n-1}{2}} e^{\mp inz} H^{\pm}(z; \boldsymbol{\lambda})$. *Let* $M - 1 \geqslant j \geqslant 0$. *We have*

$$W^{(j)}(z; \pm, \boldsymbol{\lambda}) = z^{-\frac{n-1}{2}} \left(\sum_{m=j}^{M-1} (\pm i)^{j-m} B_{m,j}(\boldsymbol{\lambda}) z^{-m} + O_{M,\vartheta,j,n}\left(\mathfrak{C}^{2M-2j} |z|^{-M}\right) \right), \tag{1.9.2}$$

for all $z \in \mathbb{S}'_{\pm 1}(\vartheta)$ *such that* $|z| \geqslant_{M,\vartheta,n} \mathfrak{C}^2$.

Observe that

$$\begin{aligned}\mathbb{H}^{\pm} &= \{z \in \mathbb{C} : 0 \leqslant \pm \arg z \leqslant \pi\} \\ \subset \mathbb{S}'_{\pm 1}(\vartheta) &= \left\{ z \in \mathbb{U} : -\left(\frac{1}{2} - \frac{1}{n}\right)\pi - \vartheta < \pm \arg z < \left(\frac{3}{2} + \frac{1}{n}\right)\pi + \vartheta \right\}.\end{aligned}$$

Fixing ϑ and restricting to the domain $\{z \in \mathbb{H}^{\pm} : |z| \geqslant_{M,n} \mathfrak{C}^2\}$, Proposition 1.9.1 improves Theorem 1.5.11.

1.9.2. Exponential decay of K-Bessel functions

Now suppose that $J(z; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is a K-Bessel function so that $0 < n_{\pm}(\boldsymbol{\varsigma}) < n$. Since $\mathbb{R}_+ \subset \mathbb{S}'_{\xi(\boldsymbol{\varsigma})}(\vartheta)$, Corollary 1.8.3 and Theorem 1.7.24 imply that $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$, as well as all its derivatives, is not only a Schwartz function at infinity, which was shown in Theorem 1.5.6, but also a function of exponential decay on $\mathbb{R}_+$.

Proposition 1.9.2. *If* $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *is a K-Bessel function, then for all* $x \geqslant_n \mathfrak{C}^2$

$$J^{(j)}(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) \ll_{j,n} x^{-\frac{n-1}{2}} e^{-\pi \Im\mathrm{m}\, \Lambda(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) - n I(\boldsymbol{\varsigma}) x},$$

where $\Lambda(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \mp \sum_{l \in L_{\pm}(\boldsymbol{\varsigma})} \lambda_l$ *and* $I(\boldsymbol{\varsigma}) = \Im\mathrm{m}\, \xi(\boldsymbol{\varsigma}) = \sin\left(\frac{n_{\pm}(\boldsymbol{\varsigma})}{n}\pi\right) > 0$. *In particular, we have*

$$J^{(j)}(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) \ll_{j,n} x^{-\frac{n-1}{2}} e^{\pi \mathfrak{I} - n \sin\left(\frac{1}{n}\pi\right) x},$$

for all K-Bessel functions $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ *with given* $\boldsymbol{\lambda}$, *where* $\mathfrak{I} = \max\{|\Im\mathrm{m}\, \lambda_l|\}$.

1.9.3. The asymptotic of the Bessel kernel $J_{(\boldsymbol{\lambda}, \boldsymbol{\delta})}$

In comparison with Theorem 1.5.13, we have the following proposition.

Theorem 1.9.3. *Let notations be as in Theorem* 1.5.13. *Then, for* $x \geqslant_{M,n} \mathfrak{C}^2$, *we have*

$$W_{\boldsymbol{\lambda}}^{\pm,(j)}(x) = \sum_{m=j}^{M-1} B^{\pm}_{m,j}(\boldsymbol{\lambda}) x^{-m} + O_{M,j,n}\left(\mathfrak{C}^{2M} x^{-M}\right),$$

and

$$E^{\pm,(j)}_{(\lambda,\delta)}(x) = O_{j,n}\left(x^{-\frac{n-1}{2}} \exp\left(\pi\mathfrak{I} - 2\pi n \sin\left(\tfrac{1}{n}\pi\right) x\right)\right),$$

with $\mathfrak{I} = \max\{|\mathfrak{Im}\,\lambda_l|\}$.

1.A. An alternative approach to asymptotic expansions

When $n = 3$, the application of Stirling's asymptotic formula in deriving the asymptotic expansion of s Hankel transform was first found in [Mil, §4]. The asymptotic was later formulated more explicitly in [Li1, Lemma 6.1], where the author attributed the arguments in her proof to [Ivi]. Furthermore, using similar ideas as in [Mil], [Blo] simplified the proof of [Li1, Lemma 6.1] (see the proof of [Blo, Lemma 6]). This method using Stirling's asymptotic formula is however the only known approach so far in the literature.

Closely following [Blo], we shall prove the asymptotic expansions of H-Bessel functions $H^{\pm}(x;\boldsymbol{\lambda})$ of any rank n by means of Stirling's asymptotic formula.

From (1.2.5, 1.2.3) we have

$$H^{\pm}(x;\boldsymbol{\lambda}) = \frac{1}{2\pi i}\int_{\mathcal{C}}\left(\prod_{\ell=1}^{n}\Gamma(s-\lambda_\ell)\right) e\left(\pm\frac{ns}{4}\right) x^{-ns} ds. \tag{1.A.1}$$

In view of the condition $\sum_{\ell=1}^{n}\lambda_\ell = 0$, Stirling's asymptotic formula yields

$$\prod_{\ell=1}^{n}\Gamma(s-\lambda_\ell) = n^{-ns}\Gamma\left(ns - \frac{n-1}{2}\right)\exp\left(\sum_{m=0}^{M} C_m(\boldsymbol{\lambda}) s^{-m}\right)(1 + R_{M+1}(s))$$

for some constants $C_m(\boldsymbol{\lambda})$ and remainder term $R_{M+1}(s) = O_{\boldsymbol{\lambda},M,n}(|s|^{-M-1})$. Using the Taylor expansion for the exponential function and some straightforward algebraic manipulations, the right hand side can be written as

$$n^{-ns}\sum_{m=0}^{M}\widetilde{C}_m(\boldsymbol{\lambda})\Gamma\left(ns - \frac{n-1}{2} - m\right)\left(1 + \widetilde{R}_{M+1,m}(s)\right)$$

for certain constants $\widetilde{C}_m(\lambda)$ and similar functions $\widetilde{R}_{M+1,m}(s) = O_{\lambda,M,n}\left(|s|^{-M-1}\right)$. Suitably choosing the contour $\mathcal{C}$, it follows from (1.2.13) that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma\left(ns-\frac{n-1}{2}-m\right)e\left(\pm\frac{ns}{4}\right)(nx)^{-ns}ds\\
&=\frac{e\left(\pm\left(\frac{n-1}{8}+\frac{1}{4}m\right)\right)}{n(nx)^{\frac{n-1}{2}+m}}\cdot\frac{1}{2\pi i}\int_{n\mathcal{C}-\frac{n-1}{2}-m}\Gamma(s)e\left(\pm\frac{s}{4}\right)(nx)^{-s}ds=\frac{(\pm i)^{\frac{n-1}{2}+m}}{n^{\frac{n+1}{2}+m}}\cdot\frac{e^{\pm inx}}{x^{\frac{n-1}{2}+m}}.
\end{aligned}$$

As for the error estimate, let us assume $x \geqslant 1$. Insert the part containing $\widetilde{R}_{M+1,m}(s)$ into (1.A.1) and shift the contour to the vertical line of real part $\frac{1}{n}(M-\frac{1}{2})+\frac{1}{2}$. By Stirling's formula, the integral remains absolutely convergent and is of size $O_{\lambda,M,n}\left(x^{-M-\frac{n-1}{2}}\right)$. Absorbing the last main term into the error, we arrive at the following asymptotic expansion

$$H^{\pm}(x;\lambda)=e^{\pm inx}x^{-\frac{n-1}{2}}\left(\sum_{m=0}^{M-1}C_m^{\pm}(\lambda)x^{-m}+O_{\lambda,M,n}\left(x^{-M}\right)\right),\quad x\geqslant 1, \tag{1.A.2}$$

where $C_m^{\pm}(\lambda)$ is some constant depending on λ.

Remark 1.A.1. *For the analytic continuation $H^{\pm}(z;\lambda)$, we have the Barnes type integral representation as in Remark* 1.7.10*. This however does not yield an asymptotic expansion of $H^{\pm}(z;\lambda)$ along with the above method. The obvious issue is with the error estimate, as $|z^{-ns}|$ is unbounded on the integral contour if $|z|\to\infty$.*

Finally, we make some comparisons between the three asymptotic expansions (1.A.2), (1.5.11) and (1.9.1) obtained from

- Stirling's asymptotic formula,
- the method of stationary phase,
- the asymptotic method of ordinary differential equations.

Recall that $\mathfrak{C} = \max\{|\lambda_\ell|\} + 1$, $\mathfrak{R} = \max\{|\mathfrak{Re}\,\lambda_\ell|\}$. Firstly, the admissible domains of these asymptotic expansions are

$$\{x \in \mathbb{R}_+ : x \geqslant 1\},$$

$$\{z \in \mathbb{C} : |z| \geqslant \mathfrak{C},\ 0 \leqslant \pm \arg z \leqslant \pi\},$$

$$\left\{z \in \mathbb{U} : |z| \geqslant_{M,\vartheta,n} \mathfrak{C}^2,\ -\left(\frac{1}{2} - \frac{1}{n}\right)\pi - \vartheta < \pm \arg z < \left(\frac{3}{2} + \frac{1}{n}\right)\pi + \vartheta\right\},$$

respectively. The range of argument is extending while that of modulus is reducing. Secondly, the error estimates are

$$O_{\lambda,M,n}\left(x^{-M-\frac{n-1}{2}}\right),\ O_{\mathfrak{R},M,n}\left(\mathfrak{C}^{2M}|z|^{-M}\right),\ O_{M,\vartheta,n}\left(\mathfrak{C}^{2M}|z|^{-M-\frac{n-1}{2}}\right),$$

respectively. Thus, in the error estimate, the dependence of the implied constant on $\boldsymbol{\lambda}$ is improving in all aspects.

Chapter 2

Hankel Transforms and Fundamental Bessel Kernels

2.1. Introduction

In this chapter, we shall study *Hankel transforms*[XI] as well as their integral kernels, called *fundamental Bessel kernels*[XII], over an archimedean field. These Hankel transforms are the archimedean constituent of the Voronoĭ summation formula over a number field.

2.1.1. Analytic theory

Let n be a positive integer. In the case $n \geqslant 3$ Hankel transforms of rank n over $\mathbb{R}$ have been investigated in the work of Miller and Schmid [MS1, MS3, MS4] on the Voronoĭ summation formula for $\mathrm{GL}_n(\mathbb{Z})$. The notion of *automorphic distributions*[XIII] is used for their proof of this formula, and is also used to derive the analytic continuation and the functional equation of the L-function of a cuspidal $\mathrm{GL}_n(\mathbb{Z})$-automorphic representation of

[XI] They are called Bessel transforms in some literatures, for instance, [IT]. However, this type of integral transforms should actually be attributed to Hermann Hankel. Moreover, we shall reserve the term *Bessel transforms* for the transforms shown in the Kuznetsov trace formula.

[XII] The adjective *fundamental* is added for the distinction from the Bessel functions for GL_n in the Kuznetsov formula, and will be dropped when no confusion occurs.

[XIII] According to Stephen Miller, the origin of automorphic distributions can be traced back to the 19th century in the work of Siméon Poisson on Poisson's integral for harmonic functions on either the unit disk or the upper half-plane.

$\mathrm{GL}_n(\mathbb{R})$. As the foundation of automorphic distributions, the harmonic analysis over $\mathbb{R}$ is studied in [MS3] from the viewpoint of the signed Mellin transforms. As explained in [MS2], the cases $n = 1, 2$ can also be incorporated into their framework. Furthermore, it is shown in Chapter 1 that all Hankel transforms over $\mathbb{R}$ admit integral kernels, which can be partitioned into combinations of the so-called *fundamental Bessel functions*. These Bessel functions are studied from two approaches via their *formal integral representations* and *Bessel differential equations*.

In §2.2 - 2.8, we shall establish the analytic theory of Hankel transforms and their Bessel kernels over $\mathbb{C}$. The study of Hankel transforms for $\mathrm{GL}_n(\mathbb{C})$ from the perspective of [MS3] is complete to some extent. On the other hand, Bessel functions in Chapter 1 play a fundamental role in our study of Bessel kernels over $\mathbb{C}$, for instance, in finding their asymptotic expansions. Although our main focus is on the theory over $\mathbb{C}$, the theory of Hankel transforms over $\mathbb{R}$ extracted from [MS3] as well as some treatments of Bessel kernels over $\mathbb{R}$ will also be included for the sake of comparison.

The sections §2.2 - 2.8 are outlined as follows.

In the preliminary section §2.2, some basic notions are introduced, such as gamma factors, Schwartz spaces, the Fourier transform and Mellin transforms. The three kinds of Mellin transforms $\mathcal{M}$, $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{C}}$ are first defined over the Schwartz spaces over $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$ respectively.

In §2.3, the definitions of the Mellin transforms $\mathcal{M}$, $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{C}}$ are extended onto certain function spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$, $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ and $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ respectively. We shall precisely characterize their image spaces $\mathscr{M}_{\mathrm{sis}}$, $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}}$ and $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$ under their corresponding Mellin transforms. In spite of their similar constructions, the analysis of the Mellin transform $\mathcal{M}_{\mathbb{C}}$ is much more elaborate than that of $\mathcal{M}_{\mathbb{R}}$ or $\mathcal{M}$.

In §2.4, based on gamma factors and Mellin transforms, we shall construct Hankel transforms upon suitable subspaces of the $\mathscr{S}_{\mathrm{sis}}$ function spaces just introduced in §2.3 and study their Bessel kernels. It turns out that all these Bessel kernels can be formulated in terms of the Bessel functions in Chapter 1.

In §2.5, we shall first introduce the Schmid-Miller transforms in companion with the Fourier transform and then use them to establish a Fourier type integral transform expression of a Hankel transform.

In §2.6, we shall introduce certain integrals, derived from the Fourier type integral transforms given in §2.5, that represents Bessel kernels. When the field is real, these integrals never absolutely converge and are closely connected to the formal integrals studied in Chapter 1. In the complex case, however, some range of index can be found where such integrals are absolutely convergent.

The last two sections §2.7 and §2.8 are devoted to Bessel kernels over $\mathbb{C}$. In §2.7, we shall prove two connection formulae that relate a Bessel kernel over $\mathbb{C}$ to the two kinds of Bessel functions of *positive* sign. These kinds of Bessel functions arise in the study of Bessel equations in §1.7. In §2.8, as a consequence of the second connection formula above, we shall derive the asymptotic expansion of a Bessel kernel over $\mathbb{C}$ from Theorem 1.7.24.

2.1.2. Representation theory

The work of Miller and Schmid is extended by Ichino and Templier [IT] to any irreducible cuspidal automorphic representation of GL_n, $n \geqslant 2$, over an arbitrary number field

$\mathbb{K}$. The Voronoĭ summation formula for GL_n follows from the global theory of $\mathrm{GL}_n \times \mathrm{GL}_1$-Rankin-Selberg L-functions. For an archimedean completion $\mathbb{K}_v$ of $\mathbb{K}$, the defining identities of the Hankel transform associated with an infinite dimensional irreducible unitary generic representation of $\mathrm{GL}_n(\mathbb{K}_v)$ are reformulations of the corresponding local functional equations for $\mathrm{GL}_n \times \mathrm{GL}_1$-Rankin-Selberg zeta integrals over $\mathbb{K}_v$.

In §2.A, we shall first recollect the definition of the Hankel transform associated with an infinite dimensional irreducible admissible generic representation of $\mathrm{GL}_n(\mathbb{F})$ for an archimedean field $\mathbb{F}$ in [IT]. We stress that this definition actually works for any irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$, including $n = 1$. We shall then give a detailed discussion on Hankel transforms of rank n over $\mathbb{F}$ using the Langlands classification for $\mathrm{GL}_n(\mathbb{F})$.

2.1.3. Distribution theory

Although such a theory can be formulated, we shall not touch here the theory of Mellin transforms over $\mathbb{C}^\times$ from the perspective of distributions as in [MS3]. It is very likely that this will lead to the theory of automorphic distributions on $\mathrm{GL}_n(\mathbb{C})$ with respect to congruence subgroups, as well as the Voronoĭ summation formula for cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{C})$. The Voronoĭ summation formula in this generality is already covered by [IT], but this approach would still be of its own interest.

2.1.4. Applications

When $n = 2$, there are numerous applications in analytic number theory of the Voronoĭ summation formula and the Kuznetsov trace formula over $\mathbb{Q}$, which include subconvexity, non-vanishing of automorphic L-functions and estimates for shifted convolution sums. The Voronoĭ summation formula for $\mathrm{GL}_3(\mathbb{Z})$ is used to establish subconvexity in [Li2] as well. In order to work over an arbitrary number field, one also needs to understand Hankel transforms and Bessel kernels at least for $\mathrm{GL}_2(\mathbb{C})$. We hope that the present paper and its sequel will make these problems over a number field more approachable from the analytic perspective.

2.2. Notations and preliminaries

2.2.1. Gamma factors

1.

We define the gamma factor

$$G(s, \pm) = \Gamma(s) e\left(\pm \frac{s}{4}\right). \tag{2.2.1}$$

For $(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = (\varsigma_1, ..., \varsigma_n, \lambda_1, ..., \lambda_n) \in \{+, -\}^n \times \mathbb{C}^n$ let

$$G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \prod_{\ell=1}^{n} G(s - \lambda_\ell, \varsigma_\ell). \tag{2.2.2}$$

2.

For $\delta \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, we define the gamma factor

$$G_\delta(s) = i^\delta \pi^{\frac{1}{2} - s} \frac{\Gamma\left(\frac{1}{2}(s + \delta)\right)}{\Gamma\left(\frac{1}{2}(1 - s + \delta)\right)} = \begin{cases} 2(2\pi)^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right), & \text{if } \delta = 0, \\ 2i(2\pi)^{-s}\Gamma(s) \sin\left(\frac{\pi s}{2}\right), & \text{if } \delta = 1. \end{cases} \tag{2.2.3}$$

Here, we have used the duplication formula and Euler's reflection formula of the Gamma function,

$$\Gamma(1 - s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

Let $(\boldsymbol{\mu}, \boldsymbol{\delta}) = (\mu_1, ..., \mu_n, \delta_1, ..., \delta_n) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ and define

$$G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(s) = \prod_{\ell=1}^{n} G_{\delta_\ell}(s - \mu_\ell). \tag{2.2.4}$$

One observes the following simple functional relation

$$G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(1 - s) G_{(-\boldsymbol{\mu}, \boldsymbol{\delta})}(s) = (-1)^{|\boldsymbol{\delta}|}. \tag{2.2.5}$$

3.

For $m \in \mathbb{Z}$, we define the gamma factor

$$G_m(s) = i^{|m|}(2\pi)^{1-2s} \frac{\Gamma\left(s + \frac{1}{2}|m|\right)}{\Gamma\left(1 - s + \frac{1}{2}|m|\right)}. \tag{2.2.6}$$

Let $(\boldsymbol{\mu}, \boldsymbol{m}) = (\mu_1, ..., \mu_n, m_1, ..., m_n) \in \mathbb{C}^n \times \mathbb{Z}^n$ and define

$$G_{(\boldsymbol{\mu},\boldsymbol{m})}(s) = \prod_{\ell=1}^{n} G_{m_\ell}(s - \mu_\ell). \tag{2.2.7}$$

We have the functional relation

$$G_{(\boldsymbol{\mu},\boldsymbol{m})}(1 - s)G_{(-\boldsymbol{\mu},\boldsymbol{m})}(s) = (-1)^{|\boldsymbol{m}|}. \tag{2.2.8}$$

Relations between the three types of gamma factors

We first observe that

$$G_\delta(s) = (2\pi)^{-s}\left(G(s, +) + (-)^\delta G(s, -)\right).$$

Hence

$$G_{(\boldsymbol{\mu},\boldsymbol{\delta})}(s) = \sum_{\boldsymbol{\varsigma}\in\{+,-\}^n} \boldsymbol{\varsigma}^{\boldsymbol{\delta}}(2\pi)^{|\boldsymbol{\mu}|-ns}G(s; \boldsymbol{\varsigma}, \boldsymbol{\mu}), \quad \boldsymbol{\varsigma}^{\boldsymbol{\delta}} = \prod_{\ell=1}^{n} \varsigma_\ell^{\delta_\ell},\ |\boldsymbol{\mu}| = \sum_{\ell=1}^{n} \mu_\ell. \tag{2.2.9}$$

Euler's reflection formula and certain trigonometric identities yield

$$\begin{aligned} iG_m(s) &= i^{|m|+1}2(2\pi)^{-2s}\Gamma\left(s + \frac{|m|}{2}\right)\Gamma\left(s - \frac{|m|}{2}\right)\sin\left(\pi\left(s - \frac{|m|}{2}\right)\right) \\ &= G_{\delta(m)+1}\left(s - \frac{|m|}{2}\right)G_0\left(s + \frac{|m|}{2}\right) \\ &= G_{\delta(m)}\left(s - \frac{|m|}{2}\right)G_1\left(s + \frac{|m|}{2}\right), \end{aligned} \tag{2.2.10}$$

with $\delta(m) = m(\mathrm{mod}\ 2)$. Therefore, $G_{(\boldsymbol{\mu},\boldsymbol{m})}(s)$ may be viewed as a certain $G_{(\boldsymbol{\eta},\boldsymbol{\delta})}(s)$ of doubled rank.

Lemma 2.2.1. *Suppose that $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbb{C}^{2n} \times (\mathbb{Z}/2\mathbb{Z})^{2n}$ are subjected to one of the following two sets of relations*

$$\eta_{2\ell-1} = \mu_\ell + \frac{|m_\ell|}{2},\ \eta_{2\ell} = \mu_\ell - \frac{|m_\ell|}{2},\ \delta_{2\ell-1} = \delta(m) + 1,\ \delta_{2\ell} = 0; \tag{2.2.11}$$

$$\eta_{2\ell-1} = \mu_\ell + \frac{|m_\ell|}{2},\ \eta_{2\ell} = \mu_\ell - \frac{|m_\ell|}{2},\ \delta_{2\ell-1} = \delta(m),\ \delta_{2\ell} = 1. \tag{2.2.12}$$

Then $i^n G_{(\boldsymbol{\mu},\boldsymbol{m})}(s) = G_{(\boldsymbol{\eta},\boldsymbol{\delta})}(s)$.

Stirling's formula

Fix $s_0 \in \mathbb{C}$, and let $|\arg s| < \pi - \epsilon$, $0 < \epsilon < \pi$. We have the following asymptotic as $|s| \to \infty$

$$\log \Gamma(s_0 + s) \sim \left(s_0 + s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi).$$

If one writes $s_0 = \rho_0 + it_0$ and $s = \rho + it$, $\rho \geqslant 0$, then the right hand side is equal to

$$\begin{aligned}&\left(\rho_0 + \rho - \frac{1}{2}\right) \log \sqrt{t^2 + \rho^2} - (t_0 + t) \arctan\left(\frac{t}{\rho}\right) - \rho + \frac{1}{2} \log(2\pi)\\ &+ i(t_0 + t) \log \sqrt{t^2 + \rho^2} - it + i\left(\rho_0 + \rho - \frac{1}{2}\right) \arctan\left(\frac{t}{\rho}\right),\end{aligned}$$

and therefore

$$|\Gamma(s_0 + s)| \sim \sqrt{2\pi} \left(t^2 + \rho^2\right)^{\frac{1}{2}\left(\rho_0 + \rho - \frac{1}{2}\right)} e^{-(t_0+t) \arctan(t/\rho) - \rho}. \tag{2.2.13}$$

Lemma 2.2.2. *We have*

$$G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) \ll_{\lambda,a,b,r} (|\Im\mathrm{m}\, s| + 1)^{n\left(\Re\mathrm{e}\, s - \frac{1}{2}\right) - \Re\mathrm{e}\, |\lambda|}, \tag{2.2.14}$$

for all $s \in \mathbb{S}[a, b] \smallsetminus \bigcup_{\ell=1}^{n} \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\lambda_\ell - \kappa)$, with small $r > 0$,

$$G_{(\boldsymbol{\mu},\boldsymbol{\delta})}(s) \ll_{\boldsymbol{\mu},a,b,r} (|\Im\mathrm{m}\, s| + 1)^{n\left(\Re\mathrm{e}\, s - \frac{1}{2}\right) - \Re\mathrm{e}\, |\boldsymbol{\mu}|}, \tag{2.2.15}$$

for all $s \in \mathbb{S}[a, b] \smallsetminus \bigcup_{\ell=1}^{n} \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\mu_\ell - \delta_\ell - 2\kappa)$, *and*

$$G_{(\boldsymbol{\mu}, \boldsymbol{m})}(s) \ll_{\boldsymbol{\mu}, a, b, r} \prod_{\ell=1}^{n} (|\mathfrak{Im}\, s| + |m_\ell| + 1)^{2\mathfrak{Re}\, s - 2\mathfrak{Re}\, \mu_\ell - 1}, \tag{2.2.16}$$

for all $2s \in \mathbb{S}[a, b] \smallsetminus \bigcup_{\ell=1}^{n} \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(2\mu_\ell - |m_\ell| - 2\kappa)$.

In other words, if $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$ *are given, then* $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$, $G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(s)$ *and* $G_{(\boldsymbol{\mu}, \boldsymbol{m})}(s)$ *are all of moderate growth with respect to* $\mathfrak{Im}\, s$, *uniformly on vertical strips* (*with bounded width*), *and moreover* $G_{(\boldsymbol{\mu}, \boldsymbol{m})}(s)$ *is also of uniform moderate growth with respect to* $\boldsymbol{m}$.

2.2.2. Basic notions for $\mathbb{R}_+$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$

Define $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$. We observe the isomorphisms $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+, -\}$ ($\cong \mathbb{R}_+ \times \mathbb{Z}/2\mathbb{Z}$) and $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$, the latter being realized via the *polar coordinate* $z = xe^{i\phi}$.

1.

Let $|\ |$ denote the ordinary absolute value on either $\mathbb{R}$ or $\mathbb{C}$, and set $\|\ \|_{\mathbb{R}} = |\ |$ for $\mathbb{R}$ and $\|\ \| = \|\ \|_{\mathbb{C}} = |\ |^2$ for $\mathbb{C}$. Let dx be the Lebesgue measure on $\mathbb{R}$, and let $d^\times x = |x|^{-1} dx$ be the standard choice of the multiplicative Haar measure on $\mathbb{R}^\times$. Similarly, let dz be *twice* the ordinary Lebesgue measure on $\mathbb{C}$, and choose the standard multiplicative Haar measure $d^\times z = \|z\|^{-1} dz$ on $\mathbb{C}^\times$. Moreover, in the polar coordinate, one has $d^\times z = 2 d^\times x d\phi$. For $x \in \mathbb{R}^\times$ the sign function $\mathrm{sgn}(x)$ is equal to $x/|x|$, whereas for $z \in \mathbb{C}^\times$ we introduce the notation $[z] = z/|z|$.

Henceforth, we shall let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, and occasionally let x, y denote elements in $\mathbb{F}$ even if $\mathbb{F} = \mathbb{C}$.

2.

For $\delta \in \mathbb{Z}/2\mathbb{Z}$, we define the space $C_\delta^\infty(\mathbb{R}^\times)$ of all smooth functions $\varphi \in C^\infty(\mathbb{R}^\times)$ satisfying the parity condition

$$\varphi(-x) = (-)^\delta \varphi(x). \tag{2.2.17}$$

Observe that a function $\varphi \in C_\delta^\infty(\mathbb{R}^\times)$ is determined by its restriction on $\mathbb{R}_+$, namely, $\varphi(x) = \operatorname{sgn}(x)^\delta \varphi(|x|)$. Therefore,

$$C_\delta^\infty(\mathbb{R}^\times) = \operatorname{sgn}(x)^\delta C^\infty(\mathbb{R}_+) = \left\{ \operatorname{sgn}(x)^\delta \varphi(|x|) : \varphi \in C^\infty(\mathbb{R}_+) \right\}. \tag{2.2.18}$$

For a smooth function $\varphi \in C^\infty(\mathbb{R}^\times)$, we define $\varphi_\delta \in C^\infty(\mathbb{R}_+)$ by

$$\varphi_\delta(x) = \frac{1}{2}\left(\varphi(x) + (-)^\delta \varphi(-x)\right), \quad x \in \mathbb{R}_+. \tag{2.2.19}$$

Clearly,

$$\varphi(x) = \varphi_0(|x|) + \operatorname{sgn}(x)\varphi_1(|x|). \tag{2.2.20}$$

For $m \in \mathbb{Z}$, we define the space $C_m^\infty(\mathbb{C}^\times)$ of all smooth functions $\varphi \in C^\infty(\mathbb{C}^\times)$ satisfying

$$\varphi\left(xe^{i\phi} \cdot e^{i\phi'}\right) = e^{im\phi'} \varphi\left(xe^{i\phi}\right). \tag{2.2.21}$$

A function $\varphi \in C_m^\infty(\mathbb{C}^\times)$ is determined by its restriction on $\mathbb{R}_+$, namely, $\varphi(z) = [z]^m \varphi(|z|)$, or, in the polar coordinate, $\varphi(xe^{i\phi}) = e^{im\phi}\varphi(x)$. Therefore,

$$C_m^\infty(\mathbb{R}^\times) = [z]^m C^\infty(\mathbb{R}_+) = \left\{ [z]^m \varphi(|z|) = e^{im\phi}\varphi(x) : \varphi \in C^\infty(\mathbb{R}_+) \right\}. \tag{2.2.22}$$

For a smooth function $\varphi \in C^\infty(\mathbb{C}^\times)$, we let $\varphi_m \in C^\infty(\mathbb{R}_+)$ denote the m-th Fourier coefficient of φ given by

$$\varphi_m(x) = \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(xe^{i\phi}\right) e^{-im\phi} d\phi. \tag{2.2.23}$$

One has the Fourier expansion of φ,

$$\varphi\left(xe^{i\phi}\right) = \sum_{m\in\mathbb{Z}} \varphi_m(x)e^{im\phi}. \tag{2.2.24}$$

3.

Subsequently, we shall encounter various subspaces of $C^\infty(\mathbb{F}^\times)$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, for instance, $\mathscr{S}(\mathbb{F})$, $\mathscr{S}(\mathbb{F}^\times)$, $\mathscr{S}_{\mathrm{sis}}(\mathbb{F}^\times)$, $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$ and $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times)$. Here, we list three central questions that will be the guidelines of our investigations of these function spaces.

For now, we let D be a subspace of $C^\infty(\mathbb{F}^\times)$. For $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$), we shall add a superscript or subscript δ (respectively m) to the notation of D, say D_δ (respectively D_m), to denote the space of $\varphi \in D$ satisfying (2.2.17) (respectively (2.2.21)). In view of (2.2.18) (respectively (2.2.22)), there is a subspace of $C^\infty(\mathbb{R}_+)$, say E_δ (respectively E_m), such that $D_\delta = \mathrm{sgn}(x)^\delta E_\delta$ (respectively $D_m = [z]^m E_m$).

Firstly, we are interested in the question,

> " How to characterize the space E_δ (respectively E_m)?".

Moreover, the subspaces $D \subset C^\infty(\mathbb{F}^\times)$ that we shall consider always satisfy the following two hypotheses,

- $\varphi \in D$ implies $\varphi_\delta \in E_\delta$ for $\mathbb{F} = \mathbb{R}$ (respectively, $\varphi \in D$ implies $\varphi_m \in E_m$ for $\mathbb{F} = \mathbb{C}$), and
- D is closed under addition.

For $\mathbb{F} = \mathbb{R}$, under these two hypotheses, it follows from (2.2.20) that

$$D = D_0 \oplus D_1 \cong E_0 \times E_1.$$

For $\mathbb{F} = \mathbb{C}$, in view of (2.2.24), the map that sends φ to the sequence $\{\varphi_m\}$ of its Fourier coefficients is injective. The second question arises,

“ What is the image of D in $\prod_{m\in\mathbb{Z}} E_m$ under this map?”, or equivalently,

“ What conditions should a sequence $\{\varphi_m\} \in \prod_{m\in\mathbb{Z}} E_m$ satisfy in order for the Fourier series defined by (2.2.24) giving a function $\varphi \in D$?”.

Finally, after introducing the Mellin transform $\mathcal{M}_{\mathbb{F}}$, we shall focus on the question,

“ What is the image of D under the Mellin transform $\mathcal{M}_{\mathbb{F}}$?”.

2.2.3. Schwartz spaces

We say that a function $\varphi \in C^\infty(\mathbb{R}_+)$ is *smooth at zero* if all of its derivatives admit asymptotics as below,

$$\varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_\alpha(x) \text{ as } x \to 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}. \tag{2.2.25}$$

Remark 2.2.3. *Consequently, one has the asymptotic expansion* $\varphi(x) \sim \sum_{\kappa=0}^{\infty} a_\kappa x^\kappa$*, which means that* $\varphi(x) = \sum_{\kappa=0}^{A} a_\kappa x^\kappa + O_A\left(x^{A+1}\right)$ *as* $x \to 0$ *for any* $A \in \mathbb{N}$*. It is not required that the series* $\sum_{\kappa=0}^{\infty} a_\kappa x^\kappa$ *be convergent for any* $x \in \mathbb{R}^\times$*.*

Actually, (2.2.25) *is equivalent to the following*

$$\varphi^{(\alpha)}(x) = \sum_{\kappa=\alpha}^{\alpha+A} a_\kappa [\kappa]_\alpha x^{\kappa-\alpha} + O_{\alpha,A}\left(x^{A+1}\right) \text{ as } x \to 0, \text{ for any } \alpha, A \in \mathbb{N}. \tag{2.2.26}$$

Another observation is that, for a given constant $1 > \rho > 0$*,* (2.2.25) *is equivalent to the following seemingly weaker statement,*

$$\varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_{\alpha,\rho}\left(x^\rho\right) \text{ as } x \to 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}. \tag{2.2.27}$$

Let $C^{\infty}(\overline{\mathbb{R}}_+)$ denote the subspace of $C^{\infty}(\mathbb{R}_+)$ consisting of smooth functions on $\mathbb{R}_+$ that are also smooth at zero.

Let $\mathscr{S}(\overline{\mathbb{R}}_+)$ denote the space of functions in $C^{\infty}(\overline{\mathbb{R}}_+)$ that rapidly decay at infinity along with all of their derivatives. Let $\mathscr{S}(\mathbb{F})$ denote the Schwartz space on $\mathbb{F}$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Let $\mathscr{S}(\mathbb{R}_+)$ denote the space of Schwartz functions on $\mathbb{R}_+$, that is, smooth functions on $\mathbb{R}_+$ whose derivatives rapidly decay at *both* zero and infinity. Similarly, we denote by $\mathscr{S}(\mathbb{F}^{\times})$ the space of Schwartz functions on $\mathbb{F}^{\times}$.

The following lemma provides criteria for characterizing functions in these Schwartz spaces, especially functions in $\mathscr{S}(\mathbb{C})$ or $\mathscr{S}(\mathbb{C}^{\times})$ in the polar coordinate. Its proof is left as an easy excise in analysis for the reader.

Lemma 2.2.4. *Let notations be as above.*

(1.1). *Let $\varphi \in C^{\infty}(\overline{\mathbb{R}}_+)$ satisfy the asymptotics* (2.2.25). *Then $\varphi \in \mathscr{S}(\overline{\mathbb{R}}_+)$ if and only if φ also satisfies*

$$x^{\alpha+\beta}\varphi^{(\alpha)}(x) \ll_{\alpha,\beta} 1 \text{ for all } \alpha, \beta \in \mathbb{N}. \tag{2.2.28}$$

(1.2). *A smooth function φ on $\mathbb{R}_+$ belongs to $\mathscr{S}(\mathbb{R}_+)$ if and only if φ satisfies* (2.2.28) *with $\beta \in \mathbb{N}$ replaced by $\beta \in \mathbb{Z}$.*

Let $\varphi \in \mathscr{S}(\overline{\mathbb{R}}_+)$ and a_α be as in (2.2.25). *Then $\varphi \in \mathscr{S}(\mathbb{R}_+)$ if and only if $a_\alpha = 0$ for all $\alpha \in \mathbb{N}$.*

(2.1). *A smooth function φ on $\mathbb{R}^{\times}$ extends to a function in $\mathscr{S}(\mathbb{R})$ if and only if*

- *φ satisfies* (2.2.28) *with $x^{\alpha+\beta}$ replaced by $|x|^{\alpha+\beta}$, and*
- *all the derivatives of φ admit asymptotics*

$$\varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_\alpha(|x|) \text{ as } x \to 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}. \tag{2.2.29}$$

(2.2). *Let φ be a smooth function on $\mathbb{R}^\times$. Then $\varphi \in \mathscr{S}(\mathbb{R}^\times)$ if and only if φ satisfies* (2.2.28) *with $x^{\alpha+\beta}$ replaced by $|x|^{\alpha+\beta}$ and $\beta \in \mathbb{N}$ by $\beta \in \mathbb{Z}$.*

Suppose $\varphi \in \mathscr{S}(\mathbb{R})$, then $\varphi \in \mathscr{S}(\mathbb{R}^\times)$ if and only if $\varphi^{(\alpha)}(0) = 0$ for all $\alpha \in \mathbb{N}$, or equivalently, $a_\alpha = 0$ for all $\alpha \in \mathbb{N}$, with a_α given in (2.2.29).

(3.1). *Write $\partial_x = \partial/\partial x$ and $\partial_\phi = \partial/\partial\phi$. In the polar coordinate, a smooth function $\varphi\left(xe^{i\phi}\right) \in C^\infty(\mathbb{C}^\times)$ extends to a function in $\mathscr{S}(\mathbb{C})$ if and only if*

- *$\varphi\left(xe^{i\phi}\right)$ satisfies*

$$x^{\alpha+\beta}\partial_x^\alpha\partial_\phi^\gamma\varphi\left(xe^{i\phi}\right) \ll_{\alpha,\beta,\gamma} 1 \text{ for all } \alpha,\beta,\gamma \in \mathbb{N}, \tag{2.2.30}$$

- *all the partial derivatives of φ admit asymptotics*

$$x^\alpha\partial_x^\alpha\partial_\phi^\beta\varphi\left(xe^{i\phi}\right) = \sum_{|m|\leqslant\alpha+\beta}\ \sum_{\substack{|m|\leqslant\kappa\leqslant\alpha+\beta\\ \kappa\equiv m(\mathrm{mod}\ 2)}} a_{m,\kappa}[\kappa]_\alpha(im)^\beta x^\kappa e^{im\phi} + O_{\alpha,\beta}\left(x^{\alpha+\beta+1}\right) \tag{2.2.31}$$

as $x \to 0$, for any $\alpha,\beta \in \mathbb{N}$, with $a_{m,\kappa} \in \mathbb{C}$ for $\kappa \geqslant |m|$ and $\kappa \equiv m(\mathrm{mod}\ 2)$.

Let $\varphi \in \mathscr{S}(\mathbb{C})$ and φ_m be the m-th Fourier coefficient of φ given by (2.2.23)*, then it follows from* (2.2.30, 2.2.4) *that*

- *φ_m satisfies*

$$x^{\alpha+\beta}\varphi_m^{(\alpha)}(x) \ll_{\alpha,\beta,A} (|m|+1)^{-A} \text{ for all } \alpha,\beta,A \in \mathbb{N}, \tag{2.2.32}$$

- *all the derivatives of φ_m admit asymptotics*

$$\varphi_m^{(\alpha)}(x) = \sum_{\kappa=\alpha}^{\alpha+A} a_{m,\kappa}[\kappa]_\alpha x^{\kappa-\alpha} + O_{\alpha,A}\left((|m|+1)^{-A}x^{A+1}\right) \tag{2.2.33}$$

as $x \to 0$, for any given $\alpha, A \in \mathbb{N}$, with $a_{m,\kappa} \in \mathbb{C}$ satisfying $a_{m,\kappa} = 0$ if either $\kappa < |m|$ or $\kappa \not\equiv m(\mathrm{mod}\ 2)$.

Observe that (2.2.4) *is equivalent to the following two conditions,*

(2.2.34) $\varphi_m^{(\alpha)}(x) = \alpha! a_{m,\alpha} + O_\alpha(x)$ *as* $x \to 0$, *for any* $\alpha \geqslant |m|$, *with* $a_{m,\alpha} \in \mathbb{C}$ *satisfying*

$a_{m,\alpha} = 0$ *if* $\alpha \not\equiv m (\mathrm{mod}\ 2)$,

(2.2.35) *for any given* $\alpha, A \in \mathbb{N}$, $\varphi_m^{(\alpha)}(x) = O_{\alpha,A}\left((|m|+1)^{-A} x^{A+1}\right)$ *as* $x \to 0$, *if* $|m| > \alpha + A$.

In particular, $\varphi_m \in \mathscr{S}(\overline{\mathbb{R}}_+)$.

Conversely, if a sequence $\{\varphi_m\}$ *of functions in* $C^\infty(\mathbb{R}_+)$ *satisfies* (2.2.32), (2.2.4) *and* (2.2.4), *then the Fourier series defined by* $\{\varphi_m\}$, *that is, the right hand side of* (2.2.24), *is a Schwartz function on* $\mathbb{C}$.

(3.2). *In the polar coordinate, a smooth function* $\varphi\left(xe^{i\phi}\right) \in C^\infty(\mathbb{C}^\times)$ *is a Schwartz function on* $\mathbb{C}^\times$ *if and only if* φ *satisfies* (2.2.30) *with* $\beta \in \mathbb{N}$ *by* $\beta \in \mathbb{Z}$.

Let $\varphi \in \mathscr{S}(\mathbb{C}^\times)$ *and* φ_m *be the m-th Fourier coefficient of* φ, *then it is necessary that* φ_m *satisfies* (2.2.32) *with* $\beta \in \mathbb{N}$ *replaced by* $\beta \in \mathbb{Z}$. *In particular,* $\varphi_m \in \mathscr{S}(\mathbb{R}_+)$.

Conversely, if a sequence $\{\varphi_m\}_{m\in\mathbb{Z}}$ *of functions in* $C^\infty(\mathbb{R}_+)$ *satisfies the condition* (2.2.32) *with* $\beta \in \mathbb{N}$ *replaced by* $\beta \in \mathbb{Z}$, *then the Fourier series defined by* $\{\varphi_m\}$ *gives rise to a Schwartz function on* $\mathbb{C}^\times$.

Let $\varphi \in \mathscr{S}(\mathbb{C})$ *and* $a_{m,\kappa}$ *be given in* (2.2.4), (2.2.4) *or* (2.2.4). $\varphi \in \mathscr{S}(\mathbb{C}^\times)$ *if and only if* $a_{m,\kappa} = 0$ *for all* $m \in \mathbb{Z}, \kappa \in \mathbb{N}$.

Some subspaces of $\mathscr{S}(\overline{\mathbb{R}}_+)$

In the following, we introduce several subspaces of $\mathscr{S}(\overline{\mathbb{R}}_+)$ which are closely related to $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}(\mathbb{C})$.

We first define for $\delta \in \mathbb{Z}/2\mathbb{Z}$ the subspace $C^\infty_\delta(\overline{\mathbb{R}}_+) \subset C^\infty(\overline{\mathbb{R}}_+)$ of functions with an asymptotic expansion of the form $\sum_{\kappa=0}^\infty a_\kappa x^{\delta+2\kappa}$ at zero.

Remark 2.2.5. *A question arises, "whether $C^\infty(\overline{\mathbb{R}}_+) = C^\infty_0(\overline{\mathbb{R}}_+) + C^\infty_1(\overline{\mathbb{R}}_+)$?".*

The answer is affirmative.

To see this, we define the space $C^\infty_\delta(\mathbb{R})$ of smooth functions φ on $\mathbb{R}$ satisfying (2.2.17)*. One has $\mathrm{sgn}(x)^\delta \varphi(|x|) \in C^\infty_\delta(\mathbb{R})$ if $\varphi \in C^\infty_\delta(\overline{\mathbb{R}}_+)$, and conversely, $\varphi\restriction_{\mathbb{R}_+} \in C^\infty_\delta(\overline{\mathbb{R}}_+)$ if $\varphi \in C^\infty_\delta(\mathbb{R})$. Thus, with the simple observation $C^\infty(\mathbb{R}) = C^\infty_0(\mathbb{R}) \oplus C^\infty_1(\mathbb{R})$, one sees that $C^\infty_0(\overline{\mathbb{R}}_+) + C^\infty_1(\overline{\mathbb{R}}_+)$ is the subspace of $C^\infty(\overline{\mathbb{R}}_+)$ consisting of functions on $\mathbb{R}_+$ that admit a smooth extension onto $\mathbb{R}$.*

On the other hand, the Borel theorem ([Nar, 1.5.4])*, which is a special case of the Whitney extension theorem* ([Nar, 1.5.5, 1.5.6])*, states that for any sequence $\{a_\alpha\}$ of constants there exists a smooth function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi^{(\alpha)}(0) = \alpha! a_\alpha$. Clearly, this theorem of Borel implies our assertion above.*

In §2.3.1*, we shall give an alternative proof of this using the Mellin transform. See Remark* 2.3.5*.*

We define $\mathscr{S}_\delta(\overline{\mathbb{R}}_+) = \mathscr{S}(\overline{\mathbb{R}}_+) \cap C^\infty_\delta(\overline{\mathbb{R}}_+)$. The following identity is obvious

$$\mathscr{S}_\delta(\overline{\mathbb{R}}_+) = x^\delta \mathscr{S}_0(\overline{\mathbb{R}}_+).$$

In view of Lemma 2.2.4 (1.2), we have $\mathscr{S}_0(\overline{\mathbb{R}}_+) \cap \mathscr{S}_1(\overline{\mathbb{R}}_+) = \mathscr{S}(\mathbb{R}_+)$.

If we let $\mathscr{S}_\delta(\mathbb{R})$ be the space of functions $\varphi \in \mathscr{S}(\mathbb{R})$ satisfying (2.2.17), then

$$\mathscr{S}_\delta(\mathbb{R}) = \mathrm{sgn}(x)^\delta \mathscr{S}_\delta(\overline{\mathbb{R}}_+) = \left\{ \mathrm{sgn}(x)^\delta \varphi(|x|) : \varphi \in \mathscr{S}_\delta(\overline{\mathbb{R}}_+) \right\}.$$

Clearly, $\mathscr{S}(\mathbb{R}) = \mathscr{S}_0(\mathbb{R}) \oplus \mathscr{S}_1(\mathbb{R})$.

We define the subspace $\mathscr{S}_m(\overline{\mathbb{R}}_+) \subset \mathscr{S}_{\delta(m)}(\overline{\mathbb{R}}_+)$, with $\delta(m) = m (\mathrm{mod}\ 2)$, of functions with an asymptotic expansion of the form $\sum_{\kappa=0}^{\infty} a_\kappa x^{|m|+2\kappa}$ at zero. We have

$$\mathscr{S}_m(\overline{\mathbb{R}}_+) = x^{|m|}\mathscr{S}_0(\overline{\mathbb{R}}_+).$$

If we define $\mathscr{S}_m(\mathbb{C})$ to be the space of $\varphi \in \mathscr{S}(\mathbb{C})$ satisfying (2.2.21), then

$$\mathscr{S}_m(\mathbb{C}) = [z]^m \mathscr{S}_m(\overline{\mathbb{R}}_+) = \left\{ [z]^m \varphi(|z|) = e^{im\phi}\varphi(x) : \varphi \in \mathscr{S}_m(\overline{\mathbb{R}}_+) \right\}.$$

The last two paragraphs in Lemma 2.2.4 (3.1) can be recapitulated as below

$$\mathscr{S}(\mathbb{C}) \xrightarrow{\cong} \left\{ \{\varphi_m\} \in \prod_{m \in \mathbb{Z}} \mathscr{S}_m(\overline{\mathbb{R}}_+) : \varphi_m \text{ satisfies (2.2.32, 2.2.4, 2.2.4)} \right\} \twoheadrightarrow \mathscr{S}_m(\overline{\mathbb{R}}_+),$$

where the first map sends $\varphi \in \mathscr{S}(\mathbb{C})$ to the sequence $\{\varphi_m\}$ of its Fourier coefficients, and the second is the m-th projection. According to Lemma 2.2.4 (3.1), the first map is an isomorphism, and the second projection is surjective.

$\mathscr{S}_\delta(\mathbb{R}^\times)$ **and** $\mathscr{S}_m(\mathbb{C}^\times)$

Let $\delta \in \mathbb{Z}/2\mathbb{Z}$ and $m \in \mathbb{Z}$. We define $\mathscr{S}_\delta(\mathbb{R}^\times) = \mathscr{S}(\mathbb{R}^\times) \cap \mathscr{S}_\delta(\mathbb{R})$ and $\mathscr{S}_m(\mathbb{C}^\times) = \mathscr{S}(\mathbb{C}^\times) \cap \mathscr{S}_m(\mathbb{C})$. Clearly, $\mathscr{S}_\delta(\mathbb{R}^\times) = \mathrm{sgn}(x)^\delta \mathscr{S}(\mathbb{R}_+)$ and $\mathscr{S}_m(\mathbb{C}^\times) = [z]^m \mathscr{S}(\mathbb{R}_+)$.

2.2.4. The Fourier transform

According to the local theory in Tate's thesis for an archimedean local field $\mathbb{F}$, the Fourier transform $\widehat{\varphi} = \mathcal{F}\varphi$ of a Schwartz function $\varphi \in \mathscr{S}(\mathbb{F})$ is defined by

$$\widehat{\varphi}(y) = \int_{\mathbb{F}} \varphi(x) e(-\Lambda(xy)) dx, \tag{2.2.36}$$

with

$$\Lambda(x) = \begin{cases} x, & \text{if } \mathbb{F} = \mathbb{R}; \\ \mathrm{Tr}(x) = x + \overline{x}, & \text{if } \mathbb{F} = \mathbb{C}. \end{cases} \tag{2.2.37}$$

The Schwartz space $\mathscr{S}(\mathbb{F})$ is invariant under the Fourier transform. Moreover, with our choice of measure in §2.2.2, the following inversion formula holds

$$\hat{\hat{\varphi}}(x) = \varphi(-x), \quad x \in \mathbb{F}. \tag{2.2.38}$$

2.2.5. The Mellin transforms $\mathcal{M}$, $\mathcal{M}_\delta$ and $\mathcal{M}_m$

Corresponding to $\mathbb{R}_+$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$, there are three kinds of Mellin transforms $\mathcal{M}$, $\mathcal{M}_\delta$ and $\mathcal{M}_m$.

Definition 2.2.6 (Mellin transforms)**.**

(1). *The* Mellin transform $\mathcal{M}\varphi$ *of a Schwartz function* $\varphi \in \mathscr{S}(\mathbb{R}_+)$ *is given by*

$$\mathcal{M}\varphi(s) = \int_{\mathbb{R}_+} \varphi(x) x^s d^\times x. \tag{2.2.39}$$

(2). *For* $\delta \in \mathbb{Z}/2\mathbb{Z}$*, the* (signed) Mellin transform $\mathcal{M}_\delta\varphi$ with order δ *of a Schwartz function* $\varphi \in \mathscr{S}(\mathbb{R}^\times)$ *is defined by*

$$\mathcal{M}_\delta\varphi(s) = \int_{\mathbb{R}^\times} \varphi(x) \operatorname{sgn}(x)^\delta |x|^s d^\times x. \tag{2.2.40}$$

Moreover, define $\mathcal{M}_\mathbb{R} = (\mathcal{M}_0, \mathcal{M}_1)$.

(3). *For* $m \in \mathbb{Z}$*, the* Mellin transform $\mathcal{M}_m\varphi$ with order m *of a Schwartz function* $\varphi \in \mathscr{S}(\mathbb{C}^\times)$ *is defined by*

$$\mathcal{M}_m\varphi(s) = \int_{\mathbb{C}^\times} \varphi(z)[z]^m \|z\|^{\frac{1}{2}s}\, d^\times z = 2\int_0^\infty \int_0^{2\pi} \varphi\left(xe^{i\phi}\right) e^{im\phi} d\phi \cdot x^s d^\times x. \tag{2.2.41}$$

Moreover, define $\mathcal{M}_\mathbb{C} = \prod_{m\in\mathbb{Z}} \mathcal{M}_{-m}$.

Observation 2.2.1. *For* $\varphi \in \mathscr{S}(\mathbb{R}^\times)$*, we have*

$$\mathcal{M}_\delta\varphi(s) = 2\mathcal{M}\varphi_\delta(s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}. \tag{2.2.42}$$

Similarly, for $\varphi \in \mathscr{S}(\mathbb{C}^\times)$, we have

$$\mathcal{M}_{-m}\varphi(s) = 4\pi\mathcal{M}\varphi_m(s), \quad m \in \mathbb{Z}. \tag{2.2.43}$$

The relations (2.2.42) *and* (2.2.43) *reflect the identities $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+,-\}$ and $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ respectively.*

Lemma 2.2.7 (Mellin inversions)**.** *Let σ be real. Denote by (σ) the vertical line from $\sigma - i\infty$ to $\sigma + i\infty$.*

(1). *For $\varphi \in \mathscr{S}(\mathbb{R}_+)$, we have*

$$\varphi(x) = \frac{1}{2\pi i}\int_{(\sigma)} \mathcal{M}\varphi(s)x^{-s}ds. \tag{2.2.44}$$

(2). *For $\varphi \in \mathscr{S}(\mathbb{R}^\times)$, we have*

$$\varphi(x) = \frac{1}{4\pi i}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}} \operatorname{sgn}(x)^\delta \int_{(\sigma)} \mathcal{M}_\delta\varphi(s)|x|^{-s}ds. \tag{2.2.45}$$

(3). *For $\varphi \in \mathscr{S}(\mathbb{C}^\times)$, we have*

$$\varphi(z) = \frac{1}{8\pi^2 i}\sum_{m\in\mathbb{Z}} [z]^{-m} \int_{(\sigma)} \mathcal{M}_m\varphi(s)\|z\|^{-\frac{1}{2}s}ds, \tag{2.2.46}$$

or, in the polar coordinate,

$$\varphi\left(xe^{i\phi}\right) = \frac{1}{8\pi^2 i}\sum_{m\in\mathbb{Z}} e^{-im\phi} \int_{(\sigma)} \mathcal{M}_m\varphi(s)x^{-s}ds. \tag{2.2.47}$$

Definition 2.2.8.

(1). *Let $\mathscr{H}_{\mathrm{rd}}$ denote the space of all entire functions $H(s)$ on the complex plane that rapidly decay along vertical lines, uniformly on vertical strips.*

(2). *Define $\mathscr{H}_{\mathrm{rd}}^{\mathbb{R}} = \mathscr{H}_{\mathrm{rd}} \times \mathscr{H}_{\mathrm{rd}}$.*

(3). *Let $\mathscr{H}_{\mathrm{rd}}^{\mathbb{C}}$ be the subset of $\prod_{\mathbb{Z}} \mathscr{H}_{\mathrm{rd}}$ consisting of sequences $\{H_m(s)\}$ of entire functions in $\mathscr{H}_{\mathrm{rd}}$ satisfying the following condition,* (2.2.48) *for any given* $\alpha, A \in \mathbb{N}$ *and vertical strip* $\mathbb{S}[a, b]$,

$$H_m(s) \lll_{\alpha,A,a,b} (|m| + 1)^{-A}(|\Im\mathrm{m}\, s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a, b].$$

Corollary 2.2.9.

(1). *The Mellin transform* $\mathcal{M}$ *and its inversion establish an isomorphism between* $\mathscr{S}(\mathbb{R}_+)$ *and* $\mathscr{H}_{\mathrm{rd}}$.

(2). *For each* $\delta \in \mathbb{Z}/2\mathbb{Z}$, $\mathcal{M}_\delta$ *establishes an isomorphism between* $\mathscr{S}_\delta(\mathbb{R}^\times)$ *and* $\mathscr{H}_{\mathrm{rd}}$. *Hence,* $\mathcal{M}_{\mathbb{R}}$ *establishes an isomorphism between* $\mathscr{S}(\mathbb{R}^\times)$ *and* $\mathscr{H}^{\mathbb{R}}_{\mathrm{rd}}$.

(3). *For each* $m \in \mathbb{Z}$, $\mathcal{M}_{-m}$ *establishes an isomorphism between* $\mathscr{S}_m(\mathbb{C}^\times)$ *and* $\mathscr{H}_{\mathrm{rd}}$. *Moreover,* $\mathcal{M}_{\mathbb{C}}$ *establishes an isomorphism between* $\mathscr{S}(\mathbb{C}^\times)$ *and* $\mathscr{H}^{\mathbb{C}}_{\mathrm{rd}}$.

Proof. (1) is a well-known consequence of Lemma 2.2.7 (1), whereas (2) directly follows from (1) and Lemma 2.2.7 (2). As for (3), in addition to (1) and Lemma 2.2.7 (3), Lemma 2.2.4 (3.2) is also required for the rapid decay in m. Q.E.D.

2.3. The function spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$, $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ and $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$

The goal of this section is to extend the definitions of the Mellin transforms $\mathcal{M}$, $\mathcal{M}_{\mathbb{F}}$ and generalize the settings in §2.2.5 to the function spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$, $\mathscr{M}_{\mathrm{sis}}$, $\mathscr{S}_{\mathrm{sis}}(\mathbb{F}^\times)$ and $\mathscr{M}^{\mathbb{F}}_{\mathrm{sis}}$. These spaces are much more sophisticated than $\mathscr{S}(\mathbb{R}_+)$, $\mathscr{H}_{\mathrm{rd}}$, $\mathscr{S}(\mathbb{F}^\times)$ and $\mathscr{H}^{\mathbb{F}}_{\mathrm{rd}}$ but most suitable for investigating Hankel transforms over $\mathbb{R}_+$ and $\mathbb{F}^\times$.

We shall first construct the function spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$, $\mathscr{M}_{\mathrm{sis}}$ and establish an isomorphism between them using the Mellin transform $\mathcal{M}$. Based on these, we shall then turn to the spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{F}^\times)$, $\mathscr{M}^{\mathbb{F}}_{\mathrm{sis}}$ and the Mellin transform $\mathcal{M}_{\mathbb{F}}$. The case $\mathbb{F} = \mathbb{R}$ has been worked out in [MS3, §6]. Since $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+, -\}$ is simply two copies of $\mathbb{R}_+$, the properties of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ and $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}}$ are in substance the same as those of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ and $\mathscr{M}_{\mathrm{sis}}$. In the case

$\mathbb{F} = \mathbb{C}$, $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ and $\mathscr{M}_{\mathrm{sis}}^{\mathbb{C}}$ can be constructed in a parallel way. The study on $\mathbb{C}^\times$ is however much more elaborate, since $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ and the analysis on the circle $\mathbb{R}/2\pi\mathbb{Z}$ is also taken into account.

2.3.1. The spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ and $\mathscr{M}_{\mathrm{sis}}$

The spaces $x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+)$ and $\mathscr{M}_{\mathrm{sis}}^{\lambda,j}$

Let $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$.

We define

$$x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+) = \left\{ x^{-\lambda}(\log x)^j \varphi(x) : \varphi \in \mathscr{S}(\overline{\mathbb{R}}_+) \right\}.$$

We say that a meromorphic function $H(s)$ has a pole of *pure order* $j+1$ at $s = \lambda$ if the principal part of $H(s)$ at $s = \lambda$ is $a(s-\lambda)^{-j-1}$ for some constant $a \in \mathbb{C}$. Of course, $H(s)$ does not have a genuine pole at $s = \lambda$ if $a = 0$. We define the space $\mathscr{M}_{\mathrm{sis}}^{\lambda,j}$ of all meromorphic functions $H(s)$ on the complex plane such that

- the only possible singularities of $H(s)$ are poles of pure order $j+1$ at the points in $\lambda - \mathbb{N} = \{\lambda - \kappa : \kappa \in \mathbb{N}\}$, and

- $H(s)$ decays rapidly along vertical lines, uniformly on vertical strips, that is,

(2.3.1) for any given $\alpha \in \mathbb{N}$, vertical strip $\mathbb{S}[a,b]$ and $r > 0$,

$$H(s) \ll_{\lambda,j,\alpha,a,b,r} (|\mathfrak{Im}\, s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a,b] \smallsetminus \textstyle\bigcup_{\kappa\in\mathbb{N}} \mathbb{B}_r(\lambda - \kappa).$$

The constructions of the Mellin transform $\mathcal{M}$ and its inversion (2.2.39, 2.2.44) identically extend from $\mathscr{S}(\mathbb{R}_+)$ onto $\mathscr{S}_{\mathrm{sis}}^{\lambda,j}(\mathbb{R}_+)$, except that the conditions $\mathfrak{Re}\, s > \mathfrak{Re}\, \lambda$ and $\sigma > \mathfrak{Re}\, \lambda$ are required to guarantee convergence.

Lemma 2.3.1. *Let $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$. The Mellin transform $\mathcal{M}$ and its inversion establish an isomorphism of between $x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+)$ and $\mathscr{M}_{\mathrm{sis}}^{\lambda, j}$.*

This lemma is essentially [MS3, Lemma 6.13, Corollary 6.17]. Nevertheless, we shall include its proof as the reference for the constructions of $\mathscr{N}_{\mathrm{sis}}^{\mathbb{C},\lambda,j}$ and $\mathscr{M}_{\mathrm{sis}}^{\mathbb{C}}$ in §2.3.3 as well as the proof of Lemma 2.3.8.

Proof. Let $\upsilon(x) = x^{-\lambda}(\log x)^j \varphi(x)$ for some $\varphi \in \mathscr{S}(\overline{\mathbb{R}}_+)$. Suppose that the derivatives of φ satisfy (2.2.26) and (2.2.28), that is, asymptotic expansions at zero and the Schwartz condition at infinity.

Claim 1. Let

$$H(s) = \mathcal{M}\upsilon(s) = \int_0^\infty \upsilon(s) x^{s-1} dx, \quad \Re\mathfrak{e}\, s > \Re\mathfrak{e}\, \lambda.$$

Then H admits a meromorphic continuation onto the whole complex plane. The only singularities of H are poles of pure order $j+1$ at the points in $\lambda - \mathbb{N}$. More precisely, $H(s)$ has a pole at $s = \lambda - \kappa$ of principal part $(-)^j j! a_\kappa (s - \lambda + \kappa)^{-j-1}$. Moreover, H decays rapidly along vertical lines, uniformly on vertical strips. To be concrete, we have

(2.3.2) for any given $\alpha, A \in \mathbb{N}, b \geqslant a > \Re\mathfrak{e}\, \lambda - \alpha - A - 1$ and $r > 0$,

$$H(s) \ll_{\lambda,j,\alpha,A,a,b,r} (|\Im\mathfrak{m}\, s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a,b] \smallsetminus \textstyle\bigcup_{\kappa=0}^{\alpha+A} \mathbb{B}_r(\lambda - \kappa).$$

We remark that (2.3.1) and (2.3.2) are equivalent.

Proof of Claim 1. In view of $\mathcal{M}\left(x^{-\lambda}(\log x)^j \varphi(x)\right)(s) = \mathcal{M}\left((\log x)^j \varphi(x)\right)(s - \lambda)$, one may assume $\lambda = 0$. As such, $\upsilon(x) = (\log x)^j \varphi(x)$.

Let $A \in \mathbb{N}$. We have for $\Re\mathfrak{e}\, s > 0$

$$\begin{aligned} \mathcal{M}\upsilon(s) = & \int_0^1 (\log x)^j \left(\varphi(x) - \sum_{\kappa=0}^{A} a_\kappa x^\kappa \right) x^{s-1} dx + \sum_{\kappa=0}^{A} \frac{(-)^j j! a_\kappa}{(s+\kappa)^{j+1}} \\ & + \int_1^\infty (\log x)^j \varphi(x) x^{s-1} dx. \end{aligned} \tag{2.3.3}$$

Here, we have used

$$\int_0^1 (\log x)^j x^{s-1} dx = \frac{(-)^j j!}{s^{j+1}}, \quad \Re\mathrm{e}\, s > 0.$$

In view of $\varphi(x) - \sum_{\kappa=0}^{A} a_\kappa x^\kappa = O_A(x^{A+1})$, the first integral in (2.3.3) converges in the half-plane $\{s : \Re\mathrm{e}\, s > -A - 1\}$. The last integral converges for all s on the whole complex plane due to the rapid decay of φ. Thus $H(s) = \mathcal{M}\upsilon(s)$ admits a meromorphic extension onto $\{s : \Re\mathrm{e}\, s > -A - 1\}$ and, since A was arbitrary, onto the whole complex plane, with poles of pure order $j + 1$ at the points in $-\mathbb{N}$.

For any given $\alpha \in \mathbb{N}$, repeating partial integration α times to the defining integral of $\mathcal{M}\upsilon(s)$ yields

$$(-)^\alpha (s)_\alpha \mathcal{M}\upsilon(s) = \mathcal{M}\upsilon^{(\alpha)}(s + \alpha).$$

In view of this, we first expand $\mathcal{M}\upsilon^{(\alpha)}(s + \alpha)$ according to the expansion of $\upsilon^{(\alpha)}(x) = (d/dx)^\alpha \big((\log x)^j \varphi(x) \big)$. We then write each term in the expansion of $\mathcal{M}\upsilon^{(\alpha)}(s + \alpha)$ in the same fashion as (2.3.3) and apply (2.2.26) and (2.2.28) to estimate the first and the last integral respectively. We conclude that

$$\mathcal{M}\upsilon(s) \ll_{j,\alpha,A,a,b} \frac{1}{|(s)_\alpha|} \left(1 + \sum_{\kappa=0}^{\alpha+A} \left(\frac{1}{|s+\kappa|} + \dots + \frac{1}{|s+\kappa|^{j+1}} \right) \right),$$

for all $s \in \mathbb{S}[a, b]$, with $b \geqslant a > -\alpha - A - 1$. In particular, (2.3.2) is proven.

Let $H \in \mathscr{M}_{\mathrm{sis}}^{\lambda, j}$. Suppose that the principal part of $H(s)$ at $s = \lambda - \kappa$ is equal to $(-)^j j! a_\kappa (s + \lambda + \kappa)^{-j-1}$ and that $H(s)$ satisfies the condition (2.3.2).

Claim 2. If we denote by $\upsilon(x)$ the following integral

$$\upsilon(x) = \frac{1}{2\pi i} \int_{(\sigma)} H(s) x^{-s} ds, \quad \sigma > \Re\mathrm{e}\, \lambda,$$

then all the derivatives of $\varphi(x) = x^\lambda (\log x)^{-j} \upsilon(x)$ satisfy the asymptotics in (2.2.27) at zero and rapidly decay at infinity.

Proof of Claim 2. Again, let us assume $\lambda = 0$.

Let $1 > \rho > 0$. We left shift the contour of integration from (σ) to $(-\rho)$. When moving across $s = 0$, we obtain $a_0(\log x)^j$ in view of Cauchy's differentiation formula[XIV]. It follows that

$$\upsilon(x) = a_0(\log x)^j + \frac{1}{2\pi i}\int_{(-\rho)} H(s)x^{-s}ds.$$

Using (2.3.2) with r small, say $r < \rho$, to estimate the above integral, we arrive at

$$\upsilon(x) = a_0(\log x)^j + O\left(x^\rho\right) = (\log x)^j\left(a_0 + O(x^\rho)\right), \quad \text{as } x \to 0.$$

Thus $\varphi(x) = (\log x)^{-j}\upsilon(x)$ satisfies the asymptotic (2.2.27) with $\alpha = 0$. For the general case $\alpha \in \mathbb{N}$, we have

$$\upsilon^{(\alpha)}(x) = (-)^\alpha\frac{1}{2\pi i}\int_{(\sigma)}(s)_\alpha H(s)x^{-s-\alpha}ds. \tag{2.3.4}$$

Shifting the contour from (σ) to $(-\alpha - \rho)$ and following the same lines of arguments as above, combined with some straightforward algebraic manipulations, one may show (2.2.27) by an induction.

We are left to show the Schwartz condition for $\varphi(x) = (\log x)^{-j}\upsilon(x)$, or equivalently, that for $\upsilon(x)$. Indeed, the bound (2.2.28) for $\upsilon^{(\alpha)}(x)$ follows from right shifting the contour of the integral in (2.3.4) to the vertical line (β) and applying the estimates in (2.3.2). Q.E.D.

[XIV] Recall Cauchy's differentiation formula,

$$f^{(j)}(\zeta) = \frac{j!}{2\pi i}\oint_{\partial\mathbb{B}_r(\zeta)}\frac{f(s)}{(s-\zeta)^{j+1}}ds,$$

where f is a holomorphic function on a neighborhood of the closed disc $\overline{\mathbb{B}}_r(\zeta)$ centered at ζ, and the integral is taken counter-clockwise on the circle $\partial\mathbb{B}_r(\zeta)$. In the present situation, this formula is applied for $f(s) = x^{-s}$.

The spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ and $\mathscr{M}_{\mathrm{sis}}$

Let $\lambda, \lambda' \in \mathbb{C}$. We write $\lambda \preccurlyeq_1 \lambda'$ if $\lambda' - \lambda \in \mathbb{N}$ and $\lambda \sim_1 \lambda'$ if $\lambda' - \lambda \in \mathbb{Z}$. Observe that "$\preccurlyeq_1$" and "$\sim_1$" define an order relation and an equivalence relation on $\mathbb{C}$ respectively.

Define

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+) = \sum_{\lambda\in\mathbb{C}} \sum_{j\in\mathbb{N}} x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+),$$

where the sum $\sum_{\lambda\in\mathbb{C}} \sum_{j\in\mathbb{N}}$ is in the *algebraic* sense. It is clear that $\lambda \preccurlyeq_1 \lambda'$ if and only if $x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+) \subseteq x^{-\lambda'}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+)$. One also observes that $x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+) \cap x^{-\lambda'}(\log x)^{j'} \mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}(\mathbb{R}_+)$ if either $j \neq j'$ or $\lambda \not\sim_1 \lambda'$. Therefore,

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)/\mathscr{S}(\mathbb{R}_+) = \bigoplus_{\omega\in\mathbb{C}/\sim_1} \bigoplus_{j\in\mathbb{N}} \varinjlim_{\lambda\in\omega} \left(x^{-\lambda}(\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+)\right) \big/ \mathscr{S}(\mathbb{R}_+). \tag{2.3.5}$$

Here the direct limit $\varinjlim_{\lambda\in\omega}$ is taken on the totally ordered set $(\omega, \preccurlyeq_1)$ and may be simply viewed as the union $\bigcup_{\lambda\in\omega}$. More precisely, each function $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ can be expressed as a sum

$$\upsilon(x) = \upsilon^0(x) + \sum_{\lambda\in\Lambda} \sum_{j=0}^{N} x^{-\lambda}(\log x)^j \upsilon_{\lambda,j}(x),$$

with $\Lambda \subset \mathbb{C}$ a finite set such that $\lambda \not\sim_1 \lambda'$ for any two distinct points $\lambda, \lambda' \in \Lambda$, $N \in \mathbb{N}$, $\upsilon^0 \in \mathscr{S}(\mathbb{R}_+)$ and $\upsilon_{\lambda,j} \in \mathscr{S}(\overline{\mathbb{R}}_+)$. This expression is unique up to addition of Schwartz functions in $\mathscr{S}(\mathbb{R}_+)$.

On the other hand, we define the space $\mathscr{M}_{\mathrm{sis}}$ of all meromorphic functions H satisfying the following conditions,

- the poles of H lie in a finite number of sets $\lambda - \mathbb{N}$,
- the orders of the poles of H are uniformly bounded, and
- H decays rapidly along vertical lines, uniformly on vertical strips.

Appealing to certain Gamma identities for the Gamma function in [MS3, Lemma 6.24], one may show, in the same way as [MS3, Lemma 6.35], that

$$\mathscr{M}_{\mathrm{sis}} = \sum_{\lambda\in\mathbb{C}} \sum_{j\in\mathbb{N}} \mathscr{M}_{\mathrm{sis}}^{\lambda,j}.$$

We have $\mathscr{M}_{\mathrm{sis}}^{\lambda,j} \subseteq \mathscr{M}_{\mathrm{sis}}^{\lambda',j}$ if and only if $\lambda \preccurlyeq_1 \lambda'$, and $\mathscr{M}_{\mathrm{sis}}^{\lambda,j} \cap \mathscr{M}_{\mathrm{sis}}^{\lambda',j'} = \mathscr{H}_{\mathrm{rd}}$ if either $j \neq j'$ or $\lambda \nsim \lambda'$. Therefore

$$\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega\in\mathbb{C}/\sim_1} \bigoplus_{j\in\mathbb{N}} \varinjlim_{\lambda\in\omega} \mathscr{M}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.6}$$

The following lemma is a direct consequence of Lemma 2.3.1.

Lemma 2.3.2. *The Mellin transform* $\mathcal{M}$ *is an isomorphism between* $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ *and* $\mathscr{M}_{\mathrm{sis}}$ *which respects their decompositions* (2.3.5) *and* (2.3.6).

More refined decompositions of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$ and $\mathscr{M}_{\mathrm{sis}}$

Alternatively, we define an order relation on $\mathbb{C}$, $\lambda \preccurlyeq_2 \lambda'$ if $\lambda' - \lambda \in 2\mathbb{N}$, as well as an equivalence relation, $\lambda \sim_2 \lambda'$ if $\lambda' - \lambda \in 2\mathbb{Z}$.

Define $\mathscr{N}_{\mathrm{sis}}^{\lambda,j}$ in the same way as $\mathscr{M}_{\mathrm{sis}}^{\lambda,j}$ with $\lambda - \mathbb{N}$ replaced by $\lambda - 2\mathbb{N}$. Under the isomorphism via $\mathcal{M}$ in Lemma 2.3.1, $\mathscr{N}_{\mathrm{sis}}^{\lambda,j}$ is then isomorphic to $x^{-\lambda}(\log x)^j \mathscr{S}_0(\overline{\mathbb{R}}_+)$.

According to [MS3, Lemma 6.35], we have the following decomposition,

$$\mathscr{M}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}} = \mathscr{N}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}} \oplus \mathscr{N}_{\mathrm{sis}}^{\lambda-1,j}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.7}$$

Inserting this into (2.3.6), we obtain the following refined decomposition of $\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}}$

$$\bigoplus_{\omega\in\mathbb{C}/\sim_1} \bigoplus_{j\in\mathbb{N}} \varinjlim_{\lambda\in\omega} \left(\mathscr{N}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}} \oplus \mathscr{N}_{\mathrm{sis}}^{\lambda-1,j}/\mathscr{H}_{\mathrm{rd}}\right) = \bigoplus_{\omega\in\mathbb{C}/\sim_2} \bigoplus_{j\in\mathbb{N}} \varinjlim_{\lambda\in\omega} \mathscr{N}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}}.$$

Under the isomorphism via $\mathcal{M}$ in Lemma 2.3.2, the reflection of this refinement on the decomposition of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)/\mathscr{S}(\mathbb{R}_+)$ is

$$\bigoplus_{\omega\in\mathbb{C}/\sim_2} \bigoplus_{j\in\mathbb{N}} \varinjlim_{\lambda\in\omega} \left(x^{-\lambda}(\log x)^j \mathscr{S}_0(\overline{\mathbb{R}}_+)\right)/\mathscr{S}(\mathbb{R}_+).$$

Lemma 2.3.3. *We have the following refinements of the decompositions* (2.3.5, 2.3.6)*,*

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)/\mathscr{S}(\mathbb{R}_+) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\left(x^{-\lambda}(\log x)^j\mathscr{S}_0(\overline{\mathbb{R}}_+)\right)/\mathscr{S}(\mathbb{R}_+). \tag{2.3.8}$$

$$\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\mathscr{N}_{\mathrm{sis}}^{\lambda,j}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.9}$$

The Mellin transform $\mathcal{M}$ *respects these two decompositions.*

Corollary 2.3.4. *Let* $\delta\in\mathbb{Z}/2\mathbb{Z}$ *and* $m\in\mathbb{Z}$*, and recall the definitions of* $\mathscr{S}_\delta(\overline{\mathbb{R}}_+)$ *and* $\mathscr{S}_m(\overline{\mathbb{R}}_+)$ *in* §2.2.3.

(1). *The Mellin transform* $\mathcal{M}$ *respects the following decompositions,*

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)/\mathscr{S}(\mathbb{R}_+) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\left(x^{-\lambda}(\log x)^j\mathscr{S}_\delta(\overline{\mathbb{R}}_+)\right)/\mathscr{S}(\mathbb{R}_+), \tag{2.3.10}$$

$$\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\mathscr{N}_{\mathrm{sis}}^{\lambda-\delta,j}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.11}$$

(2). *The Mellin transform* $\mathcal{M}$ *respects the following decompositions,*

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)/\mathscr{S}(\mathbb{R}_+) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\left(x^{-\lambda}(\log x)^j\mathscr{S}_m(\overline{\mathbb{R}}_+)\right)/\mathscr{S}(\mathbb{R}_+), \tag{2.3.12}$$

$$\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\bigoplus_{j\in\mathbb{N}}\varinjlim_{\lambda\in\omega}\mathscr{N}_{\mathrm{sis}}^{\lambda-|m|,j}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.13}$$

Proof. These follow from Lemma 2.3.3 in conjunction with $x^\delta\mathscr{S}_0(\overline{\mathbb{R}}_+)=\mathscr{S}_\delta(\overline{\mathbb{R}}_+)$ and $x^{|m|}\mathscr{S}_0(\overline{\mathbb{R}}_+)=\mathscr{S}_m(\overline{\mathbb{R}}_+)$. Q.E.D.

Remark 2.3.5. *Set* $\lambda=0$ *and* $j=0$ *in* (2.3.7)*. It follows from the isomorphism* $\mathcal{M}$ *the decomposition as below,*

$$\mathscr{S}(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+) = \mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+)\oplus x\mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+).$$

Since $x\mathscr{S}_0(\overline{\mathbb{R}}_+)=\mathscr{S}_1(\overline{\mathbb{R}}_+)$*, one obtains* $\mathscr{S}(\overline{\mathbb{R}}_+)=\mathscr{S}_0(\overline{\mathbb{R}}_+)+\mathscr{S}_1(\overline{\mathbb{R}}_+)$ *and therefore* $C^\infty(\overline{\mathbb{R}}_+)=C_0^\infty(\overline{\mathbb{R}}_+)+C_1^\infty(\overline{\mathbb{R}}_+)$*. See Remark* 2.2.5.

2.3.2. The spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ and $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}}$

Following [MS3, (6.10)], we write $(\lambda, \delta) \preccurlyeq (\lambda', \delta')$ if $\lambda' - \lambda \in \mathbb{N}$ and $\lambda' - \lambda \equiv \delta' + \delta (\mathrm{mod}\ 2)$ and $(\lambda, \delta) \sim (\lambda', \delta')$ if $\lambda' - \lambda - (\delta' + \delta) \in 2\mathbb{Z}$. Again, these define an order relation and an equivalence relation on $\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

The space $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$

According to [MS3, Definition 6.4] and [MS3, Lemma 6.35], define

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} \mathrm{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R}).$$

We have the following decomposition,

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)/\mathscr{S}(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda,\delta) \in \omega} \left(\mathrm{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R}) \right) / \mathscr{S}(\mathbb{R}^\times).$$

It follows from $\mathrm{sgn}(x)|x|\mathscr{S}(\mathbb{R}) = x\mathscr{S}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ that

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)/\mathscr{S}(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(|x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R}) \right) / \mathscr{S}(\mathbb{R}^\times).$$

We let $\mathscr{S}^\delta_{\mathrm{sis}}(\mathbb{R}^\times)$ denote the space of functions $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ satisfying the parity condition (2.2.17). Clearly, $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times) = \mathscr{S}^0_{\mathrm{sis}}(\mathbb{R}^\times) \oplus \mathscr{S}^1_{\mathrm{sis}}(\mathbb{R}^\times)$. Then,

$$\mathscr{S}^\delta_{\mathrm{sis}}(\mathbb{R}^\times)/\mathscr{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(|x|^{-\lambda} (\log |x|)^j \mathscr{S}_\delta(\mathbb{R}) \right) / \mathscr{S}_\delta(\mathbb{R}^\times),$$

where $\mathscr{S}_\delta(\mathbb{R})$ and $\mathscr{S}_\delta(\mathbb{R}^\times)$ are defined in §2.2.3 and §2.2.3 respectively. Since $\mathscr{S}_\delta(\mathbb{R}) = \mathrm{sgn}(x)^\delta \mathscr{S}_\delta(\overline{\mathbb{R}}_+)$,

$$\mathscr{S}^\delta_{\mathrm{sis}}(\mathbb{R}^\times)/\mathscr{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(\mathrm{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathscr{S}_\delta(\overline{\mathbb{R}}_+) \right) / \mathscr{S}_\delta(\mathbb{R}^\times). \tag{2.3.14}$$

Consequently, $|x|^{-\delta}\mathscr{S}_\delta(\overline{\mathbb{R}}_+) = \mathscr{S}_0(\overline{\mathbb{R}}_+)$ and $\mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+) \oplus |x|\mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+) = \mathscr{S}(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+)$ (see Remark 2.3.5) yields

$$\mathscr{S}^\delta_{\mathrm{sis}}(\mathbb{R}^\times)/\mathscr{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(\mathrm{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\overline{\mathbb{R}}_+) \right) / \mathscr{S}_\delta(\mathbb{R}^\times). \tag{2.3.15}$$

In particular,

$$\mathscr{S}^{\delta}_{\mathrm{sis}}(\mathbb{R}^{\times}) = \mathrm{sgn}(x)^{\delta}\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_{+}) = \left\{\mathrm{sgn}(x)^{\delta}\upsilon(|x|) : \upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}_{+})\right\}.$$

The space $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}}$

We simply define $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}} = \mathscr{M}_{\mathrm{sis}} \times \mathscr{M}_{\mathrm{sis}}$.

Isomorphism between $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times})$ and $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}}$ via the Mellin transform $\mathcal{M}_{\mathbb{R}}$

Let $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times})$. Since $\upsilon_{\delta} \in \mathscr{S}_{\mathrm{sis}}(\overline{\mathbb{R}}_{+})$, the identity

$$\mathcal{M}_{\delta}\upsilon(s) = 2\mathcal{M}\upsilon_{\delta}(s)$$

extends the definition of the Mellin transform $\mathcal{M}_{\delta}$ onto the space $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times})$. Therefore, as a consequence of Lemma 2.3.1, Lemma 2.3.2 and Corollary 2.3.4 (1), the following lemma is readily established.

Lemma 2.3.6. *For $\delta \in \mathbb{Z}/2\mathbb{Z}$, the Mellin transform $\mathcal{M}_{\delta}$ establishes an isomorphism between the spaces $\mathscr{S}^{\delta}_{\mathrm{sis}}(\mathbb{R}^{\times})$ and $\mathscr{M}_{\mathrm{sis}}$ which respects their decompositions* (2.3.15) *and* (2.3.6) *as well as* (2.3.14) *and* (2.3.11). *Therefore, $\mathcal{M}^{\mathbb{R}} = (\mathcal{M}_0, \mathcal{M}_1)$ establishes an isomorphism between $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times}) = \mathscr{S}^{0}_{\mathrm{sis}}(\mathbb{R}^{\times}) \oplus \mathscr{S}^{1}_{\mathrm{sis}}(\mathbb{R}^{\times})$ and $\mathscr{M}^{\mathbb{R}}_{\mathrm{sis}} = \mathscr{M}_{\mathrm{sis}} \times \mathscr{M}_{\mathrm{sis}}$.*

An alternative decomposition of $\mathscr{S}^{\delta}_{\mathrm{sis}}(\mathbb{R}^{\times})$

The following lemma follows from Corollary 2.3.4 (1) (compare [MS3, Corollary 6.17]).

Lemma 2.3.7. *Let $\delta \in \mathbb{Z}/2\mathbb{Z}$. The Mellin transform $\mathcal{M}_{\delta}$ respects the following decompositions,*

$$\begin{aligned} &\mathscr{S}^{\delta}_{\mathrm{sis}}(\mathbb{R}^{\times})/\mathscr{S}_{\delta}(\mathbb{R}^{\times}) \\ &= \bigoplus_{\omega \in \mathbb{C}\times\mathbb{Z}/2\mathbb{Z}/\sim} \bigoplus_{j\in\mathbb{N}} \varinjlim_{(\lambda,\epsilon)\in\omega} \left(\mathrm{sgn}(x)^{\epsilon}|x|^{-\lambda}(\log|x|)^{j}\mathscr{S}_{\epsilon+\delta}(\mathbb{R})\right)/\mathscr{S}_{\delta}(\mathbb{R}^{\times}), \end{aligned} \tag{2.3.16}$$

$$(2.3.17)\qquad \mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega\in\mathbb{C}\times\mathbb{Z}/2\mathbb{Z}/\sim}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{(\lambda,\epsilon)\in\omega} \mathscr{N}_{\mathrm{sis}}^{\lambda-(\epsilon+\delta),j}\Big/\mathscr{H}_{\mathrm{rd}}.$$

2.3.3. The spaces $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ and $\mathscr{M}_{\mathrm{sis}}^{\mathbb{C}}$

We write $(\lambda, m) \preccurlyeq (\lambda', m')$ if $\lambda' - \lambda \in |m' - m| + 2\mathbb{N}$ and $(\lambda, m) \sim (\lambda', m')$ if $\lambda' - \lambda - |m' - m| \in 2\mathbb{Z}$. These define an order relation and an equivalence relation on $\mathbb{C} \times \mathbb{Z}$.

The space $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$

In parallel to §2.3.2, we first define

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times) = \sum_{m\in\mathbb{Z}}\sum_{\lambda\in\mathbb{C}}\sum_{j\in\mathbb{N}} [z]^{-m}|z|^{-\lambda}(\log|z|)^j \mathscr{S}(\mathbb{C}).$$

We have the following decomposition,

$$\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}(\mathbb{C}^\times) = \bigoplus_{\omega\in\mathbb{C}\times\mathbb{Z}/\sim}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{(\lambda,m)\in\omega} \left([z]^{-m}|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\mathbb{C})\right)\big/\mathscr{S}(\mathbb{C}^\times).$$

It follows from $[z]|z|\mathscr{S}(\mathbb{C}) = z\mathscr{S}(\mathbb{C}) \subset \mathscr{S}(\mathbb{C})$ that

$$(2.3.18)\qquad \mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}(\mathbb{C}^\times) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{\lambda\in\omega} \left(|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\mathbb{C})\right)\big/\mathscr{S}(\mathbb{C}^\times).$$

We let $\mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)$ denote the space of functions $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ satisfying (2.2.21). Then,

$$\mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{\lambda\in\omega} \left(|z|^{-\lambda}(\log|z|)^j\mathscr{S}_m(\mathbb{C})\right)\big/\mathscr{S}_m(\mathbb{C}^\times),$$

where $\mathscr{S}_m(\mathbb{C})$ and $\mathscr{S}_m(\mathbb{C}^\times)$ are defined in §2.2.3 and §2.2.3 respectively. Since $\mathscr{S}_m(\mathbb{C}) = [z]^m\mathscr{S}_m(\overline{\mathbb{R}}_+)$,

$$(2.3.19)\qquad \mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega\in\mathbb{C}/\sim_2}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{\lambda\in\omega} \Big([z]^m|z|^{-\lambda}(\log|z|)^j\mathscr{S}_m(\overline{\mathbb{R}}_+)\Big)\big/\mathscr{S}_m(\mathbb{C}^\times).$$

Consequently, $|z|^{-|m|}\mathscr{S}_m(\overline{\mathbb{R}}_+) = \mathscr{S}_0(\overline{\mathbb{R}}_+)$ and $\mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+) \oplus |z|\mathscr{S}_0(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+) = \mathscr{S}(\overline{\mathbb{R}}_+)/\mathscr{S}(\mathbb{R}_+)$ yields

$$(2.3.20)\qquad \mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega\in\mathbb{C}/\sim_1}\ \bigoplus_{j\in\mathbb{N}}\ \varinjlim_{\lambda\in\omega} \Big([z]^m|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\overline{\mathbb{R}}_+)\Big)\big/\mathscr{S}_m(\mathbb{C}^\times).$$

In particular,

$$\mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times) = [z]^m \mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+) = \{[z]^m \upsilon(|z|) : \upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)\}.$$

The space $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$

For $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, we define the space $\mathscr{N}^{\mathbb{C},\lambda,j}_{\mathrm{sis}}$ of all sequences $\{H_m(s)\}$ of meromorphic functions such that

- the only singularities of H_m are poles of pure order $j+1$ at the points in $\lambda - |m| - 2\mathbb{N}$,
- Each H_m decays rapidly along vertical lines, uniformly on vertical strips (see (2.3.1)), and
- $H_m(s)$ also decays rapidly with respect to m, uniformly on vertical strips, in the sense that

(2.3.21) for any given $\alpha, A \in \mathbb{N}$ and vertical strip $\mathbb{S}[a,b]$,

$$H_m(s) \ll_{\lambda,j,\alpha,A,a,b} (|m|+1)^{-A}(|\Im\mathrm{m}\, s|+1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a,b], \text{ if } |m| > \Re\mathrm{e}\,\lambda - a.$$

Observe that the first two conditions amount to $H_m \in \mathscr{N}^{\lambda-|m|,j}_{\mathrm{sis}}$. Therefore, $\mathscr{N}^{\mathbb{C},\lambda,j}_{\mathrm{sis}} \subset \prod_{m\in\mathbb{Z}} \mathscr{N}^{\lambda-|m|,j}_{\mathrm{sis}}$.

Define the space $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$ of all sequences $\{H_m\}$ of meromorphic functions such that

- the poles of each H_m lie in $\lambda - |m| - 2\mathbb{N}$, for a finite number of λ,
- the orders of the poles of H_m are uniformly bounded,
- Each H_m decays rapidly along vertical lines, uniformly on vertical strips, and
- H_m decays rapidly with respect to m, uniformly on vertical strips.

Using the refined Stirling's asymptotic formula (2.2.13) in place of [MS3, (6.22)] and the following bound in place of [MS3, (6.23)]

$$\left| \frac{\Gamma^{(j)}\left(\frac{1}{2}(s - \lambda + |m|)\right)}{\Gamma\left(\frac{1}{2}(s + |m|)\right)} \right| \ll_{\lambda,j,a,b,r} (|\Im\mathfrak{m}\, s| + |m| + 1)^{-\frac{1}{2}\Re\mathfrak{e}\,\lambda}$$

for $\lambda \in \mathbb{C}$, $j \in \mathbb{N}$, $s \in \mathbb{S}[a,b] \smallsetminus \bigcup_{\substack{\kappa \geqslant |m| \\ \kappa \equiv m (\mathrm{mod}\ 2)}} \mathbb{B}_r(\lambda - \kappa)$, with $r > 0$, we may follow the same lines of the proofs of [MS3, Lemma 6.24] and [MS3, Lemma 6.35] to show that

$$\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}} = \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} \mathscr{N}^{\mathbb{C},\lambda,j}_{\mathrm{sis}},$$

and consequently

$$\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}} / \mathscr{H}^{\mathbb{C}}_{\mathrm{rd}} = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathscr{N}^{\mathbb{C},\lambda,j}_{\mathrm{sis}} / \mathscr{H}^{\mathbb{C}}_{\mathrm{rd}}. \tag{2.3.22}$$

Isomorphism between $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ and $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$ via the Mellin transform $\mathcal{M}_{\mathbb{C}}$

For $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$, its m-th Fourier coefficient υ_m is a function in $\mathscr{S}_{\mathrm{sis}}(\overline{\mathbb{R}}_+)$. Hence the identity

$$\mathcal{M}_{-m}\upsilon(s) = 4\pi \mathcal{M}\upsilon_m(s)$$

extends the definition of the Mellin transform $\mathcal{M}_{-m}$ onto the space $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$.

Lemma 2.3.8. *For $m \in \mathbb{Z}$, the Mellin transform $\mathcal{M}_{-m}$ establishes an isomorphism between the spaces $\mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)$ and $\mathscr{M}_{\mathrm{sis}}$ which respects their decompositions* (2.3.20) *and* (2.3.6) *as well as* (2.3.19) *and* (2.3.13). *Furthermore, $\mathcal{M}_{\mathbb{C}} = \prod_{m \in \mathbb{Z}} \mathcal{M}_{-m}$ establishes an isomorphism between $|z|^{-\lambda}(\log|z|)^j \mathscr{S}(\mathbb{C})$ and $\mathscr{N}^{\mathbb{C},\lambda,j}_{\mathrm{sis}}$ for any $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, and hence an isomorphism between $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ and $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$ which respects their decompositions* (2.3.18) *and* (2.3.22).

Proof. For $\upsilon \in \mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)$, one has $\upsilon\left(xe^{i\phi}\right) = e^{im\phi}\upsilon_m(x)$ and $\upsilon_m \in \mathscr{S}_{\mathrm{sis}}(\overline{\mathbb{R}}_+)$. Thus the first assertion follows immediately from Lemma 2.3.2 and Corollary 2.3.4 (2).

Now let $\varphi \in \mathscr{S}(\mathbb{C})$ and $\upsilon(z) = |z|^{-\lambda}(\log|z|)^j\varphi(z)$. Clearly, their m-th Fourier coefficients are related by $\upsilon_m(x) = x^{-\lambda}(\log x)^j\varphi_m(x)$. Since $\varphi_m \in \mathscr{S}_m(\overline{\mathbb{R}}_+)$, it follows from Corollary 2.3.4 (2) that $H_m = \mathcal{M}_{-m}\upsilon = 4\pi\mathcal{M}\upsilon_m$ lies in $\mathscr{N}_{\mathrm{sis}}^{\lambda-|m|,j}$, and therefore we are left to show (2.3.21). Recall that in the proof of Lemma 2.3.1 we turned to verify (2.3.2) instead of (2.3.1). Likewise, it is more convenient to verify the following equivalent statement of (2.3.21),

(2.3.23) for any given $\alpha, A \in \mathbb{N}, b \geqslant a > \mathfrak{Re}\,\lambda - \alpha - A - 1$,

$$H_m(s) \ll_{\lambda,j,\alpha,A,a,b} (|m|+1)^{-A}(|\mathfrak{Im}\,s|+1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a,b], \text{ if } |m| > \alpha + A.$$

According to Lemma 2.2.4 (3.1), φ_m satisfies the conditions (2.2.32, 2.2.4). Suppose $|m| > \alpha + A$. One directly applies (2.2.32) and (2.2.4) to bound the following integral by a constant multiple of $(|m|+1)^{-A}$,

$$(-)^\alpha(s-\lambda)_\alpha\mathcal{M}\upsilon_m(s) = \int_0^\infty \frac{d^\alpha}{dx^\alpha}\left((\log x)^j\varphi_m(x)\right) x^{s-\lambda+\alpha-1}dx.$$

This proves (2.3.23) for $H_m = 4\pi\mathcal{M}\upsilon_m$. Therefore, the sequence $\{\mathcal{M}_{-m}\upsilon\}$ belongs to $\mathscr{N}_{\mathrm{sis}}^{\mathbb{C},\lambda,j}$.

Conversely, let $\{H_m\} \in \mathscr{N}_{\mathrm{sis}}^{\mathbb{C},\lambda,j}$, and let $4\pi\upsilon_m$ be the Mellin inversion of H_m,

$$\upsilon_m(x) = \frac{1}{8\pi^2 i}\int_{(\sigma)} H_m(s)x^{-s}ds, \quad \sigma > \mathfrak{Re}\,\lambda - |m|.$$

Since $H_m \in \mathscr{N}_{\mathrm{sis}}^{\lambda-|m|,j}$, Corollary 2.3.4 (2) implies that $\upsilon_m(x) \in x^{-\lambda}(\log x)^j\mathscr{S}_m(\overline{\mathbb{R}}_+)$ and hence $\varphi_m(x) = x^\lambda(\log x)^{-j}\upsilon_m(x)$ lies in $\mathscr{S}_m(\overline{\mathbb{R}}_+)$. This proves (2.2.4). Similar to the proof of Lemma 2.3.1, right shifting of the contour of integration combined with (2.3.23) yields (2.2.32), whereas left shifting combined with (2.3.23) yields (2.2.4)[XV].

The proof of the second assertion is completed. Q.E.D.

[XV] Actually, $O_{\alpha,A}\left((|m|+1)^{-A}x^{A+1}\right)$ in (2.2.4) should be replaced by $O_{\alpha,A,\rho}\left((|m|+1)^{-A}x^{A+\rho}\right)$, $1 > \rho > 0$. Moreover, one observes that the left contour shift here does not cross any pole.

An alternative decomposition of $\mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)$

The following lemma follows from Corollary 2.3.4 (2).

Lemma 2.3.9. *Let $m \in \mathbb{Z}$. The Mellin transform $\mathcal{M}_{-m}$ respects the following decompositions,*

$$\begin{aligned} & \mathscr{S}^m_{\mathrm{sis}}(\mathbb{C}^\times)/\mathscr{S}_m(\mathbb{C}^\times) \\ & = \bigoplus_{\omega \in \mathbb{C}\times\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda,k)\in\omega} \left([z]^{-k}|z|^{-\lambda}(\log|z|)^j \mathscr{S}_{m+k}(\mathbb{C})\right)/\mathscr{S}_m(\mathbb{C}^\times), \end{aligned} \tag{2.3.24}$$

$$\mathscr{M}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}} = \bigoplus_{\omega \in \mathbb{C}\times\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda,k)\in\omega} \mathscr{N}^{\lambda-|m+k|,j}_{\mathrm{sis}}/\mathscr{H}_{\mathrm{rd}}. \tag{2.3.25}$$

2.4. Hankel transforms and their Bessel kernels

This section is arranged as follows. We start with the type of Hankel transforms over $\mathbb{R}_+$ whose kernels are the (fundamental) Bessel functions studied in Chapter 1. After this, we introduce two auxiliary Hankel transforms and Bessel kernels over $\mathbb{R}_+$. Finally, we proceed to construct and study Hankel transforms and their Bessel kernels over $\mathbb{F}^\times$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Definition 2.4.1. *Let $(\mathbb{X}, \preccurlyeq)$ be an ordered set satisfying the condition that*

$$\text{“}\lambda \preccurlyeq \lambda' \text{ or } \lambda' \preccurlyeq \lambda\text{” is an equivalence relation.} \tag{2.4.1}$$

We denote the above equivalence relation by $\lambda \sim \lambda'$. Given $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathbb{X}^n$, the set $\{1, ...n\}$ is partitioned into several pair-wise disjoint subsets L_α, $\alpha = 1, ..., A$, such that

$$\lambda_\ell \sim \lambda_{\ell'} \text{ if and only if } \ell, \ell' \text{ are in the same } L_\alpha.$$

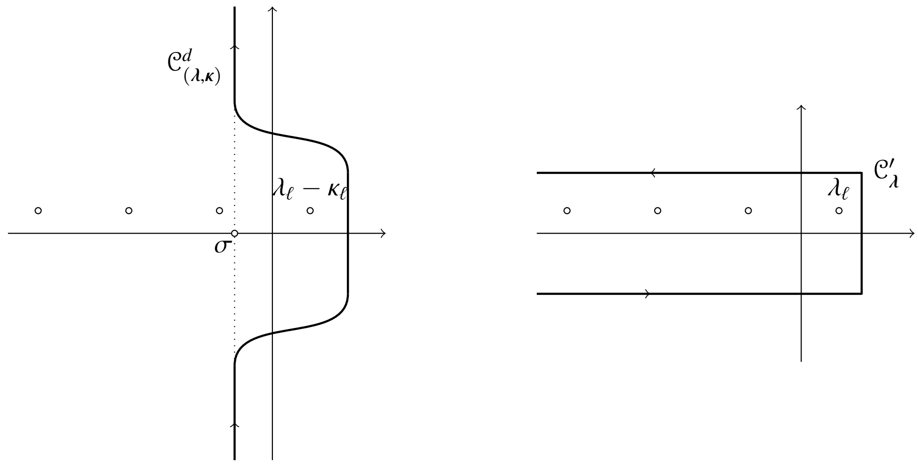


Figure 2.1: $\mathcal{C}^d_{(\boldsymbol{\lambda},\boldsymbol{\kappa})}$ and $\mathcal{C}'_{\boldsymbol{\lambda}}$

Each $\Lambda^\alpha = \{\lambda_\ell\}_{\ell \in L_\alpha}$[XVI] is a totally ordered set. Let $B_\alpha = |\Lambda^\alpha|$ and label the elements of Λ^α in the descending order, $\lambda_{\alpha,1} \succ \ldots \succ \lambda_{\alpha,B_\alpha}$. For $\lambda_{\alpha,\beta} \in \Lambda^\alpha$, let $M_{\alpha,\beta}$ denote the multiplicity of $\lambda_{\alpha,\beta}$ in $\boldsymbol{\lambda}$, that is, $M_{\alpha,\beta} = |\{\ell : \lambda_\ell = \lambda_{\alpha,\beta}\}|$, and define $N_{\alpha,\beta} = \sum_{\gamma=1}^{\beta} M_{\alpha,\gamma} = |\{\ell : \lambda_{\alpha,\beta} \preccurlyeq \lambda_\ell\}|$.

$\boldsymbol{\lambda}$ is called generic if $\lambda_\ell \not\sim \lambda_{\ell'}$ for any $\ell \neq \ell'$.

We recall that the ordered sets $(\mathbb{C}, \preccurlyeq_1)$, $(\mathbb{C}, \preccurlyeq_2)$, $(\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, \preccurlyeq)$ and $(\mathbb{C} \times \mathbb{Z}, \preccurlyeq)$ defined in §2.3 all satisfy (2.4.1).

Definition 2.4.2. *Let $d = 1$ or 2, $\boldsymbol{\lambda} \in \mathbb{C}^n$ and $\boldsymbol{\kappa} \in \mathbb{N}^n$. Put $\sigma < \frac{d}{2} + \frac{1}{n}(\mathfrak{Re}\,|\boldsymbol{\lambda}| - 1)$ and choose a contour $\mathcal{C}^d_{(\boldsymbol{\lambda},\boldsymbol{\kappa})}$ (see Figure 2.1) such that*

- *$\mathcal{C}^d_{(\boldsymbol{\lambda},\boldsymbol{\kappa})}$ is upward directed from $\sigma - i\infty$ to $\sigma + i\infty$,*
- *all the sets $\lambda_\ell - \kappa_\ell - \mathbb{N}$ lie on the left side of $\mathcal{C}_{(\boldsymbol{\lambda},\boldsymbol{\kappa})}$, and*
- *if $s \in \mathcal{C}^d_{(\boldsymbol{\lambda},\boldsymbol{\kappa})}$ and $|\mathfrak{Im}\, s|$ is sufficiently large, say $|\mathfrak{Im}\, s| - \max\{|\mathfrak{Im}\,\lambda_\ell|\} \geqslant 1$, then $\mathfrak{Re}\, s = \sigma$.*

[XVI] Here, $\{\lambda_\ell\}_{\ell \in L_\alpha}$ is considered as a set, namely, λ_ℓ are counted without multiplicity.

For $\lambda \in \mathbb{C}$, we denote $\mathcal{C}_{\lambda} = \mathcal{C}^1_{(\lambda, \mathbf{0})}$. For $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$, we denote $\mathcal{C}_{(\boldsymbol{\mu}, \boldsymbol{\delta})} = \mathcal{C}^1_{(\boldsymbol{\mu}, \boldsymbol{\delta})}$. For $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$, we denote $\mathcal{C}_{(\boldsymbol{\mu}, \boldsymbol{m})} = \frac{1}{2} \cdot \mathcal{C}^2_{(2\boldsymbol{\mu}, \|\boldsymbol{m}\|)}$.

Definition 2.4.3. *For $\boldsymbol{\lambda} \in \mathbb{C}^n$, choose a contour $\mathcal{C}'_{\boldsymbol{\lambda}}$ illustrated in Figure* 2.1 *such that*

- *$\mathcal{C}'_{\boldsymbol{\lambda}}$ starts from and returns to $-\infty$ counter-clockwise,*
- *$\mathcal{C}'_{\boldsymbol{\lambda}}$ consists two horizontal infinite half lines,*
- *$\mathcal{C}'_{\boldsymbol{\lambda}}$ encircles all the sets $\lambda_\ell - \mathbb{N}$, and*
- *$\mathfrak{Im}\, s \leqslant \max\{|\mathfrak{Im}\, \lambda_\ell|\} + 1$ for all $s \in \mathcal{C}'_{\boldsymbol{\lambda}}$.*

2.4.1. The Hankel transform $\mathcal{H}_{(\boldsymbol{\varsigma}, \boldsymbol{\lambda})}$ and the Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$

The definition of $\mathcal{H}_{(\boldsymbol{\varsigma}, \boldsymbol{\lambda})}$

Consider the ordered set $(\mathbb{C}, \leqslant_1)$. For $\boldsymbol{\lambda} \in \mathbb{C}^n$, let notations $\lambda_{\alpha,\beta}$, B_α, $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ be as in Definition 2.4.1. We define the following subspace of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$,

$$\mathscr{S}^{\boldsymbol{\lambda}}_{\mathrm{sis}}(\mathbb{R}_+) = \sum_{\alpha=1}^{A} \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}-1} x^{-\lambda_{\alpha,\beta}} (\log x)^j \mathscr{S}(\overline{\mathbb{R}}_+).$$

Proposition 2.4.4. *Let $(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \in \{+,-\}^n \times \mathbb{C}^n$. Suppose $\upsilon \in \mathscr{S}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathscr{S}^{\boldsymbol{\lambda}}_{\mathrm{sis}}(\mathbb{R}_+)$ satisfying the following identity,*

$$\mathcal{M}\Upsilon(s) = G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) \mathcal{M}\upsilon(1 - s). \tag{2.4.2}$$

We call Υ the Hankel transform of υ over $\mathbb{R}_+$ of index $(\boldsymbol{\varsigma}, \boldsymbol{\lambda})$ and write $\mathcal{H}_{(\boldsymbol{\varsigma}, \boldsymbol{\lambda})}\upsilon = \Upsilon$.

Proof. Recall the definition of $G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ given by (2.2.1, 2.2.2),

$$G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = e\left(\frac{\sum_{\ell=1}^n \varsigma_\ell (s - \lambda_\ell)}{4}\right) \prod_{\ell=1}^n \Gamma(s - \lambda_\ell).$$

The product in the above expression may be rewritten as below

$$\prod_{\alpha=1}^{A}\prod_{\beta=1}^{B_\alpha}\Gamma\left(s-\lambda_{\alpha,\beta}\right)^{M_{\alpha,\beta}}.$$

Thus the singularities of $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ are poles at the points in $\lambda_{\alpha,1}-\mathbb{N}$, $\alpha=1,...,A$. More precisely, $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ has a pole of pure order $N_{\alpha,\beta}$ at $\lambda\in\lambda_{\alpha,1}-\mathbb{N}$ if one let $\beta=\max\left\{\beta':\lambda\preccurlyeq_1\lambda_{\alpha,\beta'}\right\}$. Moreover, in view of (2.2.14) in Lemma 2.2.2, $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is of uniform moderate growth on vertical strips.

On the other hand, according to Corollary 2.2.9 (1), $\mathcal{M}\upsilon(1-s)$ uniformly rapidly decays on vertical strips.

Therefore, the product $G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})\mathcal{M}\upsilon(1-s)$ on the right hand side of (2.4.2) is a meromorphic function in the space $\sum_{\alpha=1}^{A}\sum_{\beta=1}^{B_\alpha}\sum_{j=0}^{N_{\alpha,\beta}-1}\mathscr{M}_{\mathrm{sis}}^{\lambda_{\alpha,\beta},j}$. We conclude from Lemma 2.3.2 that (2.4.2) uniquely determines a function Υ in $\mathscr{S}_{\mathrm{sis}}^{\boldsymbol{\lambda}}(\mathbb{R}_+)$. Q.E.D.

The Bessel function $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$

The integral kernel $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ of $\mathcal{H}_{(\boldsymbol{\varsigma},\boldsymbol{\lambda})}$.

Suppose $\upsilon\in\mathscr{S}(\mathbb{R}_+)$. By the Mellin inversion, we have

$$\Upsilon(x)=\frac{1}{2\pi i}\int_{(\sigma)}G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})\mathcal{M}\upsilon(1-s)x^{-s}ds,\quad \sigma>\max\left\{\Re\mathfrak{e}\,\lambda_\ell\right\}. \tag{2.4.3}$$

It is an iterated double integral as below

$$\Upsilon(x)=\frac{1}{2\pi i}\int_{(\sigma)}\int_0^\infty\upsilon(y)y^{-s}dy\cdot G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})x^{-s}ds.$$

We now shift the integral contour to $\mathcal{C}_{\boldsymbol{\lambda}}$ defined in Definition 2.4.2. Using (2.2.14) in Lemma 2.2.2, one shows that the above double integral becomes absolutely convergent after this contour shift. Therefore, on changing the order of integrals, one obtains

$$\Upsilon(x)=\int_0^\infty\upsilon(y)J\left((xy)^{\frac{1}{n}};\boldsymbol{\varsigma},\boldsymbol{\lambda}\right)dy. \tag{2.4.4}$$

Here $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is the (*fundamental*) *Bessel function* defined by the Barnes-Mellin type integral

(2.4.5)
$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \frac{1}{2\pi i} \int_{\mathcal{C}_{\boldsymbol{\lambda}}} G(s; \boldsymbol{\varsigma}, \boldsymbol{\lambda}) x^{-ns} ds,$$

which is categorized as a *Bessel function of the second kind* (see Chapter 1).

Remark 2.4.5. *The expression* (2.4.4) *of the Hankel transform together with properties of the Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ may also yield $\Upsilon \in \mathscr{S}^{\boldsymbol{\lambda}}_{\mathrm{sis}}(\mathbb{R}_+)$.*

The Schwartz condition on Υ at infinity follows from either the rapid decay or the oscillation of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ as well as its derivatives (*see* §1.5, §1.9).

As for the singularity type of Υ at zero, we first assume that $\boldsymbol{\lambda}$ is generic. We express $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ as a combination of Bessel functions of the first kind (*see* §1.7.1, 1.7.2). *Then the type of singularities of Υ at zero is reflected by the leading term in the series expansions of Bessel functions of the first kind. For nongeneric $\boldsymbol{\lambda}$ the occurrence of powers of* $\log x$ *follows from either solving the Bessel equations using the Frobenius method or taking the limit of the above expression of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ with respect to the index $\boldsymbol{\lambda}$.*

Shifting the index of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

Lemma 2.4.6. *Let $(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \in \{+, -\}^n \times \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Recall that $\boldsymbol{e}^n$ denotes the n-tuple* $(1, ..., 1)$. *Then*

(2.4.6)
$$J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda} - \lambda \boldsymbol{e}^n) = x^{n\lambda} J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda}).$$

Regularity of $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$.

According to §1.6, 1.7, $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ satisfies a differential equation with analytic coefficients. Therefore, $J(x; \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ admits an analytic continuation from $\mathbb{R}_+$ onto $\mathbb{U}$, and in particular is *real analytic*. Here, we shall take an alternative viewpoint from Remark 1.7.10, that is the following Barnes type integral representation,

$$(2.4.7)\qquad J(\zeta;\boldsymbol{\varsigma},\boldsymbol{\lambda}) = \frac{1}{2\pi i}\int_{\mathcal{C}'_{\boldsymbol{\lambda}}} G(s;\boldsymbol{\varsigma},\boldsymbol{\lambda})\zeta^{-ns}ds, \quad \zeta = xe^{i\omega}\in\mathbb{U}, x\in\mathbb{R}_+, \omega\in\mathbb{R},$$

with the integral contour given in Definition 2.4.3. One first rewrites $G(s,\pm)$ using Euler's reflection formula,

$$G(s,\pm) = \frac{\pi e\left(\pm\frac{1}{4}s\right)}{\sin(\pi s)\Gamma(1-s)},$$

Then Stirling's formula (2.2.13) yields,

$$G(-\rho+it;\boldsymbol{\varsigma},\boldsymbol{\lambda}) \ll_{\boldsymbol{\lambda},r} e^{n\rho}\rho^{-n\left(\rho+\frac{1}{2}\right)-\Re\mathrm{e}\,|\boldsymbol{\lambda}|},$$

for all $-\rho+it\notin\bigcup_{\ell=1}^{n}\bigcup_{\kappa\in\mathbb{N}}\mathbb{B}_r(\lambda_\ell-\kappa)$ satisfying $\rho\gg 1$ and $t\ll\max\{|\Im\mathrm{m}\,\lambda_\ell|\}+1$. It follows that the contour integral in (2.4.7) converges absolutely and locally uniformly in ζ, and hence $J(\zeta;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is analytic in ζ.

Moreover, given any bounded open subset of $\mathbb{C}^n$, one fixes a single contour $\mathcal{C}'=\mathcal{C}'_{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda}$ in this set and verifies the uniform convergence of the integral in the $\boldsymbol{\lambda}$ aspect. Then follows the analyticity of $J(\zeta;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$.

Lemma 2.4.7. *$J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ admits an analytic continuation $J(\zeta;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ from $\mathbb{R}_+$ onto $\mathbb{U}$. In particular, $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is a real analytic function of x on $\mathbb{R}_+$. Moreover, $J(\zeta;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ is an analytic function of $\boldsymbol{\lambda}$ on $\mathbb{C}^n$.*

The rank-one and rank-two cases.

Example 2.4.8. *According to Proposition* 1.2.4*, if $n=1$, then*

$$J(x;\pm,0) = e^{\pm ix}.$$

For $n=2$, from Proposition 1.2.7 *we have*

$$J(x;\pm,\pm,\lambda,-\lambda) = \pm\pi i e^{\pm\pi i\lambda}H^{(1,2)}_{2\lambda}(2x), \quad J(x;\pm,\mp,\lambda,-\lambda) = 2e^{\mp\pi i\lambda}K_{2\lambda}(2x),$$

where, for $\nu \in \mathbb{C}$, $H^{(1)}_\nu$, $H^{(2)}_\nu$ are the Hankel functions, and K_ν is the K-Bessel function (the modified Bessel function of the second kind).

2.4.2. The Hankel transforms $\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$, $\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{m})}$ and the Bessel kernels $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}$, $j_{(\boldsymbol{\mu},\boldsymbol{m})}$

Consider the ordered set $(\mathbb{C}, \preccurlyeq_2)$ and define $\lambda_{\alpha,\beta}$, B_α, $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ as in Definition 2.4.1 corresponding to $\boldsymbol{\lambda} \in \mathbb{C}^n$. We define the following subspace of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}_+)$

$$\mathscr{T}^{\boldsymbol{\lambda}}_{\mathrm{sis}}(\mathbb{R}_+) = \sum_{\alpha=1}^{A} \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}-1} x^{-\lambda_{\alpha,\beta}} (\log x)^j \mathscr{S}_0(\overline{\mathbb{R}}_+). \tag{2.4.8}$$

The definition of $\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

The following proposition provides the definition of the Hankel transform $\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$, which maps $\mathscr{T}^{-\boldsymbol{\mu}-\boldsymbol{\delta}}_{\mathrm{sis}}(\mathbb{R}_+)$ onto $\mathscr{T}^{\boldsymbol{\mu}-\boldsymbol{\delta}}_{\mathrm{sis}}(\mathbb{R}_+)$ bijectively.

Proposition 2.4.9. *Let $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Suppose $\upsilon \in \mathscr{T}^{-\boldsymbol{\mu}-\boldsymbol{\delta}}_{\mathrm{sis}}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathscr{T}^{\boldsymbol{\mu}-\boldsymbol{\delta}}_{\mathrm{sis}}(\mathbb{R}_+)$ satisfying the following identity,*

$$\mathcal{M}\Upsilon(s) = G_{(\boldsymbol{\mu},\boldsymbol{\delta})}(s)\mathcal{M}\upsilon(1-s). \tag{2.4.9}$$

We call Υ the Hankel transform of υ over $\mathbb{R}_+$ of index $(\boldsymbol{\mu}, \boldsymbol{\delta})$ and write $\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon = \Upsilon$. Furthermore, we have the Hankel inversion formula

$$\mathscr{h}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon = \Upsilon, \quad \mathscr{h}_{(-\boldsymbol{\mu},\boldsymbol{\delta})}\Upsilon = (-)^{|\boldsymbol{\delta}|}\upsilon. \tag{2.4.10}$$

Proof. Recall the definition of $G_{(\boldsymbol{\mu},\boldsymbol{\delta})}$ given by (2.2.3, 2.2.4),

$$G_{(\boldsymbol{\mu},\boldsymbol{\delta})}(s) = i^{|\boldsymbol{\delta}|}\pi^{n\left(\frac{1}{2}-s\right)+|\boldsymbol{\mu}|} \frac{\prod_{\ell=1}^n \Gamma\left(\frac{1}{2}(s-\mu_\ell+\delta_\ell)\right)}{\prod_{\ell=1}^n \Gamma\left(\frac{1}{2}(1-s+\mu_\ell+\delta_\ell)\right)},$$

where $|\boldsymbol{\delta}| = \sum_\ell \delta_\ell \in \mathbb{N}$, with each δ_ℓ viewed as a number in the set $\{0, 1\} \subset \mathbb{N}$.

We write $\boldsymbol{\mu}^{\pm} = \pm\boldsymbol{\mu} - \boldsymbol{\delta}$. Since $\mu_\ell^+ + \mu_\ell^- = -2\delta_\ell \in \{0, -2\}$, the partition $\{L_\alpha\}_{\alpha=1}^A$ of $\{1, ..., n\}$ and B_α in Definition 2.4.1 are the same for both $\boldsymbol{\mu}^+$ and $\boldsymbol{\mu}^-$. Let $\mu_{\alpha,\beta}^{\pm}$, $M_{\alpha,\beta}^{\pm}$ and $N_{\alpha,\beta}^{\pm}$ be the notations in Definition 2.4.1 corresponding to $\boldsymbol{\mu}^{\pm}$. Then the Gamma quotient above may be rewritten as follows,

$$\frac{\prod_{\alpha=1}^A \prod_{\beta=1}^{B_\alpha} \Gamma\left(\frac{1}{2}\left(s - \mu_{\alpha,\beta}^+\right)\right)^{M_{\alpha,\beta}^+}}{\prod_{\alpha=1}^A \prod_{\beta=1}^{B_\alpha} \Gamma\left(\frac{1}{2}\left(1 - s - \mu_{\alpha,\beta}^-\right)\right)^{M_{\alpha,\beta}^-}}.$$

Thus, at each point $\mu \in \mu_{\alpha,1}^+ - 2\mathbb{N}$ the product in the numerator contributes to $G_{(\mu,\delta)}(s)$ a pole of pure order $N_{\alpha,\beta}^+$, with $\beta = \max\left\{\beta' : \mu \preccurlyeq_2 \mu_{\alpha,\beta'}^+\right\}$, whereas at each point $\mu \in -\mu_{\alpha,1}^- + 2\mathbb{N} + 1$ the denominator contributes a zero of order $N_{\alpha,\beta}^-$, with $\beta = \max\left\{\beta' : 1 - \mu \preccurlyeq_2 \mu_{\alpha,\beta'}^-\right\}$. Moreover, (2.2.15) in Lemma 2.2.2 implies that $G_{(\mu,\delta)}(s)$ is of uniform moderate growth on vertical strips.

On the other hand, according to Lemma 2.3.3, the Mellin transform $\mathcal{M}\upsilon$ lies in the space $\sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}^- - 1} \mathscr{N}_{\text{sis}}^{\mu_{\alpha,\beta}^-, j}$. In particular, the poles of $\mathcal{M}\upsilon(1-s)$ are dominated by the zeros contributed from the denominator of the Gamma quotient. Furthermore, $\mathcal{M}\upsilon(1-s)$ uniformly rapidly decays on vertical strips.

We conclude that the product $G_{(\mu,\delta)}(s)\mathcal{M}\upsilon(1-s)$ on the right hand side of (2.4.9) lies in the space $\sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}^+ - 1} \mathscr{N}_{\text{sis}}^{\mu_{\alpha,\beta}^+, j}$, and hence $\Upsilon \in \mathscr{T}_{\text{sis}}^{\boldsymbol{\mu}-\boldsymbol{\delta}}(\mathbb{R}_+)$, with another application of Lemma 2.3.3.

Finally, the Hankel inversion formula (2.4.10) is an immediate consequence of the functional relation (2.2.5) of gamma factors. Q.E.D.

The definition of $\mathcal{h}_{(\boldsymbol{\mu},\boldsymbol{m})}$

The following proposition provides the definition of the Hankel transform $\mathcal{h}_{(\boldsymbol{\mu},\boldsymbol{m})}$, which maps $\mathscr{T}_{\text{sis}}^{-2\boldsymbol{\mu}-\|\boldsymbol{m}\|}(\mathbb{R}_+)$ onto $\mathscr{T}_{\text{sis}}^{2\boldsymbol{\mu}-\|\boldsymbol{m}\|}(\mathbb{R}_+)$ bijectively.

Proposition 2.4.10. *Let $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$. Suppose $\upsilon \in \mathscr{T}_{\mathrm{sis}}^{-2\mu - \|\boldsymbol{m}\|}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathscr{T}_{\mathrm{sis}}^{2\mu - \|\boldsymbol{m}\|}(\mathbb{R}_+)$ satisfying the following identity,*

$$\mathcal{M}\Upsilon(2s) = G_{(\boldsymbol{\mu},\boldsymbol{m})}(s)\mathcal{M}\upsilon(2(1-s)). \tag{2.4.11}$$

We call Υ the Hankel transform of υ over $\mathbb{R}_+$ of index $(\boldsymbol{\mu}, \boldsymbol{m})$ and write $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon = \Upsilon$. Moreover, we have the Hankel inversion formula

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon = \Upsilon, \quad \mathcal{H}_{(-\boldsymbol{\mu},\boldsymbol{m})}\Upsilon = (-)^{|\boldsymbol{m}|}\upsilon. \tag{2.4.12}$$

Proof. We first rewrite (2.4.11) as follows,

$$\mathcal{M}\Upsilon(s) = G_{(\boldsymbol{\mu},\boldsymbol{m})}\left(\frac{s}{2}\right)\mathcal{M}\upsilon(2-s).$$

From (2.2.6, 2.2.7), we have

$$G_{(\boldsymbol{\mu},\boldsymbol{m})}\left(\frac{s}{2}\right) = i^{|\|\boldsymbol{m}\||}\pi^{n(1-s)+2|\boldsymbol{\mu}|}\frac{\prod_{\ell=1}^n \Gamma\left(\frac{1}{2}(s - 2\mu_\ell + |m_\ell|)\right)}{\prod_{\ell=1}^n \Gamma\left(\frac{1}{2}(2 - s + 2\mu_\ell + |m_\ell|)\right)},$$

where $|\|\boldsymbol{m}\|| = \sum_{\ell=1}^n |m_\ell|$ according to our notations. We can now proceed to apply the same arguments in the proof of Proposition 2.4.9. Here, one uses (2.2.16) and (2.2.8) instead of (2.2.15) and (2.2.5) respectively. Q.E.D.

The Bessel kernel $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

The definition of $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}$.

For $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$, we define the Bessel kernel $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}$,

$$j_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}_{(\boldsymbol{\mu},\boldsymbol{\delta})}} G_{(\boldsymbol{\mu},\boldsymbol{\delta})}(s)x^{-s}ds. \tag{2.4.13}$$

We call the integral in (2.4.13) a Barnes-Mellin type integral. It is clear that

$$j_{(\boldsymbol{\mu}-\mu\boldsymbol{e}^n,\boldsymbol{\delta})}(x) = x^{\mu}j_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x). \tag{2.4.14}$$

In view of (2.2.9), we have

$$j_{(\mu,\delta)}(x) = (2\pi)^{|\boldsymbol{\mu}|} \sum_{\boldsymbol{\varsigma}\in\{+,-\}^n} \boldsymbol{\varsigma}^{\boldsymbol{\delta}} J\big(2\pi x^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\mu}\big). \tag{2.4.15}$$

Regularity of $j_{(\mu,\delta)}$.

It follows from (2.4.15) and Lemma 2.4.7 that $j_{(\mu,\delta)}(x)$ admits an analytic continuation $j_{(\mu,\delta)}(\zeta)$, which is also analytic with respect to $\boldsymbol{\mu}$. Moreover, $j_{(\mu,\delta)}(\zeta)$ has the following Barnes type integral representation,

$$j_{(\mu,\delta)}(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{C}'_{\mu-\delta}} G_{(\mu,\delta)}(s)\zeta^{-s}ds, \quad \zeta \in \mathbb{U}. \tag{2.4.16}$$

To see the convergence, the following formula is required

$$G_\delta(s) = \begin{cases} \dfrac{\pi(2\pi)^{-s}}{\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}, & \text{if } \delta = 0, \\ \dfrac{\pi i(2\pi)^{-s}}{\cos\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}, & \text{if } \delta = 1. \end{cases} \tag{2.4.17}$$

The integral kernel of $\mathcal{h}_{(\mu,\delta)}$.

Suppose $\upsilon \in \mathscr{T}_{\mathrm{sis}}^{-\mu-\delta}(\mathbb{R}_+)$. In order to proceed in the same way as in §2.4.1, one needs to assume that $(\boldsymbol{\mu}, \boldsymbol{\delta})$ satisfies the condition

$$\min\{\Re\mathrm{e}\,\mu_\ell + \delta_\ell\} + 1 > \max\{\Re\mathrm{e}\,\mu_\ell - \delta_\ell\}. \tag{2.4.18}$$

Then,

$$\mathcal{h}_{(\mu,\delta)}\upsilon(x) = \int_0^\infty \upsilon(y) j_{(\mu,\delta)}(xy)dy. \tag{2.4.19}$$

Here, it is required for the convergence of the integral over dy that the contour $\mathcal{C}_{(\mu,\delta)}$ in (2.4.13) is chosen to lie in the left half-plane $\{s : \Re\mathrm{e}\, s < \min\{\Re\mathrm{e}\,\mu_\ell + \delta_\ell\} + 1\}$. According to Definition 2.4.2, this choice of $\mathcal{C}_{(\mu,\delta)}$ is permissible due to our assumption (2.4.18).

If one assumes $\upsilon \in \mathscr{S}(\mathbb{R}_+)$, then (2.4.19) remains valid without requiring the condition (2.4.18).

The rank-one and rank-two examples.

Example 2.4.11. *If $n = 1$, we have*

$$j_{(0,0)}(x) = 2\cos(2\pi x), \quad j_{(0,1)}(x) = 2i\sin(2\pi x). \tag{2.4.20}$$

If $n = 2$, we are particularly interested in the following Bessel kernel,

$$j_{\left(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0\right)}(x) = j_{\left(\frac{1}{2}m, -\frac{1}{2}m, \delta(m), 1\right)}(x) = 2\pi i^{m+1} J_m(4\pi\sqrt{x}), \tag{2.4.21}$$

with $m \in \mathbb{N}$. For $\nu \in \mathbb{C}$, J_ν is the J-Bessel function (the Bessel function of the first kind).

The Bessel kernel $j_{(\boldsymbol{\mu}, \boldsymbol{m})}$

The definition of $j_{(\boldsymbol{\mu}, \boldsymbol{m})}$.

For $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ define the Bessel kernel $j_{(\boldsymbol{\mu}, \boldsymbol{m})}$ by

$$j_{(\boldsymbol{\mu}, \boldsymbol{m})}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{(\boldsymbol{\mu}, \boldsymbol{m})}} G_{(\boldsymbol{\mu}, \boldsymbol{m})}(s) x^{-2s} ds. \tag{2.4.22}$$

The integral in (2.4.22) is called a Barnes-Mellin type integral. We have

$$j_{(\boldsymbol{\mu} - \mu \boldsymbol{e}^n, \boldsymbol{m})}(x) = x^{2\mu} j_{(\boldsymbol{\mu}, \boldsymbol{m})}(x). \tag{2.4.23}$$

In view of Lemma 2.2.1, if $(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbb{C}^{2n} \times (\mathbb{Z}/2\mathbb{Z})^{2n}$ is related to $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ via either (2.2.11) or (2.2.12), then

$$i^n j_{(\boldsymbol{\mu}, \boldsymbol{m})}(x) = j_{(\boldsymbol{\eta}, \boldsymbol{\delta})}\left(x^2\right). \tag{2.4.24}$$

Regularity of $j_{(\boldsymbol{\mu}, \boldsymbol{m})}$.

In view of (2.4.24), the regularity of $j_{(\boldsymbol{\mu}, \boldsymbol{m})}$ follows from that of $j_{(\boldsymbol{\eta}, \boldsymbol{\delta})}$. Alternatively, this may be seen from

$$j_{(\boldsymbol{\mu}, \boldsymbol{m})}(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}'_{\boldsymbol{\mu} - \frac{1}{2}\|\boldsymbol{m}\|}} G_{(\boldsymbol{\mu}, \boldsymbol{m})}(s) \zeta^{-2s} ds, \quad \zeta \in \mathbb{U}. \tag{2.4.25}$$

To see the convergence, the following formula is required

$$G_m(s) = \frac{\pi i^{|m|}(2\pi)^{1-2s}}{\sin\left(\pi\left(s+\frac{1}{2}|m|\right)\right)\Gamma\left(1-s-\frac{1}{2}|m|\right)\Gamma\left(1-s+\frac{1}{2}|m|\right)}. \tag{2.4.26}$$

The integral kernel of $\hbar_{(\boldsymbol{\mu},\boldsymbol{m})}$.

Suppose $\upsilon \in \mathscr{T}_{\mathrm{sis}}^{-2\boldsymbol{\mu}-\|\boldsymbol{m}\|}(\mathbb{R}_+)$. We assume that $(\boldsymbol{\mu},\boldsymbol{m})$ satisfies the following condition

$$\min\left\{\Re\mathrm{e}\,\mu_\ell + \tfrac{1}{2}|m_\ell|\right\} + 1 > \max\left\{\Re\mathrm{e}\,\mu_\ell - \tfrac{1}{2}|m_\ell|\right\}. \tag{2.4.27}$$

Then

$$\hbar_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon(x) = \int_0^\infty \upsilon(y) j_{(\boldsymbol{\mu},\boldsymbol{m})}(xy)\cdot 2y dy, \tag{2.4.28}$$

It is required for convergence that the integral contour $\mathcal{C}_{(\boldsymbol{\mu},\boldsymbol{m})}$ in (2.4.22) lies in the left half-plane $\left\{s : \Re\mathrm{e}\, s < \min\left\{\Re\mathrm{e}\,\mu_\ell + \frac{1}{2}|m_\ell|\right\} + 1\right\}$. This is guaranteed by (2.4.27).

Moreover, if one assumes $\upsilon \in \mathscr{S}(\mathbb{R}_+)$, then (2.4.28) holds true for any index $(\boldsymbol{\mu},\boldsymbol{m})$.

The rank-one case.

Example 2.4.12. *If $n = 1$, in view of* (2.4.21) *and* (2.4.24)*, we have for $m \in \mathbb{Z}$*

$$j_{(0,m)}(x) = 2\pi i^{|m|} J_{|m|}(4\pi x) = 2\pi i^m J_m(4\pi x). \tag{2.4.29}$$

where the second equality follows from the identity $J_{-m}(x) = (-)^m J_m(x)$.

Auxiliary bounds for $j_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}$.

Lemma 2.4.13. *Let $(\boldsymbol{\mu},\boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Put*

$$A = n\left(\max\{\Re\mathrm{e}\,\mu_\ell\} + \tfrac{1}{2}\max\{|m_\ell|\} - \tfrac{1}{2}\right) - \Re\mathrm{e}\,|\boldsymbol{\mu}| + \tfrac{1}{2}|\|\boldsymbol{m}\||,$$

$$B_+ = -2\min\{\Re\mathrm{e}\,\mu_\ell\} + \max\{|m_\ell|\} + \max\left\{\tfrac{1}{n} - \tfrac{1}{2}, 0\right\},$$

$$B_- = -2\max\{\Re\mathrm{e}\,\mu_\ell\} - \max\{|m_\ell|\}.$$

Fix $\epsilon > 0$. Denote by $\boldsymbol{e}^n$ the n-tuple $(1, ..., 1)$. We have the following estimate

$$j_{(\boldsymbol{\mu}, \boldsymbol{m}+m\boldsymbol{e}^n)}(x) \ll_{(\boldsymbol{\mu}, \boldsymbol{m}), \epsilon, n} \left(\frac{2\pi e x^{\frac{1}{n}}}{|m|+1}\right)^{n|m|} (|m|+1)^{A+n\epsilon} \max\left\{x^{B_+ + 2\epsilon}, x^{B_- - 2\epsilon}\right\}. \tag{2.4.30}$$

Proof. Let

$$\rho_m = \max\left\{\Re\mathrm{e}\, \mu_\ell - \tfrac{1}{2}|m_\ell + m|\right\},$$
$$\sigma_m = \min\left\{\tfrac{1}{2} + \tfrac{1}{n}\left(\Re\mathrm{e}\,|\boldsymbol{\mu}| - \tfrac{1}{2}\,|\|\boldsymbol{m} + m\boldsymbol{e}^n\||-1\right), \rho_m\right\}.$$

Choose the contour $\mathcal{C}_m = \mathcal{C}_{(\boldsymbol{\mu}, \boldsymbol{m}+m\boldsymbol{e}^n)}$ (see Definition 2.4.2) such that

- if $s \in \mathcal{C}_m$ and $\Im\mathrm{m}\, s$ is sufficiently large, then $\Re\mathrm{e}\, s = \sigma_m - \epsilon$, and
- $\mathcal{C}_m$ lies in the vertical strip $\mathbb{S}[\sigma_m - \epsilon, \rho_m + \epsilon]$.

We first assume that $|m|$ is large enough so that

$$n\left(\rho_m + \epsilon - \tfrac{1}{2}\right) - \Re\mathrm{e}\,|\boldsymbol{\mu}| - \tfrac{1}{2}\,|\|\boldsymbol{m} + m\boldsymbol{e}^n\|| < 0.$$

For the sake of brevity, we write $y = (2\pi)^n x$. We first bound $\left|j_{(\boldsymbol{\mu}, \boldsymbol{m}+m\boldsymbol{e}^n)}(x)\right|$ by

$$(2\pi)^{n+\Re\mathrm{e}\,|\boldsymbol{\mu}|} \int_{\mathcal{C}_m} y^{-2\Re\mathrm{e}\, s} \prod_{\ell=1}^{n} \left|\frac{\Gamma\left(s - \mu_\ell + \frac{1}{2}|m_\ell + m|\right)}{\Gamma\left(1 - s + \mu_\ell + \frac{1}{2}|m_\ell + m|\right)}\right| |ds|.$$

With the observations that for $s \in \mathcal{C}_m$

- $\Re\mathrm{e}\, s \in [\sigma_m - \epsilon, \rho_m + \epsilon]$,
- $\left|\Re\mathrm{e}\, s - \mu_\ell + \frac{1}{2}|m_\ell + m|\right| \ll_{(\boldsymbol{\mu}, \boldsymbol{m})} 1$,
- $\left|\left(1 - \Re\mathrm{e}\, s + \mu_\ell + \frac{1}{2}|m_\ell + m|\right) - |m|\right| \ll_{(\boldsymbol{\mu}, \boldsymbol{m})} 1$,

in conjunction with Stirling's formula (2.2.13), we have the following estimate

$$
\begin{aligned}
j_{(\boldsymbol{\mu}, \boldsymbol{m}+me^n)}(x) &\ll_{(\boldsymbol{\mu}, \boldsymbol{m}), n, \epsilon} \max\left\{y^{-2\sigma_m+2\epsilon}, y^{-2\rho_m-2\epsilon}\right\} \\
&\qquad \int_{\mathcal{C}_m} \frac{(|\mathfrak{Im}\, s|+1)^{n\left(\mathfrak{Re}\, s-\frac{1}{2}\right)-\mathfrak{Re}\,|\boldsymbol{\mu}|+\frac{1}{2}|\|\boldsymbol{m}+me^n\||}}{e^{-n|m|}\left(\sqrt{(\mathfrak{Im}\, s)^2+m^2}+1\right)^{n\left(\frac{1}{2}-\mathfrak{Re}\, s\right)+\mathfrak{Re}\,|\boldsymbol{\mu}|+\frac{1}{2}|\|\boldsymbol{m}+me^n\||}} |ds| \\
&\leqslant \max\left\{y^{-2\sigma_m+2\epsilon}, y^{-2\rho_m-2\epsilon}\right\} e^{n|m|}(|m|+1)^{n\left(\rho_m+\epsilon-\frac{1}{2}\right)-\mathfrak{Re}\,|\boldsymbol{\mu}|-\frac{1}{2}|\|\boldsymbol{m}+me^n\||} \\
&\qquad \int_{\mathcal{C}_m} (|\mathfrak{Im}\, s|+1)^{n\left(\mathfrak{Re}\, s-\frac{1}{2}\right)-\mathfrak{Re}\,|\boldsymbol{\mu}|+\frac{1}{2}|\|\boldsymbol{m}+me^n\||} |ds|.
\end{aligned}
$$

For $s \in \mathcal{C}_m$, we have $\mathfrak{Re}\, s = \sigma_m - \epsilon$ if $\mathfrak{Im}\, s$ is sufficiently large, and our choice of σ_m implies $n\left(\sigma_m - \epsilon - \frac{1}{2}\right) - \mathfrak{Re}\,|\boldsymbol{\mu}| + \frac{1}{2}|\|\boldsymbol{m}+me^n\|| \leqslant -1-n\epsilon$, then it follows that the above integral converges and is of size $O_{(\boldsymbol{\mu}, \boldsymbol{m}), \epsilon, n}(1)$.

Finally, note that both $-2\sigma_m + 2\epsilon$ and $-2\rho_m - 2\epsilon$ are close to $|m|$, whereas the exponent of $(|m|+1)$, that is $n\left(\rho_m + \epsilon - \frac{1}{2}\right) - \mathfrak{Re}\,|\boldsymbol{\mu}| - \frac{1}{2}|\|\boldsymbol{m}+me^n\||$, is close to $-n|m|$. Thus the following bounds yield (2.4.30),

$$
\begin{aligned}
&|m| + B_- \leqslant -2\rho_m \leqslant -2\sigma_m \leqslant |m| + B_+, \\
&n\left(\rho_m - \tfrac{1}{2}\right) - \mathfrak{Re}\,|\boldsymbol{\mu}| - \tfrac{1}{2}|\|\boldsymbol{m}+me^n\|| \leqslant -n|m| + A.
\end{aligned}
$$

When $|m|$ is small, we have the following estimate that also implies (2.4.30),

$$
j_{(\boldsymbol{\mu}, \boldsymbol{m}+me^n)}(x) \ll_{(\boldsymbol{\mu}, \boldsymbol{m}), \epsilon, n} \max\left\{y^{-2\sigma_m+2\epsilon}, y^{-2\rho_m-2\epsilon}\right\} e^{n|m|}.
$$

Q.E.D.

Using the formula (2.4.26) of $G_m(s)$ instead of (2.2.6) and the Barnes type integral representation (2.4.25) for $j_{(\boldsymbol{\mu}, \boldsymbol{m}+me^n)}(\zeta)$ instead of the Barnes-Mellin type integral representation (2.4.22) for $j_{(\boldsymbol{\mu}, \boldsymbol{m}+me^n)}(x)$, similar arguments in the proof of Lemma 2.4.13 imply the following lemma.

Lemma 2.4.14. *Let* $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ *and* $m \in \mathbb{Z}$. *Put*

$$
A = n\left(\max\{\mathfrak{Re}\,\mu_\ell\} + \tfrac{1}{2}\max\{|m_\ell|\} - \tfrac{1}{2}\right) - \mathfrak{Re}\,|\boldsymbol{\mu}| + \tfrac{1}{2}|\|\boldsymbol{m}\||,
$$

$$B = -2\max\{\Re\mathfrak{e}\,\mu_\ell\} - \max\{|m_\ell|\}, \quad C = 2\max\{|\Im\mathfrak{m}\,\mu_\ell|\}.$$

Fix $X > 0$ and $\epsilon > 0$. Then

$$j_{(\boldsymbol{\mu}, \boldsymbol{m}+m\boldsymbol{e}^n)}\left(xe^{i\omega}\right) \ll_{(\boldsymbol{\mu},\boldsymbol{m}),\,X,\,\epsilon,\,n} \left(\frac{2\pi e x^{\frac{1}{n}}}{|m|+1}\right)^{n|m|} (|m|+1)^{A+n\epsilon} x^{B+2\epsilon} e^{|\omega|(C+2\epsilon)}$$

for all $x < X$.

2.4.3. The Hankel transform $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$ and the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

The definition of $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

Consider the ordered set $(\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, \preccurlyeq)$ and define $(\mu_{\alpha,\beta}, \delta_{\alpha,\beta}) = (\mu,\delta)_{\alpha,\beta}$, B_α, $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ as in Definition 2.4.1 corresponding to $(\boldsymbol{\mu},\boldsymbol{\delta}) \in (\mathbb{C} \times \mathbb{Z}/2\mathbb{Z})^n$. We define the following subspaces of $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$,

$$\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),\delta}(\mathbb{R}^\times) = \sum_{\alpha=1}^{A}\sum_{\beta=1}^{B_\alpha}\sum_{j=0}^{N_{\alpha,\beta}-1} \mathrm{sgn}(x)^{\delta_{\alpha,\beta}}|x|^{-\mu_{\alpha,\beta}}(\log|x|)^j \mathscr{S}_{\delta_{\alpha,\beta}+\delta}(\mathbb{R}). \tag{2.4.31}$$

$$\begin{aligned} \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times) &= \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),0}(\mathbb{R}^\times) \oplus \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),1}(\mathbb{R}^\times) \\ &= \sum_{\alpha=1}^{A}\sum_{\beta=1}^{B_\alpha}\sum_{j=0}^{N_{\alpha,\beta}-1} \mathrm{sgn}(x)^{\delta_{\alpha,\beta}}|x|^{-\mu_{\alpha,\beta}}(\log|x|)^j \mathscr{S}(\mathbb{R}). \end{aligned} \tag{2.4.32}$$

From the definition (2.4.8) of $\mathscr{T}_{\mathrm{sis}}^{\boldsymbol{\lambda}}(\mathbb{R}_+)$, together with $\mathscr{S}_\delta(\mathbb{R}) = \mathrm{sgn}(x)^\delta \mathscr{S}_\delta(\overline{\mathbb{R}}_+)$ and $\mathscr{S}_\delta(\overline{\mathbb{R}}_+) = x^\delta \mathscr{S}_0(\overline{\mathbb{R}}_+)$, we have

$$\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),\delta}(\mathbb{R}^\times) = \mathrm{sgn}(x)^\delta \mathscr{T}_{\mathrm{sis}}^{\boldsymbol{\mu}-(\boldsymbol{\delta}+\delta\boldsymbol{e}^n)}(\mathbb{R}_+). \tag{2.4.33}$$

The following theorem gives the definition of the Hankel transform $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$, which maps $\mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$ onto $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$ bijectively.

Theorem 2.4.15. *Let $(\boldsymbol{\mu},\boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Suppose $\upsilon \in \mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$. Then there exists a unique function $\Upsilon \in \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$ satisfying the following two identities,*

$$\mathcal{M}_\delta \Upsilon(s) = G_{(\boldsymbol{\mu},\boldsymbol{\delta}+\delta\boldsymbol{e}^n)}(s)\mathcal{M}_\delta \upsilon(1-s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}. \tag{2.4.34}$$

We call Υ the Hankel transform of υ over $\mathbb{R}^\times$ of index $(\boldsymbol{\mu}, \boldsymbol{\delta})$ and write $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon = \Upsilon$. Moreover, we have the Hankel inversion formula

(2.4.35) $$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon(x) = \Upsilon(x), \quad \mathcal{H}_{(-\boldsymbol{\mu},\boldsymbol{\delta})}\Upsilon(x) = (-)^{|\boldsymbol{\delta}|}\upsilon((-)^n x).$$

Proof. Recall that

$$\mathcal{M}_\delta \upsilon(s) = 2\mathcal{M}\upsilon_\delta(s).$$

In view of (2.4.33), one has $\upsilon_\delta \in \mathscr{T}_{\mathrm{sis}}^{-\boldsymbol{\mu}-(\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(\mathbb{R}_+)$. Applying Proposition 2.4.9, there is a unique function $\Upsilon_\delta \in \mathscr{T}_{\mathrm{sis}}^{\boldsymbol{\mu}-(\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(\mathbb{R}_+)$ satisfying

$$\mathcal{M}\Upsilon_\delta(s) = G_{(\boldsymbol{\mu},\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(s)\mathcal{M}\upsilon_\delta(1-s).$$

According to (2.4.33), $\Upsilon(x) = \Upsilon_0(|x|) + \mathrm{sgn}(x)\Upsilon_1(|x|)$ lies in $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),0}(\mathbb{R}^\times) \oplus \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta}),1}(\mathbb{R}^\times) = \mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{\delta})}(\mathbb{R}^\times)$. Clearly, Υ satisfies (2.4.34). Moreover, (2.4.35) follows immediately from (2.4.10) in Proposition 2.4.9. Q.E.D.

Corollary 2.4.16. *Let $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$. Suppose $\varphi \in \mathscr{T}_{\mathrm{sis}}^{-\boldsymbol{\mu}-(\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(\mathbb{R}_+)$ and $\upsilon(x) = \mathrm{sgn}(x)^\delta \varphi(|x|)$. Then*

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon(\pm x) = (\pm)^\delta \mathcal{h}_{(\boldsymbol{\mu},\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}\varphi(x), \quad x \in \mathbb{R}_+.$$

The Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

Let $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. We define

(2.4.36) $$J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(\pm x) = \frac{1}{2}\sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} (\pm)^\delta j_{(\boldsymbol{\mu},\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(x), \quad x \in \mathbb{R}_+,$$

or equivalently,

(2.4.37) $$J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x) = \frac{1}{2}\sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \mathrm{sgn}(x)^\delta j_{(\boldsymbol{\mu},\boldsymbol{\delta}+\delta \boldsymbol{e}^n)}(|x|), \quad x \in \mathbb{R}^\times.$$

Some properties of $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}$ are summarized as below.

Proposition 2.4.17. *Let* $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$.

(1). *Let* $(\mu, \delta) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$. *We have*

$$J_{(\boldsymbol{\mu}-\mu \boldsymbol{e}^n, \boldsymbol{\delta}-\delta \boldsymbol{e}^n)}(x) = \operatorname{sgn}(x)^\delta |x|^\mu J_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(x).$$

(2). $J_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(x)$ *is a real analytic function of* x *on* $\mathbb{R}^\times$ *as well as an analytic function of* $\boldsymbol{\mu}$ *on* $\mathbb{C}^n$.

(3). *Assume that* $\boldsymbol{\mu}$ *satisfies the condition*

$$\min\{\Re\mathrm{e}\, \mu_\ell\} + 1 > \max\{\Re\mathrm{e}\, \mu_\ell\}. \tag{2.4.38}$$

Then for $\upsilon \in \mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu}, \boldsymbol{\delta})}(\mathbb{R}^\times)$

$$\mathcal{H}_{(\boldsymbol{\mu}, \boldsymbol{\delta})} \upsilon(x) = \int_{\mathbb{R}^\times} \upsilon(y) J_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(xy) dy. \tag{2.4.39}$$

Moreover, if $\upsilon \in \mathscr{S}(\mathbb{R}^\times)$, *then* (2.4.39) *remains true for any index* $\boldsymbol{\mu} \in \mathbb{C}^n$.

Example 2.4.18. *For* $n = 1$, *we have*

$$J_{(0,0)}(x) = e(x).$$

For $n = 2$, (2.4.15) *and* (2.4.36) *yield*

$$J_{(\mu, -\mu, \delta, 0)}(\pm x) = J\left(2\pi\sqrt{x}; +, \pm, \mu, -\mu\right) + (-)^\delta J\left(2\pi\sqrt{x}; -, \mp, \mu, -\mu\right)$$

for $x \in \mathbb{R}_+$, $\mu \in \mathbb{C}$ *and* $\delta \in \mathbb{Z}/2\mathbb{Z}$. *In view of Example* 2.4.8, *for* $x \in \mathbb{R}_+$, *we have*

$$J_{(\mu, -\mu, \delta, 0)}(x) = \begin{cases} -\dfrac{\pi}{\sin(\pi\mu)}\left(J_{2\mu}(4\pi\sqrt{x}) - J_{-2\mu}(4\pi\sqrt{x})\right), & \text{if } \delta = 0, \\ \dfrac{\pi i}{\cos(\pi\mu)}\left(J_{2\mu}(4\pi\sqrt{x}) + J_{-2\mu}(4\pi\sqrt{x})\right), & \text{if } \delta = 1, \end{cases}$$

where the right hand side is replaced by its limit if $2\mu \in \delta + 2\mathbb{Z}$, *and*

$$J_{(\mu, -\mu, \delta, 0)}(-x) = \begin{cases} 4\cos(\pi\mu) K_{2\mu}(4\pi\sqrt{x}), & \text{if } \delta = 0, \\ -4i\sin(\pi\mu) K_{2\mu}(4\pi\sqrt{x}), & \text{if } \delta = 1. \end{cases}$$

Observe that for $m \in \mathbb{N}$

$$J_{\left(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0\right)}(x) = 2\pi i^{m+1} J_m(4\pi\sqrt{x}), \quad J_{\left(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0\right)}(-x) = 0.$$

2.4.4. The Hankel transform $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$ and the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{m})}$

The definition of $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$

Consider now the ordered set $(\mathbb{C}\times\mathbb{Z},\preccurlyeq)$ and define $(2\mu_{\alpha,\beta}, m_{\alpha,\beta}) = (2\mu, m)_{\alpha,\beta}$, B_α, $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ as in Definition 2.4.1 corresponding to $(2\boldsymbol{\mu},\boldsymbol{m})\in(\mathbb{C}\times\mathbb{Z})^n$. We define the following subspace of $\mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$,

$$\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times) = \sum_{\alpha=1}^{A}\sum_{\beta=1}^{B_\alpha}\sum_{j=0}^{N_{\alpha,\beta}-1} [z]^{-m_{\alpha,\beta}}\|z\|^{-\mu_{\alpha,\beta}}(\log|z|)^j\mathscr{S}(\mathbb{C}). \tag{2.4.40}$$

The projection via the m-th Fourier coefficient maps $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times)$ onto the space

$$\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m}),m}(\mathbb{C}^\times) = \sum_{\alpha=1}^{A}\sum_{\beta=1}^{B_\alpha}\sum_{j=0}^{N_{\alpha,\beta}-1} [z]^{-m_{\alpha,\beta}}\|z\|^{-\mu_{\alpha,\beta}}(\log|z|)^j\mathscr{S}_{m_{\alpha,\beta}+m}(\mathbb{C}). \tag{2.4.41}$$

From the definition (2.4.8) of $\mathscr{T}_{\mathrm{sis}}^{\boldsymbol{\lambda}}(\mathbb{R}_+)$, along with $\mathscr{S}_m(\mathbb{C}) = [z]^m\mathscr{S}_m(\overline{\mathbb{R}}_+)$ and $\mathscr{S}_m(\overline{\mathbb{R}}_+) = x^{|m|}\mathscr{S}_0(\overline{\mathbb{R}}_+)$, we have

$$\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m}),m}(\mathbb{C}^\times) = [z]^m\mathscr{T}_{\mathrm{sis}}^{2\boldsymbol{\mu}-\|\boldsymbol{m}+m\boldsymbol{e}^n\|}(\mathbb{R}_+). \tag{2.4.42}$$

The following theorem gives the definition of the Hankel transform $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$, which maps $\mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu},-\boldsymbol{m})}(\mathbb{C}^\times)$ onto $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times)$ bijectively.

Theorem 2.4.19. *Let $(\boldsymbol{\mu},\boldsymbol{m})\in\mathbb{C}^n\times\mathbb{Z}^n$. Suppose $\upsilon\in\mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu},-\boldsymbol{m})}(\mathbb{C}^\times)$. Then there exists a unique function $\Upsilon\in\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times)$ satisfying the following sequence of identities,*

$$\mathcal{M}_{-m}\Upsilon(2s) = G_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(s)\mathcal{M}_m\upsilon(2(1-s)),\quad m\in\mathbb{Z}. \tag{2.4.43}$$

We call Υ the Hankel transform of υ over $\mathbb{C}^\times$ of index $(\boldsymbol{\mu},\boldsymbol{m})$ and write $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon = \Upsilon$. Moreover, we have the Hankel inversion formula

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon(z) = \Upsilon(z),\quad \mathcal{H}_{(-\boldsymbol{\mu},-\boldsymbol{m})}\Upsilon(z) = (-)^{|\boldsymbol{m}|}\upsilon((-)^n z). \tag{2.4.44}$$

Proof. Recall that

$$\mathcal{M}_m \upsilon(s) = 4\pi \mathcal{M} \upsilon_{-m}(s).$$

In view of (2.4.42), we have $\upsilon_{-m} \in \mathscr{T}_{\mathrm{sis}}^{-2\boldsymbol{\mu}-\|\boldsymbol{m}+m\boldsymbol{e}^n\|}(\mathbb{R}_+)$. Applying Proposition 2.4.10, we infer that there is a unique function $\Upsilon_m \in \mathscr{T}_{\mathrm{sis}}^{2\boldsymbol{\mu}-\|\boldsymbol{m}+m\boldsymbol{e}^n\|}(\mathbb{R}_+)$ satisfying

$$\mathcal{M}\Upsilon_m(2s) = G_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(s)\mathcal{M}\upsilon_{-m}(2(1-s)).$$

According to Lemma 2.3.8, in order to show that the Fourier series $\Upsilon\left(xe^{i\phi}\right) = \sum \Upsilon_m(x)e^{im\phi}$ lies in $\mathscr{S}_{\mathrm{sis}}^{(\boldsymbol{\mu},\boldsymbol{m})}(\mathbb{C}^\times)$, it suffices to verify that $G_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(s)\mathcal{M}\upsilon_{-m}(2(1-s))$ rapidly decays with respect to m, uniformly on vertical strips. This however follows from the uniform rapid decay of $\mathcal{M}\upsilon_{-m}(2(1-s))$ along with the uniform moderate growth of $G_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(s)$ ((2.2.16) in Lemma 2.2.2) in the m aspect on vertical strips.

Finally, (2.4.12) in Proposition 2.4.10 implies (2.4.44). Q.E.D.

Corollary 2.4.20. *Let* $(\boldsymbol{\mu},\boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ *and* $m \in \mathbb{Z}$. *Suppose* $\varphi \in \mathscr{T}_{\mathrm{sis}}^{-2\boldsymbol{\mu}-\|\boldsymbol{m}+m\boldsymbol{e}^n\|}(\mathbb{R}_+)$ *and* $\upsilon(z) = [z]^{-m}\varphi(|z|)$. *Then*

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon\left(xe^{i\phi}\right) = e^{im\phi}\mathcal{h}_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}\varphi(x), \quad x \in \mathbb{R}_+, \phi \in \mathbb{R}/2\pi\mathbb{Z}.$$

The Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{m})}$

For $(\boldsymbol{\mu},\boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$, we define

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} j_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(x)e^{im\phi}, \tag{2.4.45}$$

or equivalently,

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}(z) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} j_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(|z|)[z]^m. \tag{2.4.46}$$

Lemma 2.4.13 secures the absolute convergence of this series.

Proposition 2.4.21. *Let* $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$.

(1). *Let* $(\mu, m) \in \mathbb{C} \times \mathbb{Z}$. *We have*

$$J_{(\boldsymbol{\mu}-\mu \boldsymbol{e}^n, \boldsymbol{m}-m\boldsymbol{e}^n)}(z) = [z]^m \|z\|^{\mu} J_{(\boldsymbol{\mu},\boldsymbol{m})}(z).$$

(2). $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ *is a real analytic function of* z *on* $\mathbb{C}^\times$ *as well as an analytic function of* $\boldsymbol{\mu}$ *on* $\mathbb{C}^n$.

(3). *Assume that* $\boldsymbol{\mu}$ *satisfies the following condition*

$$\min\{\Re\mathrm{e}\, \mu_\ell\} + 1 > \max\{\Re\mathrm{e}\, \mu_\ell\}. \tag{2.4.47}$$

Suppose $\upsilon \in \mathscr{S}_{\mathrm{sis}}^{(-\boldsymbol{\mu},-\boldsymbol{m})}(\mathbb{C}^\times)$. *Then*

$$\Upsilon\left(xe^{i\phi}\right) = \int_0^\infty \int_0^{2\pi} \upsilon\left(ye^{i\theta}\right) J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi} ye^{i\theta}\right) \cdot 2y d\theta dy, \tag{2.4.48}$$

or equivalently,

$$\Upsilon(z) = \int_{\mathbb{C}^\times} \upsilon(u) J_{(\boldsymbol{\mu},\boldsymbol{m})}(zu) du. \tag{2.4.49}$$

Moreover, (2.4.48) *and* (2.4.49) *still hold true for any index* $\boldsymbol{\mu} \in \mathbb{C}$ *if* $\upsilon \in \mathscr{S}(\mathbb{C}^\times)$.

Proof. (1). This is clear.

(2). In (2.4.45), with abuse of notation, we view x and ϕ as complex variables on $\mathbb{U}$ and $\mathbb{C}/2\pi\mathbb{Z}$ respectively, $j_{(\boldsymbol{\mu},\boldsymbol{m}+m\boldsymbol{e}^n)}(x)$ and $e^{im\phi}$ as analytic functions. Then Lemma 2.4.14 implies that the series in (2.4.45) is absolutely convergent, locally uniformly with respect to both x and ϕ, and therefore $J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right)$ is an analytic function of x and ϕ. In particular, $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ is a *real analytic* function of z on $\mathbb{C}^\times$.

Moreover, in Lemma 2.4.13, we may allow $\boldsymbol{\mu}$ to vary in an ϵ-ball in $\mathbb{C}^n$ and choose the implied constant in the estimate to be uniformly bounded with respect to $\boldsymbol{\mu}$. This implies that the series in (2.4.45) is convergent locally uniformly in the $\boldsymbol{\mu}$ aspect. Therefore, $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ is an analytic function of $\boldsymbol{\mu}$ on $\mathbb{C}^n$.

(3). It follows from (2.4.42) that $\upsilon_{-m} \in \mathscr{T}_{\mathrm{sis}}^{-2\boldsymbol{\mu}-\|\boldsymbol{m}+me^n\|}(\mathbb{R}_+)$. Moreover, one observes that $(\boldsymbol{\mu}, \boldsymbol{m}+me^n)$ satisfies the condition (2.4.27) due to (2.4.47). Therefore, in conjunction with Proposition 2.4.10, (2.4.28) implies

$$\Upsilon_m(x) = 2\int_0^\infty \upsilon_{-m}(y)\, j_{(\boldsymbol{\mu},\boldsymbol{m}+me^n)}(xy)\, y dy.$$

Hence

$$\Upsilon\left(xe^{i\phi}\right) = \sum_{m\in\mathbb{Z}} \Upsilon_m(x) e^{im\phi} = \sum_{m\in\mathbb{Z}} \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \upsilon\left(ye^{i\theta}\right) j_{(\boldsymbol{\mu},\boldsymbol{m}+me^n)}(xy)\, e^{im(\phi+\theta)} y d\theta dy.$$

The estimate of $j_{(\boldsymbol{\mu},\boldsymbol{m}+me^n)}$ in Lemma 2.4.13 implies that the above series of integrals converges absolutely. On interchanging the order of summation and integration, one obtains (2.4.48) in view of the definition (2.4.45) of $J_{(\boldsymbol{\mu},\boldsymbol{m})}$.

Note that in the case $\upsilon \in \mathscr{S}(\mathbb{C}^\times)$, one has $\upsilon_{-m} \in \mathscr{S}(\mathbb{R}_+)$, and therefore (2.4.28) can be applied unconditionally. Q.E.D.

Example 2.4.22. *Let $n = 1$. From* (2.4.29)*, we have*

$$j_{(0,m)}(x) = \begin{cases} (-)^d 2\pi J_{2d}(4\pi x), & \textit{if } |m| = 2d, \\ (-)^d 2\pi i J_{2d+1}(4\pi x), & \textit{if } |m| = 2d+1. \end{cases}$$

The following expansions ([Wat, 2.22 (3, 4)])

$$\begin{aligned} \cos(x\cos\phi) &= J_0(x) + 2\sum_{d=1}^\infty (-)^d J_{2d}(x)\cos(2d\phi), \\ \sin(x\cos\phi) &= \qquad\quad\; 2\sum_{d=0}^\infty (-)^d J_{2d+1}(x)\cos((2d+1)\phi), \end{aligned}$$

imply

$$J_{(0,0)}\left(xe^{i\phi}\right) = \cos(4\pi x\cos\phi) + i\sin(4\pi x\cos\phi) = e(2x\cos\phi),$$

or equivalently,

$$J_{(0,0)}(z) = e(z+\overline{z}).$$

We remark that the two expansions [Wat, 2.22 (3, 4)] *can be incorporated into*

$$e^{ix\cos\phi} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\phi}.$$

2.4.5. Concluding remarks

Connection formulae

From the various connection formulae (2.4.15, 2.4.24, 2.4.45, 2.4.46) which have been derived so far, one can connect the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ to the Bessel functions $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ of doubled rank $2n$. However, in contrast to the expression of $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(\pm x)$ by a *finite* sum of $J\big(2\pi x^{\frac{1}{n}};\boldsymbol{\varsigma},\boldsymbol{\mu}\big)$ (see (2.4.15, 2.4.36, 2.4.37)), which enables us to reduce the study of $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ to that of $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ given in Chapter 1, these connection formulae yield an expression of $J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right)$ in terms of an *infinite* series involving the Bessel functions $J\big(2\pi x^{\frac{1}{n}};\boldsymbol{\varsigma},\boldsymbol{\lambda}\big)$ of rank $2n$, so a similar reduction for $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ does not exist from this approach.

In §2.7, we shall prove two alternative connection formulae that relate $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ to the two kinds of Bessel functions of rank n and positive sign. These kinds of Bessel functions arise in §1.7 as solutions of the Bessel equation of positive sign.

Asymptotics of Bessel kernels

Using the connection formulae between the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ and Bessel functions $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$ along with the asymptotics of the latter, the asymptotic of $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ is readily established in Theorem 1.5.13 and 1.9.3. With the help of the second connection formula for $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ in §2.7.2, we shall present in §2.8 the asymptotic of $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ as an application of the asymptotic expansions of Bessel functions of the second kind (Theorem 1.7.24).

Normalizations of indices

Usually, it is convenient to normalize the indices in $J(x;\boldsymbol{\varsigma},\boldsymbol{\lambda})$, $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$, $j_{(\boldsymbol{\mu},\boldsymbol{m})}(x)$, $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ and $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ so that $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbb{L}^{n-1}$. Furthermore, without loss of generality, the assumptions $\delta_n = 0$ and $m_n = 0$ may also be imposed for $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ and $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ respectively. These normalizations are justified by Lemma 2.4.6, (2.4.14), (2.4.23), Proposition 2.4.17 (1) and 2.4.21 (1).

2.5. Fourier type integral transforms

In this section, we shall introduce an alternative perspective of Hankel transforms. We shall first show how to construct Hankel transforms from the Fourier transform and Miller-Schmid transforms. From this, we shall express the Hankel transforms $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$ and $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$ in terms of certain Fourier type integral transforms, assuming that the components of $\mathfrak{Re}\,\boldsymbol{\mu}$ are *strictly* decreasing.

2.5.1. The Fourier transform and rank-one Hankel transforms

For either $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, we have seen in Example 2.4.18 and 2.4.22 that $J_{(0,0)}$ is exactly the inverse Fourier kernel, namely

$$J_{(0,0)}(x) = e(\Lambda(x)), \quad x\in\mathbb{F},$$

with $\Lambda(x)$ defined by (2.2.37). Therefore, in view of Proposition 2.4.17 (3) and 2.4.21 (3), $\mathcal{H}_{(0,0)}$ is precisely the inverse Fourier transform over the Schwartz space $\mathscr{S}_{\mathrm{sis}}^{(0,0)}(\mathbb{F}^\times) = \mathscr{S}(\mathbb{F})$. The following lemma is a consequence of Theorem 2.4.15 and 2.4.19.

Lemma 2.5.1. *Let $\upsilon \in \mathscr{S}(\mathbb{F})$. If $\mathbb{F} = \mathbb{R}$, then the Fourier transform $\widehat{\upsilon}$ of υ can be determined by the following two identities*

$$\mathcal{M}_\delta \widehat{\upsilon}(s) = (-)^\delta G_\delta(s) \mathcal{M}_\delta \upsilon(1-s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}.$$

If $\mathbb{F} = \mathbb{C}$, then the Fourier transform $\widehat{\upsilon}$ of υ can be determined by the following sequence of identities

$$\mathcal{M}_{-m} \widehat{\upsilon}(2s) = (-)^m G_m(s) \mathcal{M}_m \upsilon(2(1-s)), \quad m \in \mathbb{Z}.$$

It is convenient for our purpose to introduce the renormalize rank-one Hankel transforms $\mathcal{S}_{(\mu,\epsilon)}$ and $\mathcal{S}_{(\mu,k)}$ as follows.

Lemma 2.5.2. *Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ and $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.*

(1). *For $\upsilon(x) \in \mathrm{sgn}(x)^\epsilon |x|^\mu \mathscr{S}(\mathbb{R})$, define $\mathcal{S}_{(\mu,\epsilon)}\upsilon(x) = |x|^\mu \mathcal{H}_{(\mu,\epsilon)}\upsilon(x)$. Then*

$$\mathcal{M}_\delta \mathcal{S}_{(\mu,\epsilon)} \upsilon(s) = G_{\epsilon+\delta}(s) \mathcal{M}_\delta \upsilon(1-s-\mu), \quad \delta \in \mathbb{Z}/2\mathbb{Z}, \tag{2.5.1}$$

and $\mathcal{S}_{(\mu,\epsilon)}$ sends $\mathrm{sgn}(x)^\epsilon |x|^\mu \mathscr{S}(\mathbb{R})$ onto $\mathrm{sgn}(x)^\epsilon \mathscr{S}(\mathbb{R})$ bijectively. Furthermore,

$$\mathcal{S}_{(\mu,\epsilon)}\upsilon(x) = \mathrm{sgn}(x)^\epsilon \int_{\mathbb{R}^\times} \mathrm{sgn}(y)^\epsilon |y|^{-\mu} \upsilon(y) e(xy) dy = \mathrm{sgn}(x)^\epsilon \mathcal{F}\varphi(-x),$$

with $\varphi(x) = \mathrm{sgn}(x)^\epsilon |x|^{-\mu} \upsilon(x) \in \mathscr{S}(\mathbb{R})$.

(2). *For $\upsilon(z) \in [z]^k \|z\|^\mu \mathscr{S}(\mathbb{C})$, define $\mathcal{S}_{(\mu,k)}\upsilon(z) = \|z\|^\mu \mathcal{H}_{(\mu,k)}\upsilon(z)$. Then*

$$\mathcal{M}_{-m} \mathcal{S}_{(\mu,k)} \upsilon(2s) = G_{k+m}(s) \mathcal{M}_m \upsilon(2(1-s-\mu)), \quad m \in \mathbb{Z}. \tag{2.5.2}$$

and $\mathcal{S}_{(\mu,k)}$ sends $[z]^k \|z\|^\mu \mathscr{S}(\mathbb{C})$ onto $[z]^{-k} \mathscr{S}(\mathbb{C})$ bijectively. Furthermore,

$$\mathcal{S}_{(\mu,k)}\upsilon(z) = [z]^{-k} \int_{\mathbb{C}^\times} [u]^{-k} \|u\|^{-\mu} \upsilon(u) e(zu + \overline{zu}) du = [z]^{-k} \mathcal{F}\varphi(-z),$$

with $\varphi(z) = [z]^{-k} \|z\|^{-\mu} \upsilon(z) \in \mathscr{S}(\mathbb{C})$.

Lemma 2.5.3. *Let* $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ *and* $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.

(1). *Let* $\delta \in \mathbb{Z}/2\mathbb{Z}$. *Suppose that* $\varphi(x) \in x^{\mu}\mathscr{S}_{\delta+\epsilon}(\overline{\mathbb{R}}_+)$ *and* $\upsilon(x) = \mathrm{sgn}(x)^{\delta}\varphi(|x|)$. *Then*

$$\begin{aligned}\mathcal{S}_{(\mu,\epsilon)}\upsilon(\pm x) &= (\pm)^{\delta} \int_{\mathbb{R}_+} y^{-\mu}\varphi(y) j_{(0,\delta+\epsilon)}(xy) dy \\ &= \begin{cases} (\pm)^{\epsilon} 2 \displaystyle\int_{\mathbb{R}_+} y^{-\mu}\varphi(y)\cos(xy)dy, & \text{if } \delta = \epsilon, \\ (\pm)^{\epsilon+1} 2i \displaystyle\int_{\mathbb{R}_+} y^{-\mu}\varphi(y)\sin(xy)dy, & \text{if } \delta = \epsilon + 1. \end{cases}\end{aligned}$$

The transform $\mathcal{S}_{(\mu,\epsilon)}$ *is a bijective map from* $\mathrm{sgn}(x)^{\epsilon}|x|^{\mu}\mathscr{S}_{\delta}(\mathbb{R})$ *onto* $\mathrm{sgn}(x)^{\epsilon}\mathscr{S}_{\delta}(\mathbb{R})$.

(2). *Let* $m \in \mathbb{Z}$. *Suppose that* $\varphi(x) \in x^{2\mu}\mathscr{S}_{-m-k}(\overline{\mathbb{R}}_+)$ *and* $\upsilon(z) = [z]^{-m}\varphi(|z|)$. *Then*

$$\begin{aligned}\mathcal{S}_{(\mu,k)}\upsilon\left(xe^{i\phi}\right) &= 2e^{im\phi} \int_{\mathbb{R}_+} y^{1-2\mu}\varphi(y) j_{(0,m+k)}(xy)dy \\ &= 4\pi i^{m+k} e^{im\phi} \int_{\mathbb{R}_+} y^{1-2\mu}\varphi(y) J_{m+k}(4\pi xy)dy.\end{aligned}$$

The transform $\mathcal{S}_{(\mu,k)}$ *is a bijective map from* $[z]^{k}\|z\|^{\mu}\mathscr{S}_{m}(\mathbb{C})$ *onto* $[z]^{-k}\mathscr{S}_{-m}(\mathbb{C})$.

2.5.2. Miller-Schmid transforms

In [MS1, §6], certain transforms over $\mathbb{R}$, which play an important role in the proof of the Voronoĭ summation formula in their subsequent work [MS3, MS4], are introduced by Miller and Schmid. Here, we shall first recollect their construction of these transforms with slight modifications, and then define similar transforms over $\mathbb{C}$ in a parallel way.

The Miller-Schmid transform $\mathcal{T}_{(\mu,\epsilon)}$

Lemma 2.5.4. *Let* $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

(1). *For any* $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times})$ *there is a unique function* $\Upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{R}^{\times})$ *satisfying the following two identities,*

$$\mathcal{M}_{\delta}\Upsilon(s) = G_{\epsilon+\delta}(s)\mathcal{M}_{\delta}\upsilon(s+\mu), \quad \delta \in \mathbb{Z}/2\mathbb{Z}. \tag{2.5.3}$$

We write $\Upsilon = \mathcal{T}_{(\mu,\epsilon)} \upsilon$ *and call* $\mathcal{T}_{(\mu,\epsilon)}$ *the Miller-Schmid transform over* $\mathbb{R}$ *of index* (μ, ϵ).

(2). *Let* $\lambda \in \mathbb{C}$. *Suppose* $\upsilon(x) \in \operatorname{sgn}(x)^{\delta} |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R})$. *If* $\mathfrak{Re}\, \lambda < \mathfrak{Re}\, \mu - \frac{1}{2}$, *then*

$$\mathcal{T}_{(\mu,\epsilon)} \upsilon(x) = \operatorname{sgn}(x)^{\epsilon} \int_{\mathbb{R}^{\times}} \operatorname{sgn}(y)^{\epsilon} |y|^{-\mu} \upsilon\left(y^{-1}\right) e(xy) d^{\times} y = \operatorname{sgn}(x)^{\epsilon} \mathcal{F}\varphi(-x), \tag{2.5.4}$$

with $\varphi(x) = \operatorname{sgn}(x)^{\epsilon} |x|^{-\mu-1} \upsilon(x^{-1})$.

(3). *Suppose that* $\mathfrak{Re}\, \lambda < \mathfrak{Re}\, \mu$. *Then the integral in* (2.5.4) *is absolutely convergent and* (2.5.4) *remains valid for any* $\upsilon(x) \in \operatorname{sgn}(x)^{\delta} |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R})$.

(4). *Suppose that* $\mathfrak{Re}\, \mu > 0$. *Define the function space*

$$\mathscr{T}_{\mathrm{sis}}(\mathbb{R}^{\times}) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \sum_{\mathfrak{Re}\, \lambda \leqslant 0} \sum_{j \in \mathbb{N}} \operatorname{sgn}(x)^{\delta} |x|^{-\lambda} (\log |x|)^j \mathscr{S}(\mathbb{R}).$$

Then the transform $\mathcal{T}_{(\mu,\epsilon)}$ *sends* $\mathscr{T}_{\mathrm{sis}}(\mathbb{R}^{\times})$ *into itself. Moreover,* (2.5.4) *also holds true for any* $\upsilon \in \mathscr{T}_{\mathrm{sis}}(\mathbb{R}^{\times})$, *wherein the integral absolutely converges.*

Proof. Following the ideas in the proofs of Proposition 2.4.9 and Theorem 2.4.15, one may prove (1). Actually, the case here is much easier!

As for (2), we have

$$\begin{aligned} \mathcal{T}_{(\mu,\epsilon)} \upsilon(x) &= \frac{1}{4\pi i} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \int_{\mathbb{R}^{\times}} \upsilon(y) |y|^{\mu} \cdot \operatorname{sgn}(xy)^{\delta} \int_{\mathcal{C}_{(0,\delta+\epsilon)}} G_{\delta+\epsilon}(s) |y|^{s} |x|^{-s} ds d^{\times} y \\ &= \int_{\mathbb{R}^{\times}} \upsilon(y) |y|^{\mu} J_{(0,\epsilon)}\left(xy^{-1}\right) d^{\times} y, \end{aligned}$$

provided that the double integral is absolutely convergent. In order to guarantee the convergence of the integral over $d^{\times} y$, the integral contour $\mathcal{C}_{(0,\delta+\epsilon)}$ is required to lie in the right half-plane $\{s : \mathfrak{Re}\, s > \mathfrak{Re}\,(\lambda - \mu)\}$. In view of Definition 2.4.2, such a choice of $\mathcal{C}_{(0,\delta+\epsilon)}$ is permissible since $\mathfrak{Re}\,(\lambda - \mu) < -\frac{1}{2}$ according to our assumption. Finally, the change of variables from y to y^{-1}, along with the formula $J_{(0,\epsilon)}(x) = \operatorname{sgn}(x)^{\epsilon} e(x)$, yields (2.5.4).

For the case $\mathfrak{Re}\, \lambda < \mathfrak{Re}\, \mu$ in (3), the absolute convergence of the integral in (2.5.4) is obvious. The validity of (2.5.4) follows from the analyticity with respect to μ.

Observe that, under the isomorphism established by $\mathcal{M}_{\mathbb{R}}$ in Lemma 2.3.6, $\mathscr{T}_{\mathrm{sis}}(\mathbb{R}^\times)$ corresponds to the subspace of $\mathscr{M}_{\mathrm{sis}}^{\mathbb{R}}$ consisting of pairs of meromorphic functions (H_0, H_1) such that the poles of both H_0 and H_1 lie in the left half-plane $\{s : \mathfrak{Re}\, s \leqslant 0\}$ (see Lemma 2.3.7). Then the first assertion in (4) is clear, since the map that corresponds to $\mathcal{T}_{(\mu,\epsilon)}$ is given by $(H_0, H_1) \mapsto (G_\epsilon(s)H_0(s+\mu), G_{\epsilon+1}(s)H_1(s+\mu))$ and sends the subspace of $\mathscr{M}_{\mathrm{sis}}^{\mathbb{R}}$ described above into itself. The second assertion in (4) immediately follows from (3). Q.E.D.

Similar to Lemma 2.5.3 (1), we have the following lemma.

Lemma 2.5.5. *Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ be such that $\mathfrak{Re}\, \mu > 0$. For $\delta \in \mathbb{Z}/2\mathbb{Z}$ define $\mathscr{T}_{\mathrm{sis}}^{\delta}(\mathbb{R}^\times)$ to be the space of functions in $\mathscr{T}_{\mathrm{sis}}(\mathbb{R}^\times)$ satisfying the condition* (2.2.17). *For $\upsilon \in \mathscr{T}_{\mathrm{sis}}^{\delta}(\mathbb{R}^\times)$, we write $\upsilon(x) = \mathrm{sgn}(x)^\delta \varphi(|x|)$. Then*

$$\begin{aligned}\mathcal{T}_{(\mu,\epsilon)}\upsilon(\pm x) &= (\pm)^\delta \int_{\mathbb{R}_+} y^{-\mu}\varphi\left(y^{-1}\right) j_{(0,\delta+\epsilon)}(xy) d^\times y \\ &= \begin{cases} (\pm)^\delta 2 \displaystyle\int_{\mathbb{R}_+} y^{-\mu}\varphi\left(y^{-1}\right)\cos(xy) d^\times y, & \text{if } \delta = \epsilon, \\ (\pm)^\delta 2i \displaystyle\int_{\mathbb{R}_+} y^{-\mu}\varphi\left(y^{-1}\right)\sin(xy) d^\times y, & \text{if } \delta = \epsilon + 1. \end{cases}\end{aligned}$$

The transform $\mathcal{T}_{(\mu,\epsilon)}$ sends $\mathscr{T}_{\mathrm{sis}}^{\delta}(\mathbb{R}^\times)$ into itself.

The Miller-Schmid transform $\mathcal{T}_{(\mu,k)}$

In parallel to Lemma 2.5.4, the following lemma defines the Miller-Schmid transform $\mathcal{T}_{(\mu,k)}$ over $\mathbb{C}$ and gives its connection to the Fourier transform over $\mathbb{C}$.

Lemma 2.5.6. *Let $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.*

(1). *For any $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ there is a unique function $\Upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ satisfying the following sequence of identities,*

(2.5.5) $$\mathcal{M}_{-m}\Upsilon(2s) = G_{m+k}(s)\mathcal{M}_{-m}\upsilon(2(s+\mu)), \quad m \in \mathbb{Z}.$$

We write $\Upsilon = \mathfrak{T}_{(\mu,k)}\upsilon$ and call $\mathfrak{T}_{(\mu,k)}$ the Miller-Schmid transform over $\mathbb{C}$ of index (μ, k).

(2). *Let $\lambda \in \mathbb{C}$. If $\mathfrak{Re}\,\lambda < 2\,\mathfrak{Re}\,\mu$, then for any $\upsilon(z) \in [z]^m|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\mathbb{C})$ we have*

$$\mathfrak{T}_{(\mu,k)}\upsilon(z) = [z]^k \int_{\mathbb{C}^\times} [u]^k \|u\|^{-\mu} \upsilon\left(u^{-1}\right) e(zu + \overline{zu}) d^\times u = [z]^k \mathfrak{F}\varphi\left(-z\right), \tag{2.5.6}$$

with $\varphi(z) = [z]^k\|z\|^{-\mu-1}\upsilon\left(z^{-1}\right)$.

(3). *When $\mathfrak{Re}\,\lambda < 2\mathfrak{Re}\,\mu$, the integral in* (2.5.6) *is absolutely convergent for any $\upsilon(z) \in [z]^m|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\mathbb{C})$.*

(4). *Suppose that $\mathfrak{Re}\,\mu > 0$. Define the function space*

$$\mathscr{T}_{\mathrm{sis}}(\mathbb{C}^\times) = \sum_{m\in\mathbb{Z}} \sum_{\mathfrak{Re}\,\lambda \leqslant 0} \sum_{j\in\mathbb{N}} [z]^m|z|^{-\lambda}(\log|z|)^j\mathscr{S}(\mathbb{C}).$$

Then the transform $\mathfrak{T}_{(\mu,k)}$ sends $\mathscr{T}_{\mathrm{sis}}(\mathbb{C}^\times)$ into itself. Moreover, (2.5.4) *also holds true for any $\upsilon \in \mathscr{T}_{\mathrm{sis}}(\mathbb{C}^\times)$, wherein the integral absolutely converges.*

Proof. Following literally the same ideas in the proof of Lemma 2.5.4, one may show this lemma without any difficulty. We only remark that, via the isomorphism $\mathcal{M}_{\mathbb{C}}$ in Lemma 2.3.8, $\mathscr{T}_{\mathrm{sis}}(\mathbb{C}^\times)$ corresponds to the subspace of $\mathscr{M}^{\mathbb{C}}_{\mathrm{sis}}$ consisting of sequences $\{H_m\}$ such that the poles of each H_m lie in the left half-plane $\left\{s : \mathfrak{Re}\,s \leqslant \min\{M - |m|, 0\}\right\}$ for some $M \in \mathbb{N}$ (see Lemma 2.3.9). Q.E.D.

Lemma 2.5.7. *Let $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$ be such that $\mathfrak{Re}\,\mu > 0$. For $m \in \mathbb{Z}$ define $\mathscr{T}^m_{\mathrm{sis}}(\mathbb{C}^\times)$ to be the space of functions in $\mathscr{T}_{\mathrm{sis}}(\mathbb{C}^\times)$ satisfying the condition* (2.2.21). *For $\upsilon \in \mathscr{T}^m_{\mathrm{sis}}(\mathbb{C}^\times)$, we write $\upsilon(z) = [z]^m\varphi(|z|)$. Then*

$$\begin{aligned}\mathfrak{T}_{(\mu,k)}\upsilon\left(xe^{i\phi}\right) &= 2e^{im\phi}\int_{\mathbb{R}_+} y^{-2\mu}\varphi\left(y^{-1}\right) j_{(0,m+k)}(xy) d^\times y\\ &= 4\pi i^{m+k}e^{im\phi}\int_{\mathbb{R}_+} y^{-2\mu}\varphi\left(y^{-1}\right) J_{m+k}(4\pi xy) d^\times y.\end{aligned}$$

The transform $\mathfrak{T}_{(\mu,k)}$ sends $\mathscr{T}^m_{\mathrm{sis}}(\mathbb{C}^\times)$ into itself.

2.5.3. Fourier type integral transforms

In the following, we shall derive the Fourier type integral transform expressions for $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$ and $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$ from the Fourier transform (more precisely, the renormalized rank-one Hankel transforms) and the Miller-Schmid transforms.

The Fourier type transform expression for $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$

Let $(\boldsymbol{\mu},\boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Following [MS3, (6.51)], for $\upsilon(x) \in \operatorname{sgn}(x)^{\delta_n}|x|^{\mu_n}\mathscr{S}(\mathbb{R})$, we consider

$$(2.5.7) \qquad \Upsilon(x) = |x|^{-\mu_1}\mathcal{T}_{(\mu_1-\mu_2,\delta_1)} \circ \ldots \circ \mathcal{T}_{(\mu_{n-1}-\mu_n,\delta_{n-1})} \circ \mathcal{S}_{(\mu_n,\delta_n)}\upsilon(x).$$

According to Lemma 2.5.2 (1) and Lemma 2.5.4 (1), $\mathcal{S}_{(\mu_n,\delta_n)}\upsilon$ lies in $\operatorname{sgn}(x)^{\delta_n}\mathscr{S}(\mathbb{R})(\subset \mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times))$, whereas each Miller-Schmid transform sends $\mathscr{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ into itself. Thus, one can apply the Mellin transform $\mathcal{M}_\delta$ to both sides of (2.5.7). Using (2.5.1) and (2.5.3), some calculations show that the application of $\mathcal{M}_\delta$ converts (2.5.7) exactly into (2.4.34) which defines $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}$. Therefore, $\Upsilon = \mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon$.

Theorem 2.5.8. [MS4, (1.3)]. *Let $(\boldsymbol{\mu},\boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ be such that $\mathfrak{Re}\,\mu_1 > \ldots > \mathfrak{Re}\,\mu_{n-1} > \mathfrak{Re}\,\mu_n$. Suppose $\upsilon(x) \in \operatorname{sgn}(x)^{\delta_n}|x|^{\mu_n}\mathscr{S}(\mathbb{R})$. Then*

$$(2.5.8) \qquad \mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon(x) = \frac{1}{|x|}\int_{\mathbb{R}^{\times n}} \upsilon\left(\frac{x_1 \ldots x_n}{x}\right)\left(\prod_{\ell=1}^{n} \operatorname{sgn}(x_\ell)^{\delta_\ell}|x_\ell|^{-\mu_\ell} e(x_\ell)\right) dx_n \ldots dx_1,$$

where the integral converges when performed as iterated integral in the indicated order $dx_n dx_{n-1} \ldots dx_1$, starting from dx_n, then dx_{n-1}, ..., and finally dx_1.

Proof. We first observe that $\mathcal{S}_{(\mu_n,\delta_n)}\upsilon \in \operatorname{sgn}(x)^{\delta_n}\mathscr{S}(\mathbb{R}) \subset \mathscr{T}_{\mathrm{sis}}(\mathbb{R}^\times)$. For each $\ell = 1, \ldots, n-1$, since $\mathfrak{Re}\,(\mu_\ell - \mu_{\ell+1}) > 0$, Lemma 2.5.4 (4) implies that the transform $\mathcal{T}_{(\mu_\ell-\mu_{\ell+1},\delta_\ell)}$ sends the space $\mathscr{T}_{\mathrm{sis}}(\mathbb{R}^\times)$ into itself. According to Lemma 2.5.2 (1) and Lemma 2.5.4 (3), $\mathcal{S}_{(\mu_n,\delta_n)}$ and all the $\mathcal{T}_{(\mu_\ell-\mu_{\ell+1},\delta_\ell)}$ in (2.5.7) may be expressed as integral transforms, which are absolutely convergent. From these, the right hand side of (2.5.7) turns into the integral,

$$\int_{\mathbb{R}^{\times n}} \operatorname{sgn}(x)^{\delta_1}|x|^{-\mu_1} e\left(xy_1\right) \left(\prod_{\ell=1}^{n-1} \operatorname{sgn}(y_\ell)^{\delta_{\ell+1}+\delta_\ell}|y_\ell|^{\mu_{\ell+1}-\mu_\ell-1} e\left(y_\ell^{-1}y_{\ell+1}\right)\right) \operatorname{sgn}(y_n)^{\delta_n}|y_n|^{-\mu_n}\upsilon(y_n)dy_n...dy_1,$$

which converges as iterated integral. Our proof is completed upon making the change of variables $x_1 = xy_1$, $x_{\ell+1} = y_\ell^{-1}y_{\ell+1}$, $\ell = 1, ..., n-1$. Q.E.D.

We have the following corollary to Theorem 2.5.8, which can also be seen from Lemma 2.5.3 (1) and Lemma 2.5.5.

Corollary 2.5.9. *Let* $(\boldsymbol{\mu},\boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ *and* $\delta \in \mathbb{Z}/2\mathbb{Z}$. *Assume that* $\Re\mathfrak{e}\,\mu_1 > ... > \Re\mathfrak{e}\,\mu_{n-1} > \Re\mathfrak{e}\,\mu_n$. *Let* $\varphi(x) \in x^{\mu_n}\mathscr{S}_{\delta+\delta_n}(\overline{\mathbb{R}}_+)$ *and* $\upsilon(x) = \operatorname{sgn}(x)^\delta\varphi(|x|)$. *Then*

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon\left(\pm x\right) = \frac{(\pm)^\delta}{x}\int_{\mathbb{R}_+^n} \varphi\left(\frac{x_1...x_n}{x}\right)\left(\prod_{\ell=1}^{n} x_\ell^{-\mu_\ell} j_{(0,\delta_\ell+\delta)}(x_\ell)\right) dx_n...dx_1, \tag{2.5.9}$$

with $x \in \mathbb{R}_+$. *Here the iterated integration is performed in the indicated order.*

The Fourier type transform expression for $\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}$

Let $(\boldsymbol{\mu},\boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$. For $\upsilon(z) \in [z]^{m_n}\|z\|^{\mu_n}\mathscr{S}(\mathbb{C})$, using Lemma 2.5.2 (2) and Lemma 2.5.6 (1), especially (2.5.2) and (2.5.5), one may show that

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon(z) = \|z\|^{-\mu_1}\mathcal{T}_{(\mu_1-\mu_2,m_1)} \circ ... \circ \mathcal{T}_{(\mu_{n-1}-\mu_n,m_{n-1})} \circ \mathcal{S}_{(\mu_n,m_n)}\upsilon(z). \tag{2.5.10}$$

Theorem 2.5.10. *Let* $(\boldsymbol{\mu},\boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ *be such that* $\Re\mathfrak{e}\,\mu_1 > ... > \Re\mathfrak{e}\,\mu_{n-1} > \Re\mathfrak{e}\,\mu_n$. *Suppose* $\upsilon(z) \in [z]^{m_n}\|z\|^{\mu_n}\mathscr{S}(\mathbb{C})$. *Then*

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon(z) = \frac{1}{\|z\|}\int_{\mathbb{C}^{\times n}}\left(\prod_{\ell=1}^{n}[z_\ell]^{-m_\ell}\|z_\ell\|^{-\mu_\ell} e\left(z_\ell + \overline{z_\ell}\right)\right)\upsilon\left(\frac{z_1...z_n}{z}\right) dz_n...dz_1, \tag{2.5.11}$$

where the integral converges when performed as iterated integral in the indicated order.

Proof. One applies the same arguments in the proof of Theorem 2.5.8 using Lemma 2.5.2 (2) and Lemma 2.5.6 (3, 4). Q.E.D.

Lemma 2.5.3 (2) and Lemma 2.5.7 yield the following corollary.

Corollary 2.5.11. *Let $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Assume that $\mathfrak{Re}\, \mu_1 > ... > \mathfrak{Re}\, \mu_{n-1} > \mathfrak{Re}\, \mu_n$. Let $\varphi(x) \in x^{2\mu_n} \mathscr{S}_{-m-m_n}(\overline{\mathbb{R}}_+)$ and $\upsilon(z) = [z]^{-m} \varphi(|z|)$. Then*

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon\left(xe^{i\phi}\right) = 2^n \frac{e^{im\phi}}{x^2} \int_{\mathbb{R}_+^n} \varphi\left(\frac{x_1 ... x_n}{x}\right) \left(\prod_{\ell=1}^n x_\ell^{-2\mu_\ell + 1} j_{(0, m_\ell + m)}(x_\ell)\right) dx_n ... dx_1, \tag{2.5.12}$$

with $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Here the iterated integration is performed in the indicated order.

2.6. Integral representations of Bessel kernels

In Chapter 1, when $n \geqslant 2$, the formal integral representation of the Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\mu})$ is obtained in symbolic manner from the Fourier type integral in Theorem 2.5.8, where the assumption $\mathfrak{Re}\, \mu_1 > ... > \mathfrak{Re}\, \mu_n$ is simply ignored. It is however more straightforward to derive the formal integral representation of the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ from Theorem 2.5.8. This should be well understood, since $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ is a finite combination of $J\left(2\pi |x|^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\mu}\right)$.

Similarly, Theorem 2.5.10 also yields a formal integral representation of $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$. It turns out that one can naturally transform this formal integral into an integral that is absolutely convergent, given that the index $\boldsymbol{\mu}$ satisfies certain conditions. The main reason for the absolute convergence is that $j_{(0,m)}(x) = 2\pi i^m J_m(4\pi x)$ (see (2.4.29)) decays proportionally to $\frac{1}{\sqrt{x}}$ at infinity (in comparison, $j_{(0,\delta)}(x)$ is equal to either $2\cos(2\pi x)$ or $2i \sin(2\pi x)$).

Assumptions and notations

Let $n \geqslant 2$. Assume that $\boldsymbol{\mu} \in \mathbb{L}^{n-1}$.

Notation 2.6.1. *Let $d = n - 1$. Let the pairs of tuples, $\boldsymbol{\mu} \in \mathbb{L}^d$ and $\boldsymbol{\nu} \in \mathbb{C}^d$, $\boldsymbol{\delta} \in (\mathbb{Z}/2\mathbb{Z})^{d+1}$ and $\boldsymbol{\epsilon} \in (\mathbb{Z}/2\mathbb{Z})^d$, $\boldsymbol{m} \in \mathbb{Z}^{d+1}$ and $\boldsymbol{k} \in \mathbb{Z}^d$, be subjected to the following relations*

$$\nu_\ell = \mu_\ell - \mu_{d+1}, \quad \epsilon_\ell = \delta_\ell + \delta_{d+1}, \quad k_\ell = m_\ell - m_{d+1},$$

for $\ell = 1, ..., d$.

Instead of Hankel transforms, we shall be interested in their Bessel kernels. Therefore, it is convenient to further assume that the weight functions are Schwartz, namely, $\varphi \in \mathscr{S}(\mathbb{R}_+)$ and $\upsilon \in \mathscr{S}(\mathbb{F}^\times)$. According to (2.4.19, 2.4.28), Proposition 2.4.17 (3) and 2.4.21 (3), for such Schwartz functions φ and υ,

$$\hbar_{(\boldsymbol{\mu},\boldsymbol{\delta})}\varphi(x) = \int_{\mathbb{R}_+} \varphi(y) j_{(\boldsymbol{\mu},\boldsymbol{\delta})}(xy)dy, \qquad \hbar_{(\boldsymbol{\mu},\boldsymbol{m})}\varphi(x) = 2\int_{\mathbb{R}_+} \varphi(y) j_{(\boldsymbol{\mu},\boldsymbol{m})}(xy)ydy, \tag{2.6.1}$$

$$\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon(x) = \int_{\mathbb{R}^\times} \upsilon(y) J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(xy)dy, \quad \mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{m})}\upsilon(z) = \int_{\mathbb{C}^\times} \upsilon(u) J_{(\boldsymbol{\mu},\boldsymbol{m})}(zu)du, \tag{2.6.2}$$

with the index $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ or $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ being arbitrary.

2.6.1. The formal integral $J_{\boldsymbol{\nu},\boldsymbol{\epsilon}}(x, \pm)$

To motivate the definition of $J_{\boldsymbol{\nu},\boldsymbol{\epsilon}}(x, \pm)$, we shall do certain operations on the Fourier type integral (2.5.8) in Theorem 2.5.8. In the meanwhile, we shall forget the assumption $\mathfrak{Re}\,\mu_1 > ... > \mathfrak{Re}\,\mu_n$, which is required for convergence.

Upon making the change of variables, $x_n = (x_1...x_{n-1})^{-1}xy$, $x_\ell = |xy|^{\frac{1}{n}} y_\ell^{-1}$, $\ell = 1, ..., n-1$, one converts (2.5.8) into

$$\begin{aligned}\mathcal{H}_{(\boldsymbol{\mu},\boldsymbol{\delta})}\upsilon(x) = \int_{\mathbb{R}^{\times n}} \upsilon(y) \operatorname{sgn}(xy)^{\delta_n} \left(\prod_{\ell=1}^{n-1} \operatorname{sgn}(y_\ell)^{\delta_\ell+\delta_n} |y_\ell|^{\mu_\ell-\mu_n-1} \right) \\ e\left(|xy|^{\frac{1}{n}} \left(\operatorname{sgn}(xy)\cdot y_1...y_{n-1} + \sum_{\ell=1}^{n-1} y_\ell^{-1} \right) \right) dydy_{n-1}...dy_1.\end{aligned}$$

In symbolic notation, moving the integral over dy to the outermost place and comparing the resulting integral with the right hand side of the first formula in (2.6.2), the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ is then represented by the following formal integral over $dy_{n-1}...dy_1$,

$$\operatorname{sgn}(x)^{\delta_n}\int_{\mathbb{R}^{\times n-1}}\left(\prod_{\ell=1}^{n-1}\operatorname{sgn}(y_\ell)^{\delta_\ell+\delta_n}|y_\ell|^{\mu_\ell-\mu_n-1}\right)$$
$$e\left(|x|^{\frac{1}{n}}\left(\operatorname{sgn}(x)\cdot y_1...y_{n-1}+\sum_{\ell=1}^{n-1}y_\ell^{-1}\right)\right)dy_{n-1}...dy_1.$$

We define the formal integral

$$J_{\boldsymbol{\nu},\boldsymbol{\epsilon}}(x,\pm)=\int_{\mathbb{R}^{\times d}}\left(\prod_{\ell=1}^{d}\operatorname{sgn}(y_\ell)^{\epsilon_\ell}|y_\ell|^{\nu_\ell-1}\right)e^{ix\left(\pm y_1...y_d+\sum_{\ell=1}^{d}y_\ell^{-1}\right)}dy_d...dy_1,\quad x\in\mathbb{R}_+. \tag{2.6.3}$$

Thus, in view of Notation 2.6.1, we have $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(\pm x)=(\pm)^{\delta_{d+1}}J_{\boldsymbol{\nu},\boldsymbol{\epsilon}}\left(2\pi x^{\frac{1}{d+1}},\pm\right)$ in symbolic notation.

2.6.2. The formal integral $j_{\boldsymbol{\nu},\boldsymbol{\delta}}(x)$

For $\boldsymbol{\nu}\in\mathbb{C}^d$ and $\boldsymbol{\delta}\in(\mathbb{Z}/2\mathbb{Z})^{d+1}$, we define the formal integral

$$j_{\boldsymbol{\nu},\boldsymbol{\delta}}(x)=\int_{\mathbb{R}_+^d}j_{(0,\delta_{d+1})}\left(xy_1...y_d\right)\prod_{\ell=1}^{d}y_\ell^{\nu_\ell-1}j_{(0,\delta_\ell)}\left(xy_\ell^{-1}\right)dy_d...dy_1,\quad x\in\mathbb{R}_+. \tag{2.6.4}$$

We may derive the symbolic identity $j_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)=j_{\boldsymbol{\nu},\boldsymbol{\delta}}\left(x^{\frac{1}{d+1}}\right)$ from Corollary 2.4.16 and 2.5.9, combined with the first formula in (2.6.1).

2.6.3. The integral $J_{\boldsymbol{\nu},\boldsymbol{k}}(x,u)$

First of all, proceeding in the same way as in §2.6.1, from the Fourier type integral (2.5.11) in Theorem 2.5.10, the symbolic equality $J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right)=e^{-im_{d+1}\phi}J_{\boldsymbol{\nu},\boldsymbol{k}}\left(2\pi x^{\frac{1}{d+1}},e^{i\phi}\right)$ can be deduced, with the definition of the formal integral,

$$J_{\boldsymbol{\nu},\boldsymbol{k}}(x,u)=\int_{\mathbb{C}^{\times d}}\left(\prod_{\ell=1}^{d}[u_\ell]^{k_\ell}\|u_\ell\|^{\nu_\ell-1}\right)e^{ix\Lambda\left(uu_1...u_d+\sum_{\ell=1}^{d}u_\ell^{-1}\right)}du_d...du_1,$$
$$x\in\mathbb{R}_+,\ u\in\mathbb{C},|u|=1. \tag{2.6.5}$$

Here, we recall that $\Lambda(z) = z + \bar{z}$.

In the polar coordinate, we write $u_\ell = y_\ell e^{i\theta_\ell}$ and $u = e^{i\phi}$. Moving the integral over the torus $(\mathbb{R}/2\pi\mathbb{Z})^d$ inside, in symbolic manner, the integral above turns into

$$2^d \int_{\mathbb{R}_+^d} \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} \left(\prod_{l=1}^{d} y_l^{2\nu_l - 1} \right) e^{i \sum_{l=1}^d k_l\theta_l + 2ix\left(y_1...y_d \cos\left(\sum_{l=1}^d \theta_l + \phi\right) + \sum_{l=1}^d y_l^{-1} \cos\theta_l\right)} d\theta_d...d\theta_1 dy_d...dy_1.$$

Let us introduce the following definitions

$$\Theta_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{y}; x, \phi) = 2xy_1...y_d \cos\left(\textstyle\sum_{\ell=1}^d \theta_\ell + \phi\right) + \sum_{\ell=1}^{d} \left(k_\ell\theta_\ell + 2xy_\ell^{-1}\cos\theta_\ell\right), \tag{2.6.6}$$

$$J_{\boldsymbol{k}}(\boldsymbol{y}; x, \phi) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{i\Theta_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{y}; x, \phi)} d\boldsymbol{\theta}, \tag{2.6.7}$$

$$p_{2\boldsymbol{\nu}}(\boldsymbol{y}) = \prod_{\ell=1}^{d} y_\ell^{2\nu_\ell - 1}, \tag{2.6.8}$$

with $\boldsymbol{y} = (y_1, ..., y_d)$, $\boldsymbol{\theta} = (\theta_1, ..., \theta_d)$. Then (2.6.5) can be symbolically rewritten as

$$J_{\boldsymbol{\nu},\boldsymbol{k}}\left(x, e^{i\phi}\right) = 2^d \int_{\mathbb{R}_+^d} p_{2\boldsymbol{\nu}}(\boldsymbol{y}) J_{\boldsymbol{k}}(\boldsymbol{y}; x, \phi) d\boldsymbol{y}, \quad x \in \mathbb{R}_+,\ \phi \in \mathbb{R}/2\pi\mathbb{Z}. \tag{2.6.9}$$

Theorem 2.6.2. *Let* $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{L}^d \times \mathbb{Z}^{d+1}$ *and* $(\boldsymbol{\nu}, \boldsymbol{k}) \in \mathbb{C}^d \times \mathbb{Z}^d$ *satisfy the relations given in Notation* 2.6.1. *Suppose* $\boldsymbol{\nu} \in \bigcup_{a \in \left[-\frac{1}{2}, 0\right]} \left\{\boldsymbol{\nu} \in \mathbb{C}^d : -\frac{1}{2} < 2\mathfrak{Re}\, \nu_\ell + a < 0 \text{ for all } \ell = 1, ..., d\right\}$.

(1). *The integral in* (2.6.9) *converges absolutely. Subsequently, we shall therefore use* (2.6.9) *as the definition of* $J_{\boldsymbol{\nu},\boldsymbol{k}}\left(x, e^{i\phi}\right)$.

(2). *We have the* (*genuine*) *identity*

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right) = e^{-im_{d+1}\phi} J_{\boldsymbol{\nu},\boldsymbol{k}}\left(2\pi x^{\frac{1}{d+1}}, e^{i\phi}\right).$$

2.6.4. The integral $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$

Let us consider the integral $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$ defined by

$$j_{\boldsymbol{\nu},\boldsymbol{m}}(x) = 2^d \int_{\mathbb{R}_+^d} j_{(0,m_{d+1})}\left(xy_1...y_d\right) \prod_{\ell=1}^{d} y_\ell^{2\nu_\ell - 1} j_{(0,m_\ell)}\left(xy_\ell^{-1}\right) dy_d...dy_1, \tag{2.6.10}$$

with $\boldsymbol{\nu} \in \mathbb{C}^d$ and $\boldsymbol{m} \in \mathbb{Z}^{d+1}$.

Absolute convergence of $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$

In contrast to the real case, where the integral $j_{\boldsymbol{\nu},\boldsymbol{\delta}}(x)$ never absolutely converges, $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$ is actually absolutely convergent, if each component of $\boldsymbol{\nu}$ lies in certain vertical strips of width at least $\frac{1}{4}$.

Definition 2.6.3. *For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ such that $a_\ell < b_\ell$ for all $\ell = 1, ..., d$, we define the open hyper-strip $\mathbb{S}^d(\boldsymbol{a}, \boldsymbol{b}) = \left\{\boldsymbol{\nu} \in \mathbb{C}^d : \Re\mathfrak{e}\, \nu_\ell \in (a_\ell, b_\ell)\right\}$. We write $\mathbb{S}^d(a, b) = \mathbb{S}^d(a\boldsymbol{e}^d, b\boldsymbol{e}^d)$ for simplicity.*

Proposition 2.6.4. *Let $(\boldsymbol{\nu}, \boldsymbol{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. The integral $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$ defined above by* (2.6.10) *absolutely converges if $\boldsymbol{\nu} \in \bigcup_{a \in \left[-\frac{1}{2}, |m_{d+1}|\right]} \mathbb{S}^d\left(\frac{1}{2}\left(-\frac{1}{2} - a\right)\boldsymbol{e}^d, \frac{1}{2}\left(\left\|\boldsymbol{m}^d\right\| - a\boldsymbol{e}^d\right)\right)$, where $\boldsymbol{m}^d = (m_1, ..., m_d)$ and $\left\|\boldsymbol{m}^d\right\| = (|m_1|, ..., |m_d|)$.*

To show this, we first recollect some well-known facts concerning $J_m(x)$, as $j_{(0,m)}(x) = 2\pi i^m J_m(4\pi x)$ in view of (2.4.29).

Firstly, for $m \in \mathbb{N}$, we have the Poisson-Lommel integral representation (see [Wat, 3.3 (1)])

$$J_m(x) = \frac{\left(\frac{1}{2}x\right)^m}{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \cos(x\cos\theta)\sin^{2m}\theta d\theta.$$

This yields the bound

$$|J_m(x)| \leqslant \frac{\sqrt{\pi}\left(\frac{1}{2}x\right)^{|m|}}{\Gamma\left(|m| + \frac{1}{2}\right)}, \tag{2.6.11}$$

for $m \in \mathbb{Z}$. Secondly, the asymptotic expansion of $J_m(x)$ (see [Wat, 7.21 (1)]) provides the estimate

$$J_m(x) \ll_m x^{-\frac{1}{2}}. \tag{2.6.12}$$

Combining these, we then arrive at the following lemma.

Lemma 2.6.5. *Let m be an integer.*

(1). *We have the estimates*

$$j_{(0,m)}(x) \ll_m x^{|m|}, \quad j_{(0,m)}(x) \ll_m x^{-\frac{1}{2}}.$$

(2). *More generally, for any* $a \in \left[-\frac{1}{2}, |m|\right]$, *we have the estimate*

$$j_{(0,m)}(x) \ll_m x^{a}.$$

Proof of Proposition 2.6.4. We divide $\mathbb{R}_+ = (0, \infty)$ into the union of two intervals, $I_- \cup I_+ = (0, 1] \cup [1, \infty)$. Accordingly, the integral in (2.6.10) is partitioned into 2^d many integrals, each of which is supported on some hyper-cube $I_{\boldsymbol{\varrho}} = I_{\varrho_1} \times ... \times I_{\varrho_d}$ for $\boldsymbol{\varrho} \in \{+, -\}^d$. For each such integral, we estimate $j_{(0,m_\ell)}\left(xy_\ell^{-1}\right)$ using the first or the second estimate in Lemma 2.6.5 (1) according as $\varrho_\ell = +$ or $\varrho_\ell = -$ and apply the bound in Lemma 2.6.5 (2) for $j_{(0,m_{d+1})}(xy_1...y_d)$. In this way, for any $a \in \left[-\frac{1}{2}, |m_{d+1}|\right]$, one has

$$\begin{aligned} & 2^d \int_{\mathbb{R}_+^d} \left| j_{(0,m_{d+1})}\left(xy_1...y_d\right)\right| \prod_{\ell=1}^{d} \left| y_\ell^{2\nu_\ell - 1} j_{(0,m_\ell)}\left(xy_\ell^{-1}\right)\right| dy_d...dy_1 \\ \ll & \sum_{\boldsymbol{\varrho} \in \{+,-\}^d} x^{\sum_{\ell \in L_+(\boldsymbol{\varrho})} |m_\ell| - \frac{1}{2}|L_-(\boldsymbol{\varrho})| + a} I_{2\boldsymbol{\nu} + a\boldsymbol{e}^d, \boldsymbol{m}^d}(\boldsymbol{\varrho}), \end{aligned}$$

with the auxiliary definition

$$I_{\boldsymbol{\lambda},\boldsymbol{k}}(\boldsymbol{\varrho}) = \int_{I_{\boldsymbol{\varrho}}} \left(\prod_{\ell \in L_+(\boldsymbol{\varrho})} y_\ell^{\Re\mathfrak{e}\, \lambda_\ell - |k_\ell| - 1} \right) \left(\prod_{\ell \in L_-(\boldsymbol{\varrho})} y_\ell^{\Re\mathfrak{e}\, \lambda_\ell - \frac{1}{2}} \right) dy_d...dy_1, \ \ (\boldsymbol{\lambda}, \boldsymbol{k}) \in \mathbb{C}^d \times \mathbb{Z}^d,$$

and $L_\pm(\boldsymbol{\varrho}) = \{\ell : \varrho_\ell = \pm\}$. The implied constant depends only on $\boldsymbol{m}$ and d. It is clear that all the integrals $I_{2\boldsymbol{\nu}+a\boldsymbol{e}^d,\boldsymbol{m}^d}(\boldsymbol{\varrho})$ absolutely converge if $-\frac{1}{2} < 2\,\Re\mathfrak{e}\, \nu_\ell + a < |m_\ell|$ for all $\ell = 1, ..., d$. The proof is then completed. Q.E.D.

Remark 2.6.6. *When $d = 1$, one may apply the two estimates in Lemma 2.6.5 (1) to $j_{(0,m_2)}(xy)$ in the similar fashion as $j_{(0,m_1)}(xy^{-1})$. Then*

$$2\int_0^\infty \left|y^{2\nu-1} j_{(0,m_1)}\left(xy^{-1}\right) j_{(0,m_2)}(xy)\right|dy$$
$$\ll_{m_1,m_2} x^{|m_1|-\frac{1}{2}}\int_1^\infty y^{2\mathfrak{Re}\,\nu-|m_1|-\frac{3}{2}}dy + x^{|m_2|-\frac{1}{2}}\int_0^1 y^{2\mathfrak{Re}\,\nu+|m_2|-\frac{1}{2}}dy.$$

Since both integrals above absolutely converge if $-|m_2| - \frac{1}{2} < 2\mathfrak{Re}\,\nu < |m_1| + \frac{1}{2}$, this also proves Proposition 2.6.4 in the case $d = 1$.

Equality between $j_{(\boldsymbol{\mu},\boldsymbol{m})}(x)$ and $j_{\boldsymbol{\nu},\boldsymbol{m}}\left(x^{\frac{1}{d+1}}\right)$

Proposition 2.6.7. *Let $(\boldsymbol{\nu}, \boldsymbol{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$ be as in Proposition 2.6.4 so that the integral $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$ absolutely converges. Suppose that $\boldsymbol{\mu} \in \mathbb{L}^d$ and $\boldsymbol{\nu} \in \mathbb{C}^d$ satisfy the relations given in Notation 2.6.1. Then we have the identity*

$$j_{(\boldsymbol{\mu},\boldsymbol{m})}(x) = j_{\boldsymbol{\nu},\boldsymbol{m}}\left(x^{\frac{1}{d+1}}\right).$$

Proof. Some change of variables turns the integral in Corollary 2.5.11 into

$$2^{d+1}e^{im\phi}\int_{\mathbb{R}_+^{d+1}} \varphi(y)\, j_{(0,m_{d+1})}\left((xy)^{\frac{1}{d+1}}y_1...y_d\right)\prod_{\ell=1}^d y_\ell^{2\nu_\ell-1} j_{(0,m_\ell)}\left((xy)^{\frac{1}{d+1}}y_\ell^{-1}\right) ydydy_d...dy_1.$$

Corollary 2.4.20 and 2.5.11, along with the second formula in (2.6.1), yield

$$2\int_{\mathbb{R}_+} \varphi(y) j_{(\boldsymbol{\mu},\boldsymbol{m})}(xy)ydy =$$
$$2^{d+1}\int_{\mathbb{R}_+^{d+1}} \varphi(y) j_{(0,m_{d+1})}\left((xy)^{\frac{1}{d+1}}y_1...y_d\right)\prod_{\ell=1}^d y_\ell^{2\nu_\ell-1} j_{(0,m_\ell)}\left((xy)^{\frac{1}{d+1}}y_\ell^{-1}\right) ydydy_d...dy_1,$$

for any $\varphi \in \mathscr{S}(\mathbb{R}_+)$, provided that $\mathfrak{Re}\,\mu_1 > ... > \mathfrak{Re}\,\mu_{d+1}$ or equivalently $\mathfrak{Re}\,\nu_1 > ... > \mathfrak{Re}\,\nu_d > 0$. In view of Proposition 2.6.4, the integral on the right hand side is absolute convergent at least when $\frac{1}{4} > \mathfrak{Re}\,\nu_1 > ... > \mathfrak{Re}\,\nu_d > 0$. Therefore, the asserted equality holds on the domain $\left\{\boldsymbol{\nu} \in \mathbb{C}^d : \frac{1}{4} > \Re\mathrm{e}\, \nu_1 > ... > \Re\mathrm{e}\, \nu_d > 0\right\}$ and remains valid on the whole domain of convergence for $j_{\boldsymbol{\nu},\boldsymbol{m}}(x)$ given in Proposition 2.6.4 due to the principle of analytic continuation. Q.E.D.

An auxiliary lemma

Lemma 2.6.8. *Let $(\boldsymbol{\nu}, \boldsymbol{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$ and $m \in \mathbb{Z}$. Set $A = \max_{\ell=1,...,d+1} \{|m_\ell|\}$. Suppose $\boldsymbol{\nu} \in \bigcup_{a \in \left[-\frac{1}{2}, 0\right]} \mathbb{S}^d\left(-\frac{1}{4} - \frac{1}{2}a, -\frac{1}{2}a\right)$. We have the estimate*

$$
\begin{aligned}
2^d \int_{\mathbb{R}_+^d} &\left|j_{(0, m_{d+1}+m)}\left(x y_1 ... y_d\right)\right| \prod_{l=1}^d \left|y_l^{2\nu_l - 1} j_{(0, m_l + m)}\left(x y_l^{-1}\right)\right| dy_d ... dy_1 \\
\ll_{\boldsymbol{m}, d} &\sum_{\boldsymbol{\varrho} \neq \boldsymbol{\varrho}_-} \left(\frac{2\pi e x}{|m|+1}\right)^{|L_+(\boldsymbol{\varrho})||m|} (|m|+1)^{2|L_-(\boldsymbol{\varrho})| + A|L_+(\boldsymbol{\varrho})|} \\
&\qquad x^{-\frac{1}{2}|L_-(\boldsymbol{\varrho})|} \max\left\{x^{|L_+(\boldsymbol{\varrho})|A}, x^{-|L_+(\boldsymbol{\varrho})|A - \frac{1}{2}}\right\} \\
&+ \left(\frac{2\pi e x}{|m|+1}\right)^{|m|} (|m|+1)^A x^{-\frac{d}{2}} \max\left\{x^A, x^{-A}\right\},
\end{aligned}
$$

where $\boldsymbol{\varrho} \in \{+, -\}^d$, $\boldsymbol{\varrho}_- = (-, ..., -)$ and $L_\pm(\boldsymbol{\varrho}) = \{\ell : \varrho_\ell = \pm\}$.

Firstly, we require the bound (2.6.11) for $J_m(x)$. Secondly, we observe that when $x \geqslant (|m|+1)^2$ the bound (2.6.12) for $J_m(x)$ can be improved so that the implied constant becomes absolute. This follows from the asymptotic expansion of $J_m(x)$ given in [Olv, §7.13.1]. Moreover, we have Bessel's integral representation (see [Wat, 2.2 (1)])

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos\left(m\theta - x \sin\theta\right) d\theta,$$

which yields the bound

$$|J_m(x)| \leqslant 1. \tag{2.6.13}$$

We then have the following lemma (compare [HM, Proposition 8]).

Lemma 2.6.9. *Let m be an integer.*

(1). *The following two estimates hold*

$$j_{(0,m)}(x) \ll \frac{(2\pi x)^{|m|}}{\Gamma\left(|m|+\frac{1}{2}\right)}, \quad j_{(0,m)}(x) \ll \frac{|m|+1}{\sqrt{x}},$$

with absolute implied constants.

(2). *For any $a \in \left[-\frac{1}{2}, 0\right]$ we have the estimate*

$$j_{(0,m)}(x) \ll \left((|m|+1)^{-2} x\right)^{a},$$

with absolute implied constant.

Proof of Lemma 2.6.8. Our proof here is similar to that of Proposition 2.6.4, except that

- Lemma 2.6.9 (1) and (2) are applied in place of Lemma 2.6.5 (1) and (2) respectively to bound $j_{(0,m_\ell+m)}\left(xy_\ell^{-1}\right)$ and $j_{(0,m_{d+1}+m)}\left(xy_1...y_d\right)$, and

- the first estimate in Lemma 2.6.9 (1) is used for $j_{(0,m_{d+1}+m)}\left(xy_1...y_d\right)$ in the case $\boldsymbol{\varrho} = \boldsymbol{\varrho}_-$.

In this way, one obtains the following estimate

$$\begin{aligned} & 2^d \int_{\mathbb{R}_+^d} \left| j_{(0,m_{d+1}+m)}\left(xy_1...y_d\right)\right| \prod_{\ell=1}^{d} \left| y_\ell^{2\nu_\ell - 1} j_{(0,m_\ell+m)}\left(xy_\ell^{-1}\right)\right| dy_d...dy_1 \\ \ll & \sum_{\boldsymbol{\varrho} \neq \boldsymbol{\varrho}_-} \frac{\prod_{\ell \in L_-(\boldsymbol{\varrho})} (|m_\ell + m| + 1)^{1-2a}}{\prod_{\ell \in L_+(\boldsymbol{\varrho})} \Gamma\left(|m_\ell + m| + \frac{1}{2}\right)(|m_\ell + m| + 1)^{2a}} \\ & \qquad\qquad (2\pi x)^{\sum_{\ell \in L_+(\boldsymbol{\varrho})} |m_\ell + m| - \frac{1}{2}|L_-(\boldsymbol{\varrho})| + a} I_{2\boldsymbol{\nu} + a\boldsymbol{e}^d, \boldsymbol{m}^d + m\boldsymbol{e}^d}(\boldsymbol{\varrho}) \\ & + \frac{\prod_{\ell=1}^{d} (|m_\ell + m| + 1)}{\Gamma\left(|m_{d+1} + m| + \frac{1}{2}\right)} (2\pi x)^{-\frac{d}{2} + |m_{d+1} + m|} I_{2\boldsymbol{\nu} + |m_{d+1} + m| \boldsymbol{e}^d}(\boldsymbol{\varrho}_-), \end{aligned}$$

[XVII]with $a \in \left[-\frac{1}{2}, 0\right]$. Now the implied constant above depends only on d. Suppose that $-\frac{1}{2} - a < 2\,\mathfrak{Re}\,\nu_\ell < -a$ for all $\ell = 1, ..., d$, then the integrals $I_{2\boldsymbol{\nu} + a\boldsymbol{e}^d, \boldsymbol{m}^d + m\boldsymbol{e}^d}(\boldsymbol{\varrho})$ and

[XVII]When $\boldsymbol{\varrho} = \boldsymbol{\varrho}_-$, $\boldsymbol{k}$ does not occur in the definition of $I_{\lambda, \boldsymbol{k}}(\boldsymbol{\varrho}_-)$ and is therefore suppressed from the subscript.

$I_{2\boldsymbol{\nu}+|m_{d+1}+m|\boldsymbol{e}^d}(\boldsymbol{\varrho}_-)$ are absolutely convergent and of size $O_d\left(\prod_{\ell\in L_+(\boldsymbol{\varrho})}(|m_\ell+m|+1)^{-1}\right)$ and $O_d\left((|m_{d+1}+m|+1)^{-d}\right)$ respectively. A final estimation using Stirling's asymptotic formula yields our asserted bound. Q.E.D.

Remark 2.6.10. *In the case $d = 1$, modifying over the ideas in Remark* 2.6.6*, one may show the slightly improved estimate*

$$2\int_0^\infty \left|y^{2\nu-1} j_{(0,m_1+m)}\left(xy^{-1}\right) j_{(0,m_2+m)}\left(xy\right)\right|dy \\ \ll_{m_1,m_2} \left(\frac{2\pi e x}{|m|+1}\right)^{|m|} (|m|+1)^A x^{-\frac{1}{2}} \max\left\{x^A, x^{-A}\right\},$$

given that $|\Re\mathfrak{e}\,\nu| < \frac{1}{4}$, with $A = \max\{|m_1|, |m_2|\}$.

2.6.5. The series of integrals $J_{\boldsymbol{\nu},\boldsymbol{m}}(x,u)$

We define the following series of integrals,

$$(2.6.14)\qquad \begin{aligned} J_{\boldsymbol{\nu},\boldsymbol{m}}(x,u) &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} u^m j_{\boldsymbol{\nu},\boldsymbol{m}+m\boldsymbol{e}^n}(x) \\ &= \frac{2^{d-1}}{\pi}\sum_{m\in\mathbb{Z}} u^m \int_{\mathbb{R}_+^d} j_{(0,m_{d+1}+m)}\left(xy_1\ldots y_d\right)\prod_{\ell=1}^d y_\ell^{2\nu_\ell-1} j_{(0,m_\ell+m)}\left(xy_\ell^{-1}\right) dy_d\ldots dy_1, \end{aligned}$$

with $x \in \mathbb{R}_+$ and $u \in \mathbb{C}$, $|u| = 1$.

Absolute convergence of $J_{\boldsymbol{\nu},\boldsymbol{m}}(x,u)$

We have the following direct consequence of Lemma 2.6.8.

Proposition 2.6.11. *Let $(\boldsymbol{\nu},\boldsymbol{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. The series of integrals $J_{\boldsymbol{\nu},\boldsymbol{m}}(x,u)$ defined by* (2.6.14) *is absolutely convergent if $\boldsymbol{\nu} \in \bigcup_{a\in\left[-\frac{1}{2},0\right]} \mathbb{S}^d\left(-\frac{1}{4}-\frac{1}{2}a, -\frac{1}{2}a\right)$.*

Equality between $J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right)$ and $J_{\boldsymbol{\nu},\boldsymbol{m}}\left(x^{\frac{1}{d+1}}, e^{i\phi}\right)$

In view of Proposition 2.6.7 along with (2.4.45) and (2.6.14), the following proposition is readily established.

Proposition 2.6.12. *Let $(\boldsymbol{\nu}, \boldsymbol{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. Suppose that $\boldsymbol{\nu}$ satisfies the condition in Proposition* 2.6.11 *so that $J_{\boldsymbol{\nu},\boldsymbol{m}}(x,u)$ is absolutely convergent. Then, given that $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ satisfy the relations in Notation* 2.6.1*, we have the identity*

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(xe^{i\phi}\right) = J_{\boldsymbol{\nu},\boldsymbol{m}}\left(x^{\frac{1}{d+1}}, e^{i\phi}\right),$$

with $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$.

2.6.6. Proof of Theorem 2.6.2

Lemma 2.6.13. *Let $\boldsymbol{k} \in \mathbb{Z}^d$ and recall the integral $J_{\boldsymbol{k}}(\boldsymbol{y}; x, \phi)$ defined by* (2.6.6, 2.6.7)*. We have the following absolutely convergent series expansion of $J_{\boldsymbol{k}}(\boldsymbol{y}; x, \phi)$*

$$J_{\boldsymbol{k}}(\boldsymbol{y}; 2\pi x, \phi) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im\phi} j_{(0,m)}\left(xy_1...y_d\right)\prod_{\ell=1}^{d} j_{(0,k_\ell+m)}\left(xy_\ell^{-1}\right). \tag{2.6.15}$$

Proof. In view of Example 2.4.22, we have the integral representation

$$j_{(0,m)}(x) = \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{im\theta+4\pi i x\cos\theta}d\theta$$

as well as the Fourier series expansion

$$e^{4\pi i x\cos\phi} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} j_{(0,m)}(x)e^{im\phi}.$$

Therefore

$$\begin{aligned}
&\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im\phi} j_{(0,m)}\left(xy_1...y_d\right)\prod_{\ell=1}^{d} j_{(0,k_\ell+m)}\left(xy_\ell^{-1}\right)\\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im\phi} j_{(0,m)}\left(xy_1...y_d\right)\int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{im\sum_{\ell=1}^d \theta_\ell} e^{i\sum_{\ell=1}^d\left(ik_\ell\theta+4\pi i xy_\ell^{-1}\cos\theta_\ell\right)}d\theta_d...d\theta_1\\
&= \int_{(\mathbb{R}/2\pi\mathbb{Z})^d}\left(\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im\left(\sum_{\ell=1}^d\theta_\ell+\phi\right)} j_{(0,m)}\left(xy_1...y_d\right)\right) e^{i\sum_{\ell=1}^d\left(ik_\ell\theta+4\pi i xy_\ell^{-1}\cos\theta_\ell\right)}d\theta_d...d\theta_1\\
&= \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{4\pi i xy_1...y_d\cos\left(\sum_{\ell=1}^d\theta_\ell+\phi\right)} e^{i\sum_{\ell=1}^d\left(ik_\ell\theta+4\pi i xy_\ell^{-1}\cos\theta_\ell\right)}d\theta_d...d\theta_1.
\end{aligned}$$

The absolute convergence required for the validity of each equality above is justified by the first estimate of $j_{(0,m)}(x)$ in Lemma 2.6.9 (1). The proof is completed, since the last line is exactly the definition of $J_{\boldsymbol{k}}(\boldsymbol{y}; 2\pi x, \phi)$. Q.E.D.

Inserting the series expansion of $J_{\boldsymbol{k}}(\boldsymbol{y}; 2\pi x, \phi)$ in Lemma 2.6.13 into the integral in (2.6.9) and interchanging the order of integration and summation, one arrives exactly at the series of integrals $J_{\boldsymbol{\nu},(\boldsymbol{k},0)}\left(x, e^{i\phi}\right) = e^{-im_{d+1}\phi} J_{\boldsymbol{\nu},\boldsymbol{m}}\left(x, e^{i\phi}\right)$. The first assertion on absolute convergence in Theorem 2.6.2 follows immediately from Proposition 2.6.11, whereas the identity in the second assertion is a direct consequence of Proposition 2.6.12.

2.6.7. The rank-two case ($d = 1$)

The real case

The formal integral representation $J_{\nu,\epsilon}(2\pi\sqrt{x}, \pm)$ of the Bessel kernel $J_{\left(\frac12\nu,-\frac12\nu\right),(\epsilon,0)}(\pm x)$ is reduced to the following integral representations of classical Bessel functions

$$\pm\pi i e^{\pm\frac12\pi i\nu} H_\nu^{(1,2)}(2x) = \int_0^\infty y^{\nu-1} e^{\pm ix(y+y^{-1})} dy, \ \ 2e^{\pm\frac12\pi i\nu} K_\nu(2x) = \int_0^\infty y^{\nu-1} e^{\pm ix(y-y^{-1})} dy,$$

which are only (conditionally) convergent when $|\Re\mathfrak{e}\,\nu| < 1$ (see §1.2.3).

The complex case

Lemma 2.6.14. *Let $k \in \mathbb{Z}$. Recall from* (2.6.6, 2.6.7) *the definition*

$$J_k(y; x, \phi) = \int_0^{2\pi} e^{ik\theta + 2ixy^{-1}\cos\theta + 2ixy\cos(\theta+\phi)} d\theta, \quad x, y \in (0,\infty),\ \phi \in [0, 2\pi).$$

Define $Y(y,\phi) = |y^{-1} + ye^{i\phi}| = \sqrt{y^{-2} + 2\cos\phi + y^2}$, $\Phi(y,\phi) = \arg(y^{-1} + ye^{i\phi})$ and $E(y,\phi) = e^{i\Phi(y,\phi)}$. Then

$$J_k(y; x, \phi) = 2\pi i^k E(y,\phi)^{-k} J_k\left(2xY(y,\phi)\right). \tag{2.6.16}$$

Proof. (2.6.16) follows immediately from the identity

$$2\pi i^k J_k(x) = \int_0^{2\pi} e^{ik\theta + ix\cos\theta} d\theta,$$

along with the observation

$$y^{-1}\cos\theta + y\cos(\theta+\phi) = \mathfrak{Re}\,\left(y^{-1}e^{i\theta} + ye^{i(\theta+\phi)}\right) = Y(y,\phi)\cos\left(\theta + \Phi(y,\phi)\right).$$

Q.E.D.

Proposition 2.6.15. *Let $\nu \in \mathbb{C}$ and $k \in \mathbb{Z}$. Recall the definition of $J_{\nu,k}\left(x, e^{i\phi}\right)$ given by* (2.6.9)*. Then*

$$J_{\nu,k}\left(x, e^{i\phi}\right) = 4\pi i^k \int_0^\infty y^{2\nu-1}\left[y^{-1} + ye^{i\phi}\right]^{-k} J_k\left(2x\left|y^{-1} + ye^{i\phi}\right|\right) dy, \tag{2.6.17}$$

with $x \in (0,\infty)$ and $\phi \in [0, 2\pi)$. Here, we recall the notation $[z] = z/|z|$. The integral in (2.6.17) *converges when $|\mathfrak{Re}\,\nu| < \frac{3}{4}$ and the convergence is absolute if and only if $|\mathfrak{Re}\,\nu| < \frac{1}{4}$. Moreover, it is analytic with respect to ν on the open vertical strip $\mathbb{S}\left(-\frac{3}{4}, \frac{3}{4}\right)$.*

Proof. (2.6.17) follows immediately from Lemma 2.6.14.

As for the convergence, since one arrives at an integral of the same form with ν, ϕ replaced by $-\nu, -\phi$ if the variable is changed from y to y^{-1}, it suffices to consider the integral

$$\int_2^\infty y^{2\nu-1} e^{-ik\Phi(y,\phi)} J_k\left(2xY(y,\phi)\right) dy,$$

for $\mathfrak{Re}\,\nu < \frac{3}{4}$. We have the following asymptotic of $J_k(x)$ (see [Wat, 7.21 (1)])

$$J_k(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \tfrac{1}{2}k\pi - \tfrac{1}{4}\pi\right) + O_k\left(x^{-\frac{3}{2}}\right).$$

The error term contributes an absolutely convergent integral when $\mathfrak{Re}\,\nu < \frac{3}{4}$, whereas the integral coming from the main term absolutely converges if and only if $\mathfrak{Re}\,\nu < \frac{1}{4}$. We are now reduced to the integral

$$\int_2^\infty y^{2\nu-1} e^{-ik\Phi(y,\phi)} \left(xY(y,\phi)\right)^{-\frac{1}{2}} e^{\pm 2ixY(y,\phi)} dy.$$

In order to see the convergence, we split out $e^{\pm 2ixy}$ from $e^{\pm 2ixY(y,\phi)}$ and put $f_{\nu,k}(y;x,\phi) = y^{2\nu-1} e^{-ik\Phi(y,\phi)} \left(xY(y,\phi)\right)^{-\frac{1}{2}} e^{\pm 2ix(Y(y,\phi)-y)}$. Partial integration turns the above integral into

$$\mp \frac{1}{2ix^{\frac{3}{2}}} \left(2^{2\nu-1} e^{-ik\Phi(2,\phi)} Y(2,\phi)^{-\frac{1}{2}} e^{\pm 2ixY(2,\phi)} + \int_2^\infty \left(\partial f_{\nu,k}/\partial y\right)(y;x,\phi) e^{\pm 2ixy} dy \right).$$

Some calculations show that $\left(\partial f_{\nu,k}/\partial y\right)(y;x,\phi) \ll_{\nu,k,x} y^{2\Re\mathrm{e}\,\nu-\frac{5}{2}}$ for $y \geqslant 2$, and hence the integral in the second term is absolutely convergent when $\Re\mathrm{e}\,\nu < \frac{3}{4}$. With the above arguments, the analyticity with respect to ν is obvious. Q.E.D.

Corollary 2.6.16. *Let $\mu \in \mathbb{S}\left(-\frac{3}{8}, \frac{3}{8}\right)$ and $m \in \mathbb{Z}$. We have*

$$J_{(\mu,-\mu,m,0)}\left(xe^{i\phi}\right) = 4\pi i^m \int_0^\infty y^{4\mu-1} \left[y^{-1} + ye^{i\phi}\right]^{-m} J_m\left(4\pi\sqrt{x}\left|y^{-1} + ye^{i\phi}\right|\right) dy, \tag{2.6.18}$$

with $x \in (0,\infty)$ and $\phi \in [0, 2\pi)$. The integral in (2.6.18) *converges if $|\Re\mathrm{e}\,\mu| < \frac{3}{8}$ and absolutely converges if and only if $|\Re\mathrm{e}\,\mu| < \frac{1}{8}$.*

Proof. From Theorem 2.6.2, we see that (2.6.18) holds for $\mathbb{S}\left(-\frac{1}{8}, \frac{1}{8}\right)$. In view of Proposition 2.6.15, the right hand side of (2.6.18) is analytic in μ on $\mathbb{S}\left(-\frac{3}{8}, \frac{3}{8}\right)$, and therefore it is allowed to extend the domain of equality from $\mathbb{S}\left(-\frac{1}{8}, \frac{1}{8}\right)$ onto $\mathbb{S}\left(-\frac{3}{8}, \frac{3}{8}\right)$. Q.E.D.

2.7. Two connection formulae for $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$

In this section, we shall prove two formulae for $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ in connection with the two kinds of Bessel functions of rank n and positive sign. These Bessel functions arise as solutions of Bessel equations in §1.7 and their relations have been unraveled in §1.8.2. Our motivation is based on the following self-evident identity for the rank-one example

$$e(z + \overline{z}) = e(z)e(\overline{z}).$$

2.7.1. The first connection formula

For $\varsigma \in \{+, -\}$, $\boldsymbol{\lambda} \in \mathbb{C}^n$ and $\ell = 1, ..., n$, we define the following series of ascending powers of z (see §1.7.1)

$$J_\ell(z; \varsigma, \boldsymbol{\lambda}) = \sum_{m=0}^{\infty} \frac{(\varsigma i^n)^m z^{n(-\lambda_\ell + m)}}{\prod_{k=1}^{n} \Gamma(\lambda_k - \lambda_\ell + m + 1)}, \quad z \in \mathbb{U}. \tag{2.7.1}$$

$J_\ell(z; \varsigma, \boldsymbol{\lambda})$ is called a *Bessel function of the first kind*, n , ς and $\boldsymbol{\lambda}$ its rank, sign and index, respectively. Since the definition (2.7.1) is valid for any $\boldsymbol{\lambda} \in \mathbb{C}^n$, the assumption $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ that we imposed in Chapter 1 is rather superfluous. Also, we have the following formula in the same fashion as (2.4.6) in Lemma 2.4.6,

$$J_\ell(z; \varsigma, \boldsymbol{\lambda} - \lambda \boldsymbol{e}^n) = z^{n\lambda} J_\ell(z; \varsigma, \boldsymbol{\lambda}). \tag{2.7.2}$$

Theorem 2.7.1. *Let $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$. We have*

$$J_{(\boldsymbol{\mu}, \boldsymbol{m})}(z) = \left(2\pi^2\right)^{n-1} \sum_{\ell=1}^{n} S_\ell(\boldsymbol{\mu}, \boldsymbol{m}) J_\ell\left(2\pi z^{\frac{1}{n}}; +, \boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{m}\right) J_\ell\left(2\pi \overline{z}^{\frac{1}{n}}; +, \boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{m}\right), \tag{2.7.3}$$

with $S_\ell(\boldsymbol{\mu}, \boldsymbol{m}) = \prod_{k \neq \ell} (\pm i)^{m_\ell - m_k} / \sin\left(\pi\left(\mu_\ell - \mu_k \pm \frac{1}{2}(m_\ell - m_k)\right)\right)$. Here, $z^{\frac{1}{n}}$ is the principal n-th root of z, that is $\left(xe^{i\phi}\right)^{\frac{1}{n}} = x^{\frac{1}{n}} e^{\frac{1}{n} i\phi}$. The expression on the right hand side of (2.7.3) *is independent on the choice of the argument of z modulo 2π. It is understood that the right hand side should be replaced by its limit if $(\boldsymbol{\mu}, \boldsymbol{m})$ is not generic with respect to the order $\preccurlyeq$ on $\mathbb{C} \times \mathbb{Z}$ in the sense of Definition* 2.4.1.

Proof. Recall from (2.2.6, 2.2.7, 2.4.22, 2.4.45) that

$$J_{(\boldsymbol{\mu}, \boldsymbol{m})}\left(xe^{i\phi}\right) = (2\pi)^{n-1} \sum_{m=-\infty}^{\infty} i^{\sum_{k=1}^{n} |m_k + m|} e^{im\phi}$$
$$\frac{1}{2\pi i} \int_{\mathcal{C}_{(\boldsymbol{\mu}, \boldsymbol{m} + m\boldsymbol{e}^n)}} \left(\prod_{\ell=1}^{n} \frac{\Gamma\left(s - \mu_\ell + \frac{1}{2}|m_\ell + m|\right)}{\Gamma\left(1 - s + \mu_\ell + \frac{1}{2}|m_\ell + m|\right)} \right) \left((2\pi)^n x\right)^{-2s} ds.$$

Assume first that $(\boldsymbol{\mu}, \boldsymbol{m})$ is generic with respect to the order $\preccurlyeq$ on $\mathbb{C} \times \mathbb{Z}$. The sets of poles of the gamma factors in the above integral are $\left\{\mu_\ell - \frac{1}{2}|m_\ell + m| - \alpha\right\}_{\alpha \in \mathbb{N}}$, $\ell = 1, ..., n$.

With the generic assumption, the integrand has only *simple* poles. We left shift the integral contour of each integral in the series and pick up the residues from these poles. The contribution from the residues at the poles of the ℓ-th gamma factor is the following absolutely convergent double series,

$$(2\pi)^{n-1}\sum_{m=-\infty}^{\infty} i^{\sum_{k=1}^{n}|m_k+m|}e^{im\phi}\sum_{\alpha=0}^{\infty}\frac{(-)^{\alpha}\left((2\pi)^n x\right)^{-2\mu_\ell+|m_\ell+m|+2\alpha}}{\alpha!(\alpha+|m_\ell+m|)!}$$
$$\prod_{k\neq\ell}\frac{\Gamma\left(\mu_\ell-\mu_k-\frac12(|m_\ell+m|-|m_k+m|)-\alpha\right)}{\Gamma\left(1-\mu_\ell+\mu_k+\frac12(|m_\ell+m|+|m_k+m|)+\alpha\right)}.$$

Euler's reflection formula of the Gamma function turns this into

$$\frac{(2\pi^2)^{n-1}}{\prod_{k\neq\ell} i^{m_\ell-m_k}\sin\left(\pi\left(\mu_\ell-\mu_k-\frac12(m_\ell-m_k)\right)\right)}$$
$$\sum_{m=-\infty}^{\infty} i^{n|m_\ell+m|}e^{im\phi}\sum_{\alpha=0}^{\infty}\frac{(-)^{n\alpha}\left((2\pi)^n x\right)^{-2\mu_\ell+|m_\ell+m|+2\alpha}}{\prod_{k=1}^{n}\prod_{\pm}\Gamma\left(1-\mu_\ell+\mu_k+\frac12(|m_\ell+m|\pm|m_k+m|)+\alpha\right)}.$$

We now interchange the order of summations, truncate the sum over m between $-m_\ell$ and $-m_\ell+1$ and make the change of indices $\beta=\alpha+|m_\ell+m|$. With the observation that, no matter what m_k is, one of $\frac12(|m_\ell+m|+|m_k+m|)$ and $\frac12(|m_\ell+m|-|m_k+m|)$ is equal to $\frac12(m_\ell-m_k)$ and the other to $|m_\ell+m|-\frac12(m_\ell-m_k)$ if $m\geqslant -m_\ell+1$, whereas the signs in front of the two $\frac12(m_\ell-m_k)$ are changed if $m\leqslant -m_\ell$, the double series in the expression above turns into

$$\sum_{\alpha=0}^{\infty}\sum_{\beta=\alpha+1}^{\infty}\frac{i^{n(\alpha+\beta)}e^{i(\beta-\alpha-m_\ell)\phi}\left((2\pi)^n x\right)^{-2\mu_\ell+\alpha+\beta}}{\prod_{k=1}^{n}\Gamma\left(1-\mu_\ell+\mu_k+\frac12(m_\ell-m_k)+\alpha\right)\Gamma\left(1-\mu_\ell+\mu_k-\frac12(m_\ell-m_k)+\beta\right)}$$
$$+\sum_{\alpha=0}^{\infty}\sum_{\beta=\alpha}^{\infty}\frac{i^{n(\alpha+\beta)}e^{i(\alpha-\beta-m_\ell)\phi}\left((2\pi)^n x\right)^{-2\mu_\ell+\alpha+\beta}}{\prod_{k=1}^{n}\Gamma\left(1-\mu_\ell+\mu_k-\frac12(m_\ell-m_k)+\alpha\right)\Gamma\left(1-\mu_\ell+\mu_k+\frac12(m_\ell-m_k)+\beta\right)},$$

which is then equal to

$$\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{i^{n(\alpha+\beta)}e^{i(\beta-\alpha-m_\ell)\phi}\left((2\pi)^n x\right)^{-2\mu_\ell+\alpha+\beta}}{\prod_{k=1}^{n}\Gamma\left(1-\mu_\ell+\mu_k+\frac12(m_\ell-m_k)+\alpha\right)\Gamma\left(1-\mu_\ell+\mu_k-\frac12(m_\ell-m_k)+\beta\right)}.$$

This double series is clearly independent on the choice of ϕ modulo 2π, and splits exactly as the product

$$J_\ell\big(2\pi x^{\frac{1}{n}} e^{\frac{1}{n}i\phi}; +, \boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{m}\big) J_\ell\big(2\pi x^{\frac{1}{n}} e^{-\frac{1}{n}i\phi}; +, \boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{m}\big).$$

This proves (2.7.3) in the case when $(\boldsymbol{\mu}, \boldsymbol{m})$ is generic. As for the nongeneric case, one just passes to the limit. Q.E.D.

2.7.2. The second connection formula

According to §1.7.3, *Bessel functions of the second kind* are solutions of Bessel equations defined according to their asymptotics at infinity. To remove the restriction $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ on the definition of $J(z; \boldsymbol{\lambda}; \xi)$, with ξ a $2n$-th root of unity, we simply impose the additional condition

$$J\left(z; \boldsymbol{\lambda} - \lambda \boldsymbol{e}^n; \xi\right) = z^{n\lambda} J(z; \boldsymbol{\lambda}; \xi). \tag{2.7.4}$$

Remark 2.7.2. *Let ξ be an n-th root of $\varsigma 1$. We may also use the following formula as an alternative definition of $J\left(z; \boldsymbol{\lambda}; \xi\right)$ (compare Corollary 1.8.4)*

$$J(z; \boldsymbol{\lambda}; \xi) = \sqrt{n}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} (-i\xi)^{\frac{n-1}{2}+|\boldsymbol{\lambda}|} \sum_{\ell=1}^{n} \left(i\bar{\xi}\right)^{n\lambda_\ell} S_\ell(\boldsymbol{\lambda}) J_\ell(z; \varsigma, \boldsymbol{\lambda}). \tag{2.7.5}$$

where $(-i\xi)^{\frac{n-1}{2}+|\boldsymbol{\lambda}|} = e^{\left(\frac{n-1}{2}+|\boldsymbol{\lambda}|\right)\left(-\frac{1}{2}\pi i + i \arg \xi\right)}$ and $\left(i\bar{\xi}\right)^{n\lambda_\ell} = e^{\frac{1}{2}\pi i n\lambda_\ell - i n \lambda_\ell \arg \xi}$ by convention, and $S_\ell(\boldsymbol{\lambda}) = 1/\prod_{k\neq\ell} \sin\left(\pi(\lambda_\ell - \lambda_k)\right)$.

Given an integer a, define $\xi_{a,j} = e^{2\pi i \frac{j+a-1}{n}}$, $j = 1, ..., n$. Let $\sigma_{\ell,d}(\boldsymbol{\lambda})$, $d = 0, 1, ..., n-1$, $\ell = 1, ..., n$, denote the elementary symmetric polynomial in $e^{-2\pi i\lambda_1}, ..., \widehat{e^{-2\pi i\lambda_\ell}}, ..., e^{-2\pi i\lambda_n}$ of degree d. It follows from Corollary 1.8.6 that

$$\begin{aligned} J_\ell(z; +, \boldsymbol{\lambda}) = &\frac{e^{\frac{3}{4}\pi i((n-1)+2|\boldsymbol{\lambda}|)}}{\sqrt{n}(2\pi)^{\frac{n-1}{2}}} e^{\pi i\left(\frac{1}{2}n+2a-2\right)\lambda_\ell} \\ &\sum_{j=1}^{n} (-)^{n-j} \xi_{a,j}^{-\frac{n-1}{2}-|\boldsymbol{\lambda}|} \sigma_{\ell,n-j}(\boldsymbol{\lambda}) J\left(z; \boldsymbol{\lambda}; \xi_{a,j}\right). \end{aligned} \tag{2.7.6}$$

In addition, we shall require the definition

$$\tau_\ell(\boldsymbol{\lambda}) = \prod_{k\neq\ell}\left(e^{-2\pi i\lambda_m} - e^{-2\pi i\lambda_k}\right) = (-2i)^{n-1} e^{-\pi i|\boldsymbol{\lambda}|} e^{-\pi i(n-2)\lambda_\ell} \prod_{k\neq\ell} \sin\left(\pi(\lambda_\ell - \lambda_k)\right).$$

We introduce the column vectors of the two kinds of Bessel functions

$$X(z;\boldsymbol{\lambda}) = \left(J_\ell(z;+,\boldsymbol{\lambda})\right)_{\ell=1}^n, \quad Y_a(z;\boldsymbol{\lambda}) = \left(J(z;\boldsymbol{\lambda};\xi_{a,j})\right)_{j=1}^n,$$

and the matrices

$$\begin{aligned}
&\Sigma(\boldsymbol{\lambda}) = \left(\sigma_{\ell,n-j}(\boldsymbol{\lambda})\right)_{\ell,j=1}^n, \\
&E_a(\boldsymbol{\lambda}) = \operatorname{diag}\left(e^{\pi i\left(\frac{1}{2}n+2a-2\right)\lambda_\ell}\right)_{\ell=1}^n, \quad D_a(\boldsymbol{\lambda}) = \operatorname{diag}\left((-)^{n-j}\xi_{a,j}^{-\frac{n-1}{2}-|\boldsymbol{\lambda}|}\right)_{j=1}^n.
\end{aligned}$$

Then the formula (2.7.6) may be written as

$$X(z;\boldsymbol{\lambda}) = \frac{e^{\frac{3}{4}\pi i((n-1)+2|\boldsymbol{\lambda}|)}}{\sqrt{n}(2\pi)^{\frac{n-1}{2}}} \cdot E_a(\boldsymbol{\lambda})\Sigma(\boldsymbol{\lambda})D_a(\boldsymbol{\lambda})Y_a(z;\boldsymbol{\lambda}). \tag{2.7.7}$$

We now formulate (2.7.3) as

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}(z) = (-)^{|\boldsymbol{m}|} e^{-\frac{1}{2}\pi i(n-1)} \left(4\pi^2\right)^{n-1} \cdot {}^tX\left(2\pi z^{\frac{1}{n}};\boldsymbol{\lambda}^+_{(\boldsymbol{\mu},\boldsymbol{m})}\right) S_{(\boldsymbol{\mu},\boldsymbol{m})} X\left(2\pi \bar{z}^{\frac{1}{n}};\boldsymbol{\lambda}^-_{(\boldsymbol{\mu},\boldsymbol{m})}\right), \tag{2.7.8}$$

with $\boldsymbol{\lambda}^\pm_{(\boldsymbol{\mu},\boldsymbol{m})} = \boldsymbol{\mu} \pm \frac{1}{2}\boldsymbol{m}$ and

$$S_{(\boldsymbol{\mu},\boldsymbol{m})} = \operatorname{diag}\left(\tau_\ell\left(\boldsymbol{\lambda}^\pm_{(\boldsymbol{\mu},\boldsymbol{m})}\right)^{-1} e^{-\pi i((n-2)\mu_\ell \mp m_\ell)}\right)_{\ell=1}^n.$$

We insert into (2.7.8) the formulae of $X\left(2\pi z^{\frac{1}{n}};\boldsymbol{\lambda}^+_{(\boldsymbol{\mu},\boldsymbol{m})}\right)$ and $X\left(2\pi \bar{z}^{\frac{1}{n}};\boldsymbol{\lambda}^-_{(\boldsymbol{\mu},\boldsymbol{m})}\right)$ given by (2.7.7), with $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+_{(\boldsymbol{\mu},\boldsymbol{m})}$, $a = 0$ in the former and $\boldsymbol{\lambda} = \boldsymbol{\lambda}^-_{(\boldsymbol{\mu},\boldsymbol{m})}$, $a = 1-r$, for $r = 0, 1, ..., n$, in the latter. Then follows the formula

$$\begin{aligned}
&J_{(\boldsymbol{\mu},\boldsymbol{m})}(z) = (-)^{(n-1)+|\boldsymbol{m}|}\frac{(2\pi)^{n-1}}{n} \\
&{}^tY_0\left(2\pi z^{\frac{1}{n}};\boldsymbol{\lambda}^+_{(\boldsymbol{\mu},\boldsymbol{m})}\right) D_0\left(\boldsymbol{\lambda}^+_{(\boldsymbol{\mu},\boldsymbol{m})}\right) {}^t\Sigma_{(\boldsymbol{\mu},\boldsymbol{m})} R_{(\boldsymbol{\mu},\boldsymbol{m})} \Sigma_{(\boldsymbol{\mu},\boldsymbol{m})} D_{1-r}\left(\boldsymbol{\lambda}^-_{(\boldsymbol{\mu},\boldsymbol{m})}\right) Y_{1-r}\left(2\pi \bar{z}^{\frac{1}{n}};\boldsymbol{\lambda}^-_{(\boldsymbol{\mu},\boldsymbol{m})}\right),
\end{aligned} \tag{2.7.9}$$

where

$$
\begin{aligned}
\Sigma_{(\mu,m)} &= \Sigma\left(\lambda^{+}_{(\mu,m)}\right) = \Sigma\left(\lambda^{-}_{(\mu,m)}\right),\\
R_{(\mu,m)} &= E_0\left(\lambda^{+}_{(\mu,m)}\right) S_{(\mu,m)} E_{1-r}\left(\lambda^{-}_{(\mu,m)}\right) = \operatorname{diag}\left(\tau_\ell\left(\lambda^{\pm}_{(\mu,m)}\right)^{-1} e^{-2\pi i r \lambda^{\pm}_{(\mu,m),\ell}}\right)_{\ell=1}^{n}.
\end{aligned}
$$

We are therefore reduced to computing the matrix ${}^t\Sigma_{(\mu,m)}R_{(\mu,m)}\Sigma_{(\mu,m)}$. For this, we have the following lemma.

Lemma 2.7.3. *Let $x = (x_1, ..., x_n) \in \mathbb{C}^n$ be a generic n-tuple in the sense that all its components are distinct. Let $\sigma_{\ell,d}$, respectively σ_d, denote the elementary symmetric polynomial in $x_1, ..., \widehat{x}_\ell, ..., x_n$, respectively $x_1, ..., x_n$, of degree d, and let $\tau_\ell = \prod_{h\neq\ell}(x_\ell - x_h)$. Define the matrices $\Sigma = (\sigma_{\ell,n-j})_{\ell,j=1}^{n}$, $X = \operatorname{diag}(x_\ell)_{\ell=1}^{n}$ and $T = \operatorname{diag}\left(\tau_\ell^{-1}\right)_{\ell=1}^{n}$. Then, for any $r = 0, 1, ..., n$, the matrix ${}^t\Sigma X^r T\Sigma$ can be written as*

$$
\begin{pmatrix} (-)^{n-r}A & 0 \\ 0 & (-)^{n-r+1}B \end{pmatrix},
$$

where

$$
A = \begin{pmatrix} 0 & \cdots & 0 & \sigma_n \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & \iddots & \sigma_{n-r+2} \\ \sigma_n & \cdots & \sigma_{n-r+2} & \sigma_{n-r+1} \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_{n-r-1} & \sigma_{n-r-2} & \cdots & \sigma_0 \\ \sigma_{n-r-2} & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ \sigma_0 & 0 & \cdots & 0 \end{pmatrix}.
$$

More precisely, the (k, j)-th entry $a_{k,j}$, $k, j = 1, ..., r$, of A is given by

$$
a_{k,j} = \begin{cases} \sigma_{n+r-k-j+1} & \textit{if } k + j \geqslant r + 1, \\ 0 & \textit{if otherwise}, \end{cases}
$$

whereas the (k, j)-th entry $b_{k,j}$, $k, j = 1, ..., n - r$, of B is given by

$$
b_{k,j} = \begin{cases} \sigma_{n-r-k-j+1} & \textit{if } k + j \leqslant n - r + 1, \\ 0 & \textit{if otherwise}. \end{cases}
$$

Proof of Lemma 2.7.3. Appealing to the Lagrange interpolation formula, we find in Lemma 1.8.5 that the inverse of $T\Sigma$ is equal to the matrix $U = \left((-)^{n-j} x_\ell^{j-1} \right)_{j,\ell=1}^{n}$. Therefore, it suffices to show that

$$ {}^t\Sigma X^r = \begin{pmatrix} (-)^{n-r} A & 0 \\ 0 & (-)^{n-r+1} B \end{pmatrix} U. $$

This is equivalent to the following two collections of identities,

$$ \begin{aligned} \sum_{j=r-k+1}^{r} (-)^{r+j} \sigma_{n+r-k-j+1} x_\ell^{j-1} &= \sigma_{\ell,n-k} x_\ell^r, \quad k = 1, ..., r, \\ \sum_{j=1}^{n-r-k+1} (-)^{j-1} \sigma_{n-r-k-j+1} x_\ell^{r+j-1} &= \sigma_{\ell,n-r-k} x_\ell^r, \quad k = 1, ..., n-r, \end{aligned} $$

which are further equivalent to

$$ \begin{aligned} \sum_{j=1}^{k} (-)^{k+j} \sigma_{n-j+1} x_\ell^{j-k-1} &= \sigma_{\ell,n-k}, \quad k = 1, ..., r, \\ \sum_{j=1}^{k} (-)^{j-1} \sigma_{k-j} x_\ell^{j-1} &= \sigma_{\ell,k-1}, \quad k = 1, ..., n-r. \end{aligned} $$

The last two identities can be easily seen, actually for all $k = 1, ..., n$, from computing the coefficients of x^{k-1} and x^{2n-k} on the two sides of

$$ \begin{aligned} \prod_{h \neq \ell} (x - x_h) &= \left(\sum_{p=0}^{\infty} x_\ell^p x^{-p-1} \right) \prod_{h=1}^{n} (x - x_h), \\ (x^n - x_\ell^n) \prod_{h \neq \ell} (x - x_h) &= \left(\sum_{p=1}^{n} x_\ell^{p-1} x^{n-p} \right) \prod_{h=1}^{n} (x - x_h), \end{aligned} $$

respectively. Q.E.D.

Applying Lemma 2.7.3 with $x_\ell = e^{-2\pi i \lambda^{\pm}_{(\mu,m),\ell}} = (-)^{m_\ell} e^{-2\pi i \mu_\ell}$ to the formula (2.7.9), we arrive at the following theorem.

Theorem 2.7.4. *Let* $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ *and* $r \in \{0, 1, .., n\}$. *Define* $\xi_j = e^{2\pi i \frac{j-1}{n}}$, $\zeta_j = e^{2\pi i \frac{j-r}{n}}$, *and denote by* $\sigma^d_{(\boldsymbol{\mu},\boldsymbol{m})}$ *the elementary symmetric polynomial in* $(-)^{m_1} e^{-2\pi i \mu_1}$, ..., $(-)^{m_n} e^{-2\pi i \mu_n}$

of degree d, with $j = 1, ..., n$ and $d = 0, 1, ..., n$. Then we have

$$
\begin{aligned}
J_{(\boldsymbol{\mu},\boldsymbol{m})}(z) = (-)^{|\boldsymbol{m}|}\frac{(2\pi)^{n-1}}{n} & \sum\sum_{\substack{k,j=1,...,r\\ k+j\geqslant r+1}} C_{k,j}(\boldsymbol{\mu},\boldsymbol{m}) \\
& \qquad J\left(2\pi z^{\frac{1}{n}};\boldsymbol{\mu}+\tfrac{1}{2}\boldsymbol{m};\xi_k\right) J\left(2\pi \overline{z}^{\frac{1}{n}};\boldsymbol{\mu}-\tfrac{1}{2}\boldsymbol{m};\zeta_j\right) \\
+ (-)^{|\boldsymbol{m}|}\frac{(2\pi)^{n-1}}{n} & \sum\sum_{\substack{k,j=1,...,n-r\\ k+j\leqslant n-r+1}} D_{k,j}(\boldsymbol{\mu},\boldsymbol{m}) \\
& \qquad J\left(2\pi z^{\frac{1}{n}};\boldsymbol{\mu}+\tfrac{1}{2}\boldsymbol{m};\xi_{r+k}\right) J\left(2\pi \overline{z}^{\frac{1}{n}};\boldsymbol{\mu}-\tfrac{1}{2}\boldsymbol{m};\zeta_{r+j}\right).
\end{aligned} \tag{2.7.10}
$$

with

$$C_{k,j}(\boldsymbol{\mu},\boldsymbol{m}) = (-)^{r+k+j+1}\xi_k^{-\frac{n-1}{2}-\frac{1}{2}|\boldsymbol{m}|}\zeta_j^{-\frac{n-1}{2}+\frac{1}{2}|\boldsymbol{m}|}\sigma_{(\boldsymbol{\mu},\boldsymbol{m})}^{n+r-k-j+1}, \tag{2.7.11}$$

$$D_{k,j}(\boldsymbol{\mu},\boldsymbol{m}) = (-)^{r+k+j}\xi_{r+k}^{-\frac{n-1}{2}-\frac{1}{2}|\boldsymbol{m}|}\zeta_{r+j}^{-\frac{n-1}{2}+\frac{1}{2}|\boldsymbol{m}|}\sigma_{(\boldsymbol{\mu},\boldsymbol{m})}^{n-r-k-j+1}. \tag{2.7.12}$$

Lemma 2.7.5. *We retain the notations in Theorem 2.7.4. Moreover, we define* $\Im(\boldsymbol{\mu}) = \max\{|\Im\mathfrak{m}\,\mu_\ell|\}$.

(1.1). *For $k = 1, ..., r$, we have $C_{k,r-k+1}(\boldsymbol{\mu},\boldsymbol{m}) = (-\overline{\xi}_k)^{|\boldsymbol{m}|}$.*

(1.2). *Let $k, j = 1, ..., r$ be such that $k + j \geqslant r + 2$. Denote $p = k + j - r - 1$. We have the estimate*

$$|C_{k,j}(\boldsymbol{\mu},\boldsymbol{m})| \leqslant \binom{n}{p}\exp\left(2\pi\min\{n-p,p\}\,\Im(\boldsymbol{\mu})\right).$$

(2.1). *For $k = 1, ..., n - r$, we have $D_{k,n-r-k+1}(\boldsymbol{\mu},\boldsymbol{m}) = (-\overline{\xi}_{k+r})^{|\boldsymbol{m}|}$.*

(2.2). *Let $k, j = 1, ..., n - r$ be such that $k + j \leqslant n - r$. Denote $p = n - r - k - j + 1$. We have the estimate*

$$|D_{k,j}(\boldsymbol{\mu},\boldsymbol{m})| \leqslant \binom{n}{p}\exp\left(2\pi\min\{n-p,p\}\,\Im(\boldsymbol{\mu})\right).$$

2.7.3. The rank-two case

Example 2.7.6. *Let $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}$.*

If we define

$$J_{\mu,m}(z) = J_{-2\mu-\frac{1}{2}m}\,(z)\, J_{-2\mu+\frac{1}{2}m}\,(\overline{z})\,, \tag{2.7.13}$$

then

$$J_{\mu,m}(z) = J_{-2\mu-\frac{1}{2}m}\,(z)\, J_{-2\mu+\frac{1}{2}m}\,(\overline{z})\,, \tag{2.7.14}$$

then

$$J_{(\mu,-\mu,m,0)}\,(z) = \begin{cases} \dfrac{2\pi^2}{\sin(2\pi\mu)}[\sqrt{z}]^{-m}\left(J_{\mu,m}(4\pi\sqrt{z}) - J_{-\mu,-m}(4\pi\sqrt{z})\right) & \textit{if } m \textit{ is even,} \\ \dfrac{2\pi^2 i}{\cos(2\pi\mu)}[\sqrt{z}]^{-m}\left(J_{\mu,m}(4\pi\sqrt{z}) + J_{-\mu,-m}(4\pi\sqrt{z})\right) & \textit{if } m \textit{ is odd,} \end{cases} \tag{2.7.15}$$

which should be interpreted in the way as in Theorem 2.7.1. *We remark that the generic case is when* $4\mu \in 2\mathbb{Z} + m$*.*

On the other hand, using the connection formulae ([Wat, 3.61 (1, 2)])

$$J_\nu(z) = \frac{H^{(1)}_\nu(z) + H^{(2)}_\nu(z)}{2}, \quad J_{-\nu}(z) = \frac{e^{\pi i\nu}H^{(1)}_\nu(z) + e^{-\pi i\nu}H^{(2)}_\nu(z)}{2},$$

one obtains

$$J_{(\mu,-\mu,m,0)}(z) = \pi^2 i[\sqrt{z}]^{-m}\left(e^{2\pi i\mu}H^{(1)}_{\mu,m}\,(4\pi\sqrt{z}) + (-)^{m+1}e^{-2\pi i\mu}H^{(2)}_{\mu,m}\,(4\pi\sqrt{z})\right), \tag{2.7.16}$$

with the definition

$$H^{(1,2)}_{\mu,m}(z) = H^{(1,2)}_{2\mu+\frac{1}{2}m}\,(z)\, H^{(1,2)}_{2\mu-\frac{1}{2}m}\,(\overline{z})\,. \tag{2.7.17}$$

2.8. The asymptotic expansion of $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$

The asymptotic of $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}(x)$ has already been established in [Qi1, Theorem 5.13, 9.4]. In the following, we shall present the asymptotic expansion of $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$.

First of all, we have the following proposition on the asymptotic expansion of $J(z;\boldsymbol{\lambda};\xi)$, which is in substance [Qi1, Theorem 7.27].

Proposition 2.8.1. *Let $\boldsymbol{\lambda} \in \mathbb{C}^n$ and define $\mathfrak{C}(\boldsymbol{\lambda}) = \max\left\{\left|\lambda_l - \frac{1}{n}|\boldsymbol{\lambda}|\right| + 1\right\}$. Let ξ be a $2n$-th root of unity. For a small positive constant ϑ, say $0 < \vartheta < \frac{1}{2}\pi$, we define the sector*

$$\mathbb{S}'_\xi(\vartheta) = \left\{z : \left|\arg z - \arg(i\overline{\xi})\right| < \pi + \frac{\pi}{n} - \vartheta\right\}.$$

For a positive integer A, we have the asymptotic expansion

$$J(z;\boldsymbol{\lambda};\xi) = e^{in\xi z} z^{-\frac{n-1}{2}-|\boldsymbol{\lambda}|}\left(\sum_{\alpha=0}^{A-1}(i\xi)^{-\alpha}B_\alpha\left(\boldsymbol{\lambda} - \tfrac{1}{n}|\boldsymbol{\lambda}|\boldsymbol{e}^n\right)z^{-\alpha} + O_{A,\vartheta,n}\left(\mathfrak{C}(\boldsymbol{\lambda})^{2A}|z|^{-A}\right)\right)$$

for all $z \in \mathbb{S}'_\xi(\vartheta)$ such that $|z| \gg_{A,\vartheta,n} \mathfrak{C}(\boldsymbol{\lambda})^2$. Here $B_\alpha(\boldsymbol{\lambda})$ is a certain symmetric polynomial function in $\boldsymbol{\lambda} \in \mathbb{L}^{n-1}$ of degree 2α, with $B_0(\boldsymbol{\lambda}) = 1$.

Lemma 2.8.2. *Let r be a positive integer. Suppose that either $n = 2r$ or $n = 2r - 1$. Put $\vartheta_n = \frac{1}{n}\pi$ if $n = 2r$ and $\vartheta_n = \frac{1}{2n}\pi$ if $n = 2r - 1$. For a given constant $0 < \vartheta < \vartheta_n$ define the sector*

$$\mathbb{S}_n(\vartheta) = \begin{cases} \left\{z : -\dfrac{\pi}{2} - \dfrac{\pi}{n} + \vartheta < \arg z < -\dfrac{\pi}{2} + \dfrac{3\pi}{n} - \vartheta\right\} & \text{if } n = 2r, \\ \left\{z : -\dfrac{\pi}{2} - \dfrac{\pi}{n} + \vartheta < \arg z < -\dfrac{\pi}{2} + \dfrac{2\pi}{n} - \vartheta\right\} & \text{if } n = 2r - 1, \end{cases}$$

Let $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ and define $\mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m}) = \max\left\{|\mu_l| + 1, \left|m_l - \frac{1}{n}|\boldsymbol{m}|\right| + 1\right\}$. Define $\xi_j = e^{2\pi i \frac{j-1}{n}}$ and $\zeta_j = e^{2\pi i \frac{j-r}{n}}$ for $j = 1, ..., n$. Then, for any $z \in \mathbb{S}_n(\vartheta)$ such that $|z| \gg_{A,\vartheta,n} \mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m})^2$, we have

$$J\left(2\pi z; \boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{m}; \xi_k\right) J\left(2\pi\overline{z}; \boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{m}; \zeta_j\right) = \frac{e\left(n\left(\xi_k z + \zeta_j \overline{z}\right)\right)}{(2\pi)^{n-1}|z|^{n-1}[z]^{|\boldsymbol{m}|}}$$
$$\left(\sum\sum_{\substack{\alpha,\beta=0,\dots,A-1 \\ \alpha+\beta\leqslant A-1}} (i\xi_k)^{-\alpha}(i\zeta_j)^{-\beta}B_{\alpha,\beta}(\boldsymbol{\mu},\boldsymbol{m})z^{-\alpha}\overline{z}^{-\beta} + O_{A,\vartheta,n}\left(\mathfrak{C}(\boldsymbol{\mu},\boldsymbol{m})^{2A}|z|^{-A}\right)\right),$$

with

$$B_{\alpha,\beta}(\boldsymbol{\mu},\boldsymbol{m}) = B_\alpha\left(\boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{m} - \tfrac{1}{2n}|\boldsymbol{m}|\boldsymbol{e}^n\right)B_\beta\left(\boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{m} + \tfrac{1}{2n}|\boldsymbol{m}|\boldsymbol{e}^n\right), \quad \alpha,\beta \in \mathbb{N},$$

where $B_\alpha(\boldsymbol{\lambda})$ is the polynomial function in $\boldsymbol{\lambda}$ of degree 2α given in Proposition 2.8.1.

Proof. Recall that, for an integer a, we defined $\xi_{a,j} = e^{2\pi i \frac{j+a-1}{n}}$. Note that $\xi_j = \xi_{0,j}$ and $\zeta_j = \xi_{1-r,j}$. It is clear that

$$\bigcap_{j=1}^{n} \mathbb{S}'_{\xi_{a,j}}(\vartheta) = \left\{ z : -\frac{\pi}{2} - \frac{2a+1}{n}\pi + \vartheta < \arg z < -\frac{\pi}{2} - \frac{2a-3}{n}\pi - \vartheta \right\}.$$

We denote this sector by $\mathbb{S}'_a(\vartheta)$. Observe that, when $n = 2r$ or $2r-1$, the intersection $\mathbb{S}'_0(\vartheta) \cap \overline{\mathbb{S}'_{1-r}(\vartheta)}$ is exactly the sector $\mathbb{S}_n(\vartheta)$. In other words, for all $j = 1, ..., n$, $z \in \mathbb{S}'_{\xi_j}(\vartheta)$ and $\overline{z} \in \mathbb{S}'_{\zeta_j}(\vartheta)$ both hold if $z \in \mathbb{S}_n(\vartheta)$. Therefore, Proposition 2.8.1 can be applied to yield the asymptotic expansion of $J\left(2\pi z; \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{m}; \xi_k\right) J\left(2\pi \overline{z}; \boldsymbol{\mu} - \frac{1}{2}\boldsymbol{m}; \zeta_j\right)$ as above. Q.E.D.

Remark 2.8.3. *In view of our choice of ϑ, the sector $\mathbb{S}_n(\vartheta)$ is of angle at least $\frac{2}{n}\pi$, and therefore the sector $\mathbb{S}_n(\vartheta)^n = \{z^n : z \in \mathbb{S}_n(\vartheta)\}$ covers the whole $\mathbb{C} \smallsetminus \{0\}$.*

Lemma 2.8.4. *Let notations be as in Lemma* 2.8.2*.*

(1.1). *For $k = 1, ..., r$, we have*

$$\Im\mathfrak{m}\ (\xi_k z + \zeta_{r-k+1}\overline{z}) = 0.$$

(1.2). *Let $k, j = 1, ..., r$ be such that $k + j \geqslant r + 2$. For any $z \in \mathbb{S}_n(\vartheta)$, we have*

$$\Im\mathfrak{m}\ (\xi_k z + \zeta_j \overline{z}) \geqslant 2\sin\left(\frac{k+j-r-1}{n}\pi\right)\sin\vartheta \cdot |z|.$$

(2.1). *For $k = 1, ..., n-r$, we have*

$$\Im\mathfrak{m}\ (\xi_{k+r} z + \zeta_{n-k+1}\overline{z}) = 0.$$

(2.2). *Let $k, j = 1, ..., n-r$ be such that $k + j \leqslant n - r$. For any $z \in \mathbb{S}_n(\vartheta)$, we have*

$$\Im\mathfrak{m}\ (\xi_{k+r} z + \zeta_{j+r}\overline{z}) \geqslant \begin{cases} 2\sin\left(\dfrac{n-r-k-j+1}{n}\pi\right)\sin\vartheta \cdot |z|, & \textit{if } n = 2r, \\ 2\sin\left(\dfrac{n-r-k-j+1}{n}\pi\right)\sin\left(\dfrac{\pi}{n} + \vartheta\right) \cdot |z|, & \textit{if } n = 2r-1. \end{cases}$$

Proof. We shall only prove (1.1) and (1.2) in the case $n = 2r$. The other cases follow in exactly the same way.

Write $z = xe^{i\phi}$. Since

$$\begin{aligned} \xi_k z + \zeta_j \bar{z} &= xe^{2\pi i \frac{k-1}{2r} + i\phi} + xe^{2\pi i \frac{j-r}{2r} - i\phi} \\ &= xe^{\pi i \frac{k+j-r-1}{2r}} \left(e^{\pi i \frac{k-j+r-1}{2r} + i\phi} + e^{-\pi i \frac{k-j+r-1}{2r} - i\phi} \right), \end{aligned}$$

(1.1) is then obvious (we also note that $\zeta_{r-k+1} = \bar{\xi}_k$), whereas (1.2) is equivalent to

$$\cos\left(\frac{k-j+r-1}{2r}\pi + \phi \right) \geqslant \sin\vartheta. \tag{2.8.1}$$

Observe that the condition $z \in \mathbb{S}_{2r}(\vartheta)$ amounts to

$$\left| \phi + \frac{\pi}{2} - \frac{\pi}{2r} \right| < \frac{\pi}{r} - \vartheta.$$

Moreover, under our assumptions on k and j in (1.2), one has $|k-j| \leqslant r-2$. Consequently, these yield the following estimate

$$\left| \frac{k-j+r-1}{2r}\pi + \phi \right| \leqslant \frac{r-2}{2r}\pi + \frac{\pi}{r} - \vartheta = \frac{\pi}{2} - \vartheta.$$

Thus (2.8.1) is proven. Q.E.D.

Remark 2.8.5. *In cases other than those listed in Lemma* 2.8.4, $\Im\mathrm{m}\ (\xi_k z + \zeta_j \bar{z})$ *can not always be nonnegative for all* $z \in \mathbb{S}_n(\vartheta)$. *Fortunately, these cases are excluded from the second connection formula for* $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ *in Theorem* 2.7.4.

Now the asymptotic expansion of $J_{(\boldsymbol{\mu},\boldsymbol{m})}(z)$ can be readily established using Theorem 2.7.4 along with Lemma 2.7.5, 2.8.2 and 2.8.4.

Theorem 2.8.6. *Denote by $\mathbb{X}_n$ the set of n-th roots of unity. Let $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ and define $\mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m}) = \max\left\{|\mu_l| + 1, \left|m_l - \frac{1}{n}|\boldsymbol{m}|\right| + 1\right\}$. Let A be a positive integer. Then*

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(z^n\right) = \sum_{\xi \in \mathbb{X}_n} \frac{e\big(n\big(\xi z + \overline{\xi z}\big)\big)}{n|z|^{n-1}[\xi z]^{|\boldsymbol{m}|}} \left(\mathop{\sum\sum}_{\substack{\alpha, \beta = 0, \dots, A-1 \\ \alpha + \beta \leqslant A-1}} i^{-\alpha-\beta} \xi^{-\alpha+\beta} B_{\alpha,\beta}(\boldsymbol{\mu}, \boldsymbol{m}) z^{-\alpha} \overline{z}^{-\beta} \right. \\ \left. + O_{A,n}\left(\mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m})^{2A} |z|^{-A-n+1}\right) \right),$$

if $|z| \geqslant_{A,n} \mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m})^2$, with the coefficient $B_{\alpha,\beta}(\boldsymbol{\mu}, \boldsymbol{m})$ given in Lemma 2.8.2.

We may also prove the following elaborate version of Theorem 2.8.6.

Theorem 2.8.7. *Let notations be as in Lemma* 2.8.2 *and Theorem* 2.8.6. *Let $\mathfrak{I}(\boldsymbol{\mu}) = \max\{|\mathfrak{Im}\, \mu_l|\}$. Then we may write*

$$J_{(\boldsymbol{\mu},\boldsymbol{m})}\left(z^n\right) = \sum_{\xi \in \mathbb{X}_n} \frac{e\big(n\big(\xi z + \overline{\xi z}\big)\big)}{n|z|^{n-1}[\xi z]^{|\boldsymbol{m}|}} W_{(\boldsymbol{\mu},\boldsymbol{m})}(z, \xi) + E_{(\boldsymbol{\mu},\boldsymbol{m})}(z),$$

such that

$$W_{(\boldsymbol{\mu},\boldsymbol{m})}(z, \xi) = \mathop{\sum\sum}_{\substack{\alpha, \beta = 0, \dots, A-1 \\ \alpha + \beta \leqslant A-1}} i^{-\alpha-\beta} \xi^{-\alpha-\beta} B_{\alpha,\beta}(\boldsymbol{\mu}, \boldsymbol{m}) z^{-\alpha} \overline{z}^{-\beta} + O_{A,n}\left(\mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m})^{2A} |z|^{-A}\right),$$

and

$$E_{(\boldsymbol{\mu},\boldsymbol{m})}(z) = O_n\Big(|z|^{-n+1} \exp\left(2\pi \mathfrak{I}(\boldsymbol{\mu}) - 4\pi n \sin\left(\tfrac{1}{n}\pi\right) \sin\vartheta |z|\right)\Big),$$

for $z \in \mathbb{S}_n(\vartheta)$ with $|z| \geqslant_{A,n} \mathfrak{C}(\boldsymbol{\mu}, \boldsymbol{m})^2$. Moreover, $E_{(\boldsymbol{\mu},\boldsymbol{m})}(z) \equiv 0$ when $n = 1, 2$.

2.A. Hankel transforms from the representation theoretic viewpoint

We shall start with a brief review of Hankel transforms over an *archimedean* local field[XVIII] in the work of Ichino and Templier [IT] on the Voronoĭ summation formula. For

[XVIII]For a nonarchimedean local field, Hankel transforms can also be constructed in the same way.

the theory of L-functions and local functional equations over a local field the reader is referred to Cogdell's survey [Cog]. We shall then discuss Hankel transforms using the Langlands classification. For this, Knapp's article [Kna] is used as our reference, with some change of notations for our convenience.

Let $\mathbb{F}$ be an archimedean local field with normalized absolute value $\| \ \| = \| \ \|_{\mathbb{F}}$ defined as in §2.2.2, and let ψ be a given additive character on $\mathbb{F}$. Let $\eta(x) = \mathrm{sgn}(x)$ for $x \in \mathbb{R}^\times$ and $\eta(z) = [z]$ for $z \in \mathbb{C}^\times$. We view η as a unitary character on $\mathbb{F}^\times$.

Suppose for the moment $n \geqslant 2$. Let π be an infinite dimensional irreducible admissible generic representation of $\mathrm{GL}_n(\mathbb{F})$[XIX], and $\mathcal{W}(\pi, \psi)$ be the ψ-Whittaker model of π. Denote by ω_π the central character of π. Recall that the γ-factor $\gamma(s, \pi, \psi)$ of π is given by

$$\gamma(s, \pi, \psi) = \epsilon(s, \pi, \psi) \frac{L(1 - s, \widetilde{\pi})}{L(s, \pi)}$$

where $\widetilde{\pi}$ is the contragradient representation of π, $\epsilon(s, \pi, \psi)$ and $L(s, \pi)$ are the ϵ-factor and the L-function of π respectively.

To a smooth compactly supported function w on $\mathbb{F}^\times$ we associate a dual function $\widetilde{w}$ on $\mathbb{F}^\times$ defined by [IT, (1.1)],

$$\begin{aligned} \int_{\mathbb{F}^\times} & \widetilde{w}(x) \chi(x)^{-1} \|x\|^{s - \frac{n-1}{2}} d^\times x \\ & = \chi(-1)^{n-1} \gamma(1 - s, \pi \otimes \chi, \psi) \int_{\mathbb{F}^\times} w(x) \chi(x) \|x\|^{1 - s - \frac{n-1}{2}} d^\times x, \end{aligned} \tag{2.A.1}$$

for all s of real part sufficiently large and all unitary multiplicative characters χ of $\mathbb{F}^\times$. (2.A.1) is independent of the chosen Haar measure $d^\times x$ on $\mathbb{F}^\times$, and uniquely defines $\widetilde{w}$ in terms of π, ψ and w. We shall let the Haar measure be given as in §2.2.2. We call $\widetilde{w}$ the Hankel transform of w associated with π.

[XIX] Since π is a local component of an irreducible cuspidal automorphic representation in [IT], [IT] also assumes that π is unitary. However, if one only considers the local theory, this assumption is not necessary.

According to [IT, Lemma 5.1], there exists a smooth Whitaker function $W \in \mathcal{W}(\pi, \psi)$ so that

$$w(x) = W\left(\begin{pmatrix} x & \\ & I_{n-1} \end{pmatrix}\right), \tag{2.A.2}$$

for all $x \in \mathbb{F}^\times$. Denote by w_n the n-by-n permutation matrix whose anti-diagonal entries are 1, that is, the longest Weyl element of rank n, and define

$$w_{n,1} = \begin{pmatrix} 1 & \\ & w_{n-1} \end{pmatrix}.$$

In the theory of integral representations of Rankin-Selberg L-functions, (2.A.1) amounts to the local functional equations of zeta integrals for $\pi \otimes \chi$, with

$$\widetilde{w}(x) = \widetilde{W}\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) = W\left(w_2 \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix}\right), \tag{2.A.3}$$

if $n = 2$, and

$$\widetilde{w}(x) = \int_{\mathbb{F}^{n-2}} \widetilde{W}\left(\begin{pmatrix} x & & \\ y & I_{n-2} & \\ & & 1 \end{pmatrix} w_{n,1}\right) dy_\psi, \tag{2.A.4}$$

if $n \geqslant 3$, where $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$ is the dual Whittaker function defined by $\widetilde{W}(g) = W(w_n \cdot {}^t g^{-1})$, for $g \in \mathrm{GL}_n(\mathbb{F})$, and dx_ψ denotes the self-dual additive Haar measure on $\mathbb{F}$ with respect to ψ. See [IT, Lemma 2.3].

The constraint that π be *infinite dimensional* and *generic* is actually dispensable for defining the Hankel transform via (2.A.1). In the following, we shall assume that π is any irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$. Moreover, we shall also include the case $n = 1$. It will be seen that, after renormalizing the functions w and $\widetilde{w}$, the Hankel transform defined by (2.A.1) turns into the Hankel transform given by (2.4.34) or (2.4.43). For this, we shall apply the Langlands classification for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{F})$.

2.A.1. Hankel transforms over $\mathbb{R}$

Suppose $\mathbb{F} = \mathbb{R}$. Recall that $\| \|_{\mathbb{R}} = | |$ is the ordinary absolute value. For $r \in \mathbb{R}^\times$ let $\psi(x) = \psi_r(x) = e(rx)$.

According to [Kna, §3, Lemma], every finite dimensional semisimple representation φ of the Weil group of $\mathbb{R}$ may be decomposed into irreducible representations of dimension one or two. The one-dimensional representations are parametrized by $(\mu, \delta) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$. We denote by $\varphi_{(\mu,\delta)}$ the representation given by (μ, δ). $\varphi_{(\mu,\delta)}$ corresponds to the representation $\chi_{(\mu,\delta)} = \eta^\delta | |^\mu$ of $\mathrm{GL}_1(\mathbb{R})$ under the Langlands correspondence over $\mathbb{R}$. The irreducible two-dimensional representations are parametrized by $(\mu, m) \in \mathbb{C} \times \mathbb{N}_+$. We denote by $\varphi_{(\mu,m)}$ the representation given by (μ, m). $\varphi_{(\mu,m)}$ corresponds to the representation $D_m \otimes |\det|_{\mathbb{R}}^\mu$ of $\mathrm{GL}_2(\mathbb{R})$, where D_m denotes the discrete series representation of weight m.

In view of the formulae [Kna, (3.6, 3.7)][XX] of L-functions and ϵ-factors, the definitions of G_δ and G_m in (2.2.3) and (2.2.6), along with the formula (2.2.10), we deduce that

$$\gamma(s, \varphi_{(\mu,\delta)}, \psi) = \mathrm{sgn}(r)^\delta |r|^{s+\mu-\frac{1}{2}} G_\delta(1 - s - \mu), \tag{2.A.5}$$

whereas

$$\gamma(s, \varphi_{(\mu,m)}, \psi) = \mathrm{sgn}(r)^{\delta(m)+1} |r|^{2s+2\mu-1} i G_m(1 - s - \mu), \tag{2.A.6}$$

and

$$\begin{aligned} \gamma(s, \varphi_{(\mu,m)}, \psi) &= \gamma(s, \varphi_{(\mu+\frac{1}{2}m, \delta(m)+1)}, \psi)\gamma(s, \varphi_{(\mu-\frac{1}{2}m, 0)}, \psi) \\ &= \gamma(s, \varphi_{(\mu+\frac{1}{2}m, \delta(m))}, \psi)\gamma(s, \varphi_{(\mu-\frac{1}{2}m, 1)}, \psi). \end{aligned} \tag{2.A.7}$$

To $\varphi_{(\mu,m)}$ we shall attach either one of the following two parameters

$$\left(\mu + \tfrac{1}{2}m, \mu - \tfrac{1}{2}m, \delta(m) + 1, 0\right), \ \left(\mu + \tfrac{1}{2}m, \mu - \tfrac{1}{2}m, \delta(m), 1\right). \tag{2.A.8}$$

[XX] The formulae in [Kna, (3.6, 3.7)] are for ψ_1. The relation between the epsilon factors $\epsilon(s, \pi, \psi_r)$ and $\epsilon(s, \pi, \psi)$ is given in [Tat, §3] (see in particular [Tat, (3.6.6)]).

Remark 2.A.1. (2.A.7) *reflects the isomorphism* $\varphi_{(0,m)} \otimes \varphi_{(0,1)} \cong \varphi_{(0,m)}$ *of representations of the Weil group* (*here* $(0,1)$ *is an element of* $\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$), *as well as the isomorphism* $D_m \otimes \eta \cong D_m$ *of representations of* $\mathrm{GL}_2(\mathbb{R})$.

For φ reducible, $\gamma(s, \varphi, \psi)$ is the product of the γ-factors of the irreducible constituents of φ. Suppose φ is n-dimensional. It follows from (2.A.5, 2.A.6, 2.A.7) that there is a parameter $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ attached to φ such that

$$\gamma(s, \varphi, \psi) = \mathrm{sgn}(r)^{|\boldsymbol{\delta}|} |r|^{n\left(s - \frac{1}{2}\right) + |\boldsymbol{\mu}|} G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(1 - s). \tag{2.A.9}$$

The irreducible constituents of φ are unique up to permutation, but, in view of the two different parameters attached to $\varphi_{(\mu,m)}$ in (2.A.8), the parameter $(\boldsymbol{\mu}, \boldsymbol{\delta})$ attached to φ may not.

Suppose that π corresponds to φ under the Langlands correspondence over $\mathbb{R}$. We have $\gamma(s, \pi, \psi) = \gamma(s, \varphi, \psi)$. It is known that π is an irreducible constituent of the principal series representation unitarily induced from the character $\bigotimes_{\ell=1}^{n} \chi_{(\mu_\ell, \delta_\ell)}$ of the Borel subgroup. In particular,

$$\omega_\pi(x) = \mathrm{sgn}(x)^{|\boldsymbol{\delta}|} |x|^{|\boldsymbol{\mu}|}. \tag{2.A.10}$$

Now let $\chi = \chi_{(0,\delta)} = \eta^\delta$ in (2.A.1), $\delta \in \mathbb{Z}/2\mathbb{Z}$. In view of (2.A.9) and (2.A.10), one has the following expression of the γ-factor in (2.A.1),

$$\gamma(1 - s, \pi \otimes \eta^\delta, \psi) = \omega_\pi(r) \left(\mathrm{sgn}(r)^\delta |r|^{\frac{1}{2} - s} \right)^n G_{(\boldsymbol{\mu}, \boldsymbol{\delta} + \delta \boldsymbol{e}^n)}(s). \tag{2.A.11}$$

Some calculations show that (2.A.1) is exactly translated into (2.4.34) if one let

$$\begin{aligned} \upsilon(x) &= \omega_\pi(r) w\left(|r|^{-\frac{n}{2}} x\right) |x|^{-\frac{n-1}{2}}, \\ \Upsilon(x) &= \widetilde{w}\left((-)^{n-1} \mathrm{sgn}(r)^n |r|^{-\frac{n}{2}} x\right) |x|^{-\frac{n-1}{2}}. \end{aligned} \tag{2.A.12}$$

Then, (2.4.39) can be reformulated as

$$\widetilde{w}\left((-)^{n-1} x\right) = \omega_\pi(r) |r|^{\frac{n}{2}} |x|^{\frac{n-1}{2}} \int_{\mathbb{R}^\times} w(y) J_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(r^n x y) |y|^{1 - \frac{n-1}{2}} d^\times y. \tag{2.A.13}$$

2.A.2. Hankel transforms over $\mathbb{C}$

Suppose $\mathbb{F} = \mathbb{C}$. Recall that $\| \|_{\mathbb{C}} = \| \| = | |^2$, where $| |$ denotes the ordinary absolute value. For $r \in \mathbb{C}^\times$ let $\psi(z) = \psi_r(z) = e(rz + \overline{rz})$.

The Langlands classification and correspondence for $\mathrm{GL}_n(\mathbb{C})$ are less complicated. First of all, the Weil group of $\mathbb{C}$ is simply $\mathbb{C}^\times$. Any n-dimensional semisimple representation φ of the Weil group $\mathbb{C}^\times$ is the direct sum of one-dimensional representations. The one-dimensional representations are of the form $\chi_{(\mu,m)} = \eta^m \| \|^\mu$, with $(\mu, m) \in \mathbb{C} \times \mathbb{Z}$. In view of the formulae [Kna, (4.6, 4.7)] of $L(s, \chi_{(\mu,m)})$ and $\epsilon(s, \chi_{(\mu,m)}, \psi)$ as well as the definition of G_m in (2.2.6), we have

$$\gamma(s, \chi_{(\mu,m)}, \psi) = [r]^m \|r\|^{s+\mu-\frac{1}{2}} G_m(1 - s - \mu). \tag{2.A.14}$$

Thus φ is parametrized by some $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and

$$\gamma(s, \varphi, \psi) = [r]^{|\boldsymbol{m}|} \|r\|^{n\left(s-\frac{1}{2}\right)+|\boldsymbol{\mu}|} G_{(\boldsymbol{\mu},\boldsymbol{m})}(1 - s). \tag{2.A.15}$$

This parametrization is unique up to permutation, in contrast to the case $\mathbb{F} = \mathbb{R}$.

If π corresponds to φ under the Langlands correspondence over $\mathbb{C}$, then $\gamma(s, \pi, \psi) = \gamma(s, \varphi, \psi)$. Moreover, π is an irreducible constituent of the principal series representation unitarily induced from the character $\bigotimes_{\ell=1}^n \chi_{(\mu_\ell, m_\ell)}$ of the Borel subgroup. Note that

$$\omega_\pi(z) = [z]^{|\boldsymbol{m}|} \|z\|^{|\boldsymbol{\mu}|}. \tag{2.A.16}$$

Now let $\chi = \chi_{(0,m)} = \eta^m$ in (2.A.1), $m \in \mathbb{Z}$. Then (2.A.15) and (2.A.16) imply

$$\gamma(1 - s, \pi \otimes \eta^m, \psi) = \omega_\pi(r) \left([r]^m \|r\|^{\frac{1}{2}-s} \right)^n G_{(\boldsymbol{\mu}, \boldsymbol{m}+m\boldsymbol{e}^n)}(s). \tag{2.A.17}$$

By putting

$$\begin{aligned} \upsilon(z) &= \omega_\pi(r) w \left(\|r\|^{-\frac{n}{2}} z \right) \|z\|^{-\frac{n-1}{2}}, \\ \Upsilon(z) &= \widetilde{w} \left((-)^{n-1} [r]^{-n} \|r\|^{-\frac{n}{2}} z \right) \|z\|^{-\frac{n-1}{2}}, \end{aligned} \tag{2.A.18}$$

the identity (2.A.1) is translated into (2.4.43), and (2.4.49) can be reformulated as

$$\widetilde{w}\left((-)^{n-1} z\right) = \omega_\pi(r)\|r\|^{\frac{n}{2}}\|z\|^{\frac{n-1}{2}} \int_{\mathbb{C}^\times} w(u)\, J_{(\boldsymbol{\mu},\boldsymbol{m})}(r^n z u)\|u\|^{1-\frac{n-1}{2}} d^\times u. \tag{2.A.19}$$

2.A.3. Some new notations

Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$. For $\mathbb{F} = \mathbb{R}$, respectively $\mathbb{F} = \mathbb{C}$, if π is parametrized by $(\boldsymbol{\mu}, \boldsymbol{\delta})$, respectively $(\boldsymbol{\mu}, \boldsymbol{m})$, we shall denote simply by J_π the Bessel kernel $J_{(\boldsymbol{\mu},\boldsymbol{\delta})}$, respectively $J_{(\boldsymbol{\mu},\boldsymbol{m})}$. Thus, (2.A.13) and (2.A.19) can be uniformly combined into one formula

$$\widetilde{w}\left((-)^{n-1} x\right) = \omega_\pi(r)\|r\|^{\frac{n}{2}}\|x\|^{\frac{n-1}{2}} \int_{\mathbb{F}^\times} w(y)\, J_\pi(r^n x y)\|y\|^{1-\frac{n-1}{2}} d^\times y. \tag{2.A.20}$$

Proposition 2.4.17 (1) and 2.4.21 (1) are translated into the following lemma.

Lemma 2.A.2. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$, and let χ be a character on $\mathbb{F}^\times$. We have $J_{\chi\otimes\pi}(x) = \chi^{-1}(x) J_\pi(x)$.*

Remark 2.A.3. *Let Z_n denote the center of GL_n. In view of Lemma 2.A.2, no generality will be lost if one only considers J_π for irreducible admissible representations π of $\mathrm{GL}_n(\mathbb{F})/\mathrm{Z}_n(\mathbb{R}_+)$.*

Let φ be the n-dimensional semisimple representation of the Weil group of $\mathbb{F}$ corresponding to π under the Langlands correspondence over $\mathbb{F}$.

If $\mathbb{F} = \mathbb{R}$, the function space $\mathscr{S}^{(\boldsymbol{\mu},\boldsymbol{\delta})}_{\mathrm{sis}}(\mathbb{R}^\times)$ depends on the choice of the parameter $(\boldsymbol{\mu}, \boldsymbol{\delta})$ attached to φ, if some discrete series $\varphi_{(\mu,m)}$ occurs in its decomposition. Thus, one needs to redefine the function spaces for Hankel transforms according to the Langlands classification rather than the above parametrization. For this, let $n_1, n_2 \in \mathbb{N}$, $(\boldsymbol{\mu}^1, \boldsymbol{\delta}^1) \in \mathbb{C}^{n_1} \times (\mathbb{Z}/2\mathbb{Z})^{n_1}$ and $(\boldsymbol{\mu}^2, \boldsymbol{m}^2) \in \mathbb{C}^{n_2} \times \mathbb{N}_+^{n_2}$ be such that $n_1 + 2n_2 = n$ and $\varphi = \bigoplus_{\ell=1}^{n_1} \varphi_{(\mu^1_\ell, \delta^1_\ell)} \oplus \bigoplus_{\ell=1}^{n_2} \varphi_{(\mu^1_\ell, m^2_\ell)}$.

We define the function space $\mathscr{S}^{\pi}_{\mathrm{sis}}(\mathbb{R}^\times) = \mathscr{S}^{\varphi}_{\mathrm{sis}}(\mathbb{R}^\times)$ to be

$$\mathscr{S}^{(-\boldsymbol{\mu}^1, \boldsymbol{\delta}^1)}_{\mathrm{sis}}(\mathbb{R}^\times) + \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \operatorname{sgn}(x)^\delta \mathscr{S}^{(-\boldsymbol{\mu}^2 + \frac{1}{2}\boldsymbol{m}^2, \boldsymbol{0})}_{\mathrm{sis}}(\mathbb{R}^\times), \tag{2.A.21}$$

where $\mathscr{S}^{(\boldsymbol{\mu}, \boldsymbol{\delta})}_{\mathrm{sis}}(\mathbb{R}^\times)$ is defined by (2.4.32).

Lemma 2.A.4. *$\mathscr{S}^{\pi}_{\mathrm{sis}}(\mathbb{R}^\times)$ is the sum of $\mathscr{S}^{(-\boldsymbol{\mu},\boldsymbol{\delta})}_{\mathrm{sis}}(\mathbb{R}^\times)$ for all the parameters $(\boldsymbol{\mu}, \boldsymbol{\delta})$ attached to π.*

Proof. For $\delta \in \mathbb{Z}/2\mathbb{Z}$ and $j \in \mathbb{N}$, we have the inclusion

$$\operatorname{sgn}(x)^{\delta + \delta(m)} |x|^{\mu + \frac{1}{2} m} (\log|x|)^j \mathscr{S}(\mathbb{R}) \subset \operatorname{sgn}(x)^{\delta} |x|^{\mu - \frac{1}{2} m} (\log|x|)^j \mathscr{S}(\mathbb{R}).$$

It follows that

$$\begin{aligned} &\sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \Big(\operatorname{sgn}(x)^{\delta + \delta(m)} |x|^{\mu + \frac{1}{2} m} (\log|x|)^j \mathscr{S}(\mathbb{R}) + \operatorname{sgn}(x)^{\delta + 1} |x|^{\mu - \frac{1}{2} m} (\log|x|)^j \mathscr{S}(\mathbb{R}) \Big) \\ &= \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \operatorname{sgn}(x)^{\delta} |x|^{\mu - \frac{1}{2} m} (\log|x|)^j \mathscr{S}(\mathbb{R}). \end{aligned}$$

Then it is easy to verify this lemma by definitions. Q.E.D.

If $\mathbb{F} = \mathbb{C}$, we put

$$\mathscr{S}^{\pi}_{\mathrm{sis}}(\mathbb{C}^\times) = \mathscr{S}^{\varphi}_{\mathrm{sis}}(\mathbb{C}^\times) = \mathscr{S}^{(-\boldsymbol{\mu}, -\boldsymbol{m})}_{\mathrm{sis}}(\mathbb{C}^\times). \tag{2.A.22}$$

Let $d = [\mathbb{F} : \mathbb{R}]$. For each character χ on $\mathbb{F}^\times/\mathbb{R}_+$ we define the Mellin transform $\mathcal{M}_\chi$ of a function $\upsilon \in \mathscr{S}_{\mathrm{sis}}(\mathbb{F}^\times)$ by

$$\mathcal{M}_\chi \upsilon(s) = \int_{\mathbb{F}^\times} \upsilon(x) \chi(x) \|x\|^{\frac{1}{d} s} d^\times x. \tag{2.A.23}$$

Theorem 2.A.5. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$. Suppose $\upsilon \in \mathscr{S}^{\pi}_{\mathrm{sis}}(\mathbb{F}^\times)$. Then there exists a unique $\widetilde{\upsilon} \in \mathscr{S}^{\widetilde{\pi}}_{\mathrm{sis}}(\mathbb{F}^\times)$ satisfying the following identity*

$$\mathcal{M}_{\chi^{-1}} \widetilde{\upsilon}(ds) = \gamma(1 - s, \pi \otimes \chi, \psi_1) \mathcal{M}_\chi \upsilon(d(1 - s))$$

for all characters χ on $\mathbb{F}^\times/\mathbb{R}_+$. We write $\mathcal{H}_\pi \upsilon = \widetilde{\upsilon}$ and call $\widetilde{\upsilon}$ the normalized Hankel transform of υ over $\mathbb{F}^\times$ associated with π. Moreover, we have the Hankel inversion formula

$$\mathcal{H}_\pi \upsilon(x) = \widetilde{\upsilon}(x), \quad \mathcal{H}_{\widetilde{\pi}} \widetilde{\upsilon}(x) = \omega_\pi(-1) \upsilon((-)^n x).$$

Proof. If $\mathbb{F} = \mathbb{R}$, this follows from Theorem 2.4.15, combined with Lemma 2.A.4. If $\mathbb{F} = \mathbb{C}$, this is simply a translation of Theorem 2.4.19. Q.E.D.

Chapter 3

Bessel functions for $\mathrm{GL}_2(\mathbb{F})$ and $\mathrm{GL}_3(\mathbb{F})$

In this chapter, we shall retain the notations from §2.A.

3.1. Introduction

In §3.2, according to the theory of local functional equations for $\mathrm{GL}_2 \times \mathrm{GL}_1$-Rankin-Selberg zeta integrals over $\mathbb{F}$, we shall show that the action of the long Weyl element on the Kirillov model of an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(\mathbb{F})$ is essentially a Hankel transform over $\mathbb{F}$. It follows the consensus that for $\mathrm{GL}_2(\mathbb{F})$ the Bessel functions occurring in the Kuznetsov trace formula should coincide with those in the Voronoĭ summation formula. This will let us prove and generalize the Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{Z}[i])\backslash\mathrm{PSL}_2(\mathbb{C})$ in [BM][XXI], in the same way that [CPS] does for the Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})$ in [Kuz].[XXII] Finally, summarizing and combining several inversion formulae due to Kontorovich-Lebedev, Kuznetsov [Kuz], Bruggeman-Motohashi [BM] and Lokvenec-Guleska [LG], we shall formulate the Bessel-Plancherel formula for the Bessel functions attached to tempered representations of $\mathrm{GL}_2(\mathbb{F})$.

[XXI] In an entirely different way, the formula and the integral representation of the Bessel function associated with a principal series representation of $\mathrm{PGL}_2(\mathbb{C})$ is discovered in [BM].

[XXII] In the framework of representation theory, we shall present in a subsequent article the Kuznetsov trace formula for $\Gamma\backslash\mathrm{PGL}_2(\mathbb{C})$ for an arbitrary discrete subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$ that is cofinite but not cocompact.

In §3.3, we shall derive a formula for Bessel functions for $\mathrm{GL}_3(\mathbb{F})$ in terms of certain fundamental Bessel kernels. This is based on the local functional equations for $\mathrm{GL}_3 \times \mathrm{GL}_2$-Rankin-Selberg zeta integrals along with the Bessel-Plancherel formula for $\mathrm{GL}_2(\mathbb{F})$. We stress however that our derivation is rather formal, so a lot of work remains in giving this formula a rigorous proof. Moreover, we are not concerned here with the analytic properties of these Bessel functions. We shall leave these for future work.

3.2. Bessel functions for $\mathrm{GL}_2(\mathbb{F})$

Let π be an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(\mathbb{F})$[XXIII]. Using (2.A.2, 2.A.3), one may rewrite (2.A.20) as follows,

$$W\left(\begin{pmatrix} & 1 \\ x^{-1} & \end{pmatrix}\right) = \omega_\pi(r)\|r\| \int_{\mathbb{F}^\times} \|xy\|^{\frac{1}{2}} J_\pi(-r^2 xy) W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) d^\times y, \tag{3.2.1}$$

for $W \in \mathcal{W}(\pi, \psi_r)$. We define

$$\mathcal{J}_{\pi,\psi_r}(x) = \omega_\pi(r)\|r\| \sqrt{\|x\|} J_\pi(-r^2 x). \tag{3.2.2}$$

We call $\mathcal{J}_{\pi,\psi}(x)$ the *Bessel function associated with π and ψ*. The formula (3.2.1) then reads

$$W\left(\begin{pmatrix} & 1 \\ x^{-1} & \end{pmatrix}\right) = \int_{\mathbb{F}^\times} \mathcal{J}_{\pi,\psi}(xy) W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) d^\times y. \tag{3.2.3}$$

Moreover, with the observation

$$W\left(\begin{pmatrix} & 1 \\ x^{-1} & \end{pmatrix}\right) = \omega_\pi(x)^{-1} W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_2\right),$$

(3.2.3) turns into

$$W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_2\right) = \omega_\pi(x) \int_{\mathbb{F}^\times} \mathcal{J}_{\pi,\psi}(xy) W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) d^\times y. \text{[XXIV]} \tag{3.2.4}$$

[XXIII] It is well-known that a representation of $\mathrm{GL}_2(\mathbb{F})$ satisfying these conditions is generic.

[XXIV] In the real case, this identity is given in [CPS, Theorem 4.1]. Observe the different choice of long Weyl element $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ in [CPS, Theorem 4.1].

Thus (3.2.4) indicates that the action of the Weyl element w_2 on the Kirillov model

$$\mathcal{K}(\pi, \psi) = \left\{ w(x) = W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} \right) : W \in \mathcal{W}(\pi, \psi) \right\}$$

is essentially a Hankel transform. From this perspective, the Hankel inversion formula follows from the simple identity $w_2^2 = I_2$. This may be seen from the following lemma.

Lemma 3.2.1. *Let π be an irreducible admissible representation of* $\mathrm{GL}_2(\mathbb{F})$. *Then we have* $J_{\tilde{\pi}}(x) = \omega_\pi(x) J_\pi(x)$.

Proof. This follows from some straightforward calculations using Proposition 2.4.17 (1) and 2.4.21 (1). Q.E.D.

Remark 3.2.2. *The representation theoretic viewpoint of Lemma* 3.2.1 *is the isomorphism* $\tilde{\pi} \cong \omega^{-1} \otimes \pi$. *With this, Lemma* 3.2.1 *is a direct consequence of Lemma* 2.A.2.

Finally, we shall summarize the formulae of the Bessel functions associated with infinite dimensional irreducible *unitary* representations of $\mathrm{GL}_2(\mathbb{F})$. First of all, in view of Lemma 2.A.2 and Remark 2.A.3, one may assume without loss of generality that π is trivial on $\mathrm{Z}_2(\mathbb{R}_+)$. Moreover, with the simple observation

$$\mathcal{J}_{\pi,\psi_r}(x) = \omega_\pi(r) \mathcal{J}_{\pi,\psi_1}(r^2 x), \tag{3.2.5}$$

it is sufficient to consider the Bessel function $\mathcal{J}_\pi = \mathcal{J}_{\pi,\psi_1}$ associated with ψ_1.

3.2.1. Bessel functions for $\mathrm{GL}_2(\mathbb{R})$

Under the Langlands correspondence, we have the following classification of infinite dimensional irreducible unitary representations of $\mathrm{GL}_2(\mathbb{R})/\mathrm{Z}_2(\mathbb{R}_+)$.

- (principal series and the limit of discrete series) $\varphi_{(it,\epsilon+\delta)} \oplus \varphi_{(-it,\epsilon)}$, with $t \in [0, \infty)$ and $\epsilon, \delta \in \mathbb{Z}/2\mathbb{Z}$,

- (complementary series) $\varphi_{(t,\epsilon)} \oplus \varphi_{(-t,\epsilon)}$, with $t \in \left(0, \frac{1}{2}\right)$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$,

- (discrete series) $\varphi_{(0,m)}$, with $m \in \mathbb{N}_+$.

Here, in the first case, the corresponding representation is a limit of discrete series if $t = 0$ and $\delta = 1$ and a principal series representation if otherwise. We shall write the corresponding representations as $\eta^\epsilon \otimes \pi^+(it)$ if $\delta = 0$, $\eta^\epsilon \otimes \pi^-(it)$ if $\delta = 1$, $\eta^\epsilon \otimes \pi(t)$ and $\sigma(m)$, respectively. We have

$$\omega_{\pi^+(it)} = 1, \ \ \omega_{\pi^-(it)} = \eta, \ \ \omega_{\pi(t)} = 1, \ \ \omega_{\sigma(m)} = \eta^{m+1}. \tag{3.2.6}$$

As a consequence of Example 2.4.18, we have the following proposition.

Proposition 3.2.3.

(1). *Let $t \in [0, \infty)$. We have for $x \in \mathbb{R}_+$*

$$\begin{aligned} \mathcal{J}_{\pi^+(it)}(x) &= 4\cosh(\pi t)\sqrt{x}K_{2it}(4\pi\sqrt{x}), \\ \mathcal{J}_{\pi^+(it)}(-x) &= \frac{\pi i}{\sinh(\pi t)}\sqrt{x}\left(J_{2it}(4\pi\sqrt{x}) - J_{-2it}(4\pi\sqrt{x})\right), \end{aligned}$$

where it is understood that when $t = 0$ the right hand side of the first formula should be replaced by its limit, and

$$\begin{aligned} \mathcal{J}_{\pi^-(it)}(x) &= 4\sinh(\pi t)\sqrt{x}K_{2it}(4\pi\sqrt{x}), \\ \mathcal{J}_{\pi^-(it)}(-x) &= \frac{\pi i}{\cosh(\pi t)}\sqrt{x}\left(J_{2it}(4\pi\sqrt{x}) + J_{-2it}(4\pi\sqrt{x})\right). \end{aligned}$$

(2). *Let $t \in \left(0, \frac{1}{2}\right)$. We have for $x \in \mathbb{R}_+$*

$$\begin{aligned} \mathcal{J}_{\pi(t)}(x) &= 4\cos(\pi t)\sqrt{x}K_{2t}(4\pi\sqrt{x}), \\ \mathcal{J}_{\pi(t)}(-x) &= -\frac{\pi}{\sin(\pi t)}\sqrt{x}\left(J_{2t}(4\pi\sqrt{x}) - J_{-2t}(4\pi\sqrt{x})\right). \end{aligned}$$

(3). *Let $m \in \mathbb{N}_+$. We have for $x \in \mathbb{R}_+$*

$$\mathcal{J}_{\sigma(m)}(x) = 0, \quad \mathcal{J}_{\sigma(m)}(-x) = 2\pi i^{m+1}\sqrt{x}J_m\left(4\pi\sqrt{x}\right).$$

Remark 3.2.4. *$\pi^+(it)$, $\pi(t)$ and $\sigma(2d-1)$ exhaust all the infinite dimensional irreducible unitary representations of* $\mathrm{PGL}_2(\mathbb{R})$. *Their Bessel functions are also given in* [CPS, Proposition 6.1].

3.2.2. Bessel functions for $\mathrm{GL}_2(\mathbb{C})$

Under the Langlands correspondence, we have the following classification of infinite dimensional irreducible unitary representations of $\mathrm{GL}_2(\mathbb{C})/\mathrm{Z}_2(\mathbb{R}_+)$.

- (principal series) $\chi_{(it,k+d+\delta)} \oplus \chi_{(-it,k-d)}$, with $t \in [0,\infty)$, $k, d \in \mathbb{Z}$ and $\delta \in \mathbb{Z}/2\mathbb{Z} = \{0,1\}$,
- (complementary series) $\chi_{(t,k+d)} \oplus \chi_{(-t,k-d)}$, with $t \in \left(0, \frac{1}{2}\right)$, $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$.

We write the corresponding representations as $\eta^k \otimes \pi_d^+(it)$ if $\delta = 0$, $\eta^k \otimes \pi_d^-(it)$ if $\delta = 1$ and $\eta^k \otimes \pi_d(t)$, respectively. We have

$$\omega_{\pi_d^+(it)} = 1, \ \ \omega_{\pi_d^-(it)} = \eta, \ \ \omega_{\pi_d(t)} = 1. \tag{3.2.7}$$

According to Example 2.7.6, we have the following proposition.

Proposition 3.2.5. *Recall the definitions* (2.7.14, 2.7.17) *of $J_{\mu,m}(z)$ and $H^{(1,2)}_{\mu,m}(z)$ in Example* 2.7.6.

(1). *Let $t \in [0,\infty)$ and $d \in \mathbb{Z}$. We have for $z \in \mathbb{C}^\times$*

$$\begin{aligned}
\mathcal{J}_{\pi_d^+(it)}(z) &= -\frac{2\pi^2 i}{\sinh(2\pi t)}|z| \left(J_{it,2d}(4\pi i\sqrt{z}) - J_{-it,-2d}(4\pi i\sqrt{z})\right) \\
&= \pi^2 i|z| \left(e^{-2\pi t} H^{(1)}_{it,2d}(4\pi i\sqrt{z}) - e^{2\pi t} H^{(2)}_{it,2d}(4\pi i\sqrt{z})\right), \\
\mathcal{J}_{\pi_d^-(it)}(z) &= \frac{2\pi^2}{\cosh(2\pi t)} \sqrt{|z|\,\overline{z}} \left(J_{it,2d+1}(4\pi i\sqrt{z}) + J_{-it,-2d-1}(4\pi i\sqrt{z})\right) \\
&= \pi^2 \sqrt{|z|\,\overline{z}} \left(e^{-2\pi t} H^{(1)}_{it,2d+1}(4\pi i\sqrt{z}) + e^{2\pi t} H^{(2)}_{it,2d+1}(4\pi i\sqrt{z})\right).
\end{aligned}$$

(2). *Let $t \in \left(0, \frac{1}{2}\right)$ and $d \in \mathbb{Z}$. We have for $z \in \mathbb{C}^\times$*

$$\begin{aligned}\mathcal{J}_{\pi_d(t)}(z) &= \frac{2\pi^2}{\sin(2\pi t)} |z| \left(J_{t,2d}(4\pi i\sqrt{z}) - J_{-t,-2d}(4\pi i\sqrt{z})\right) \\ &= \pi^2 i |z| \left(e^{2\pi i t} H^{(1)}_{t,2d}(4\pi i\sqrt{z}) - e^{-2\pi i t} H^{(2)}_{t,2d}(4\pi i\sqrt{z})\right).\end{aligned}$$

In view of Corollary 2.6.16, we have the following integral representations of $\mathcal{J}_\pi\left(xe^{i\phi}\right)$ unless $\pi = \pi_d(t)$ and $t \in \left[\frac{3}{8}, \frac{1}{2}\right)$.

Proposition 3.2.6.

(1). *Let $t \in [0, \infty)$ and $d \in \mathbb{Z}$. We have for $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$*

$$\mathcal{J}_{\pi_d^+(it)}\left(xe^{i\phi}\right) = 4\pi x e^{id\phi} \int_0^\infty y^{4it-1} \left[y^{-1} - ye^{i\phi}\right]^{-2d} J_{2d}\left(4\pi\sqrt{x}\left|y^{-1} - ye^{i\phi}\right|\right) dy,$$

$$\mathcal{J}_{\pi_d^-(it)}\left(xe^{i\phi}\right) = 4\pi i x e^{id\phi} \int_0^\infty y^{4it-1} \left[y^{-1} - ye^{i\phi}\right]^{-2d-1} J_{2d+1}\left(4\pi\sqrt{x}\left|y^{-1} - ye^{i\phi}\right|\right) dy.$$

(2). *Let $t \in \left(0, \frac{3}{8}\right)$ and $d \in \mathbb{Z}$. We have for $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$*

$$\mathcal{J}_{\pi_d(t)}\left(xe^{i\phi}\right) = 4\pi x e^{id\phi} \int_0^\infty y^{4t-1} \left[y^{-1} - ye^{i\phi}\right]^{-2d} J_{2d}\left(4\pi\sqrt{x}\left|y^{-1} - ye^{i\phi}\right|\right) dy.$$

The integral on the right hand side converges absolutely only for $t \in \left(0, \frac{1}{8}\right)$.

Remark 3.2.7. *$\pi_d^+(it)$ and $\pi_d(t)$ exhaust all the infinite dimensional irreducible unitary representations of* $\mathrm{PGL}_2(\mathbb{C})$. *Proposition* 3.2.5 *shows that the Bessel function for $\pi_d^+(it)$ actually coincide with that given in* [BM]. *More precisely, we have the equality $\mathcal{J}_{\pi_d^+(it)}(z) = 2\pi^2|z|\mathcal{K}_{2it,-d}(4\pi i\sqrt{z})$, with $\mathcal{K}_{\nu,p}$ given by* [BM, (6.21), (7.21)]. *Furthermore, the integral representation of $\mathcal{J}_{\pi_d^+(it)}$ in Proposition* 3.2.6 (1) *is tantamount to* [BM, Theorem 12.1].

3.2.3. Comments on the Kuznetsov trace formula for $\mathrm{PGL}_2(\mathbb{F})$

In [Kuz], Kuznetsov proved his formula for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2 \cong \mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/K$, where $\mathbb{H}^2$ denotes the hyperbolic upper half-plane and $K = \mathrm{SO}(2)/\{\pm 1\}$. In the framework of representation theory, Cogdell and Piatetski-Shapiro [CPS] proved this formula for an arbitrary Fuchsian group of the first kind $\Gamma \subset \mathrm{PGL}_2(\mathbb{R})$. Their computations use the Whittaker and Kirillov models of an irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$. They observe that the Bessel function occurring in the Kuznetsov trace formula should be identified with the Bessel function for an irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{R})$ given in [CPS, Theorem 4.1]. Note that the approach to Bessel functions for $\mathrm{GL}_2(\mathbb{R})$ using local functional equations for $\mathrm{GL}_2 \times \mathrm{GL}_1$-Rankin-Selberg zeta integrals over $\mathbb{R}$ is already shown in [CPS, §8]. The Kuznetsov trace formula is derived in [CPS] from computing the Fourier coefficients of a *single* Poincaré series in two different ways, first unfolding, and second spectral decomposing in $L^2(\Gamma\backslash \mathrm{PGL}_2(\mathbb{R}))$. On the other hand, many authors, including Kuznetsov, approach this through a formula for the inner product of *two* Poincaré series.

The Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{Z}[i])\backslash \mathrm{PSL}_2(\mathbb{C})$ was given in [BM]. Let $K = \mathrm{SU}(2)/\{\pm 1\}$ and let $\mathbb{H}^3$ denote the three dimensional hyperbolic upper half space. Their analysis is on the space $\mathbb{H}^3 \times K$, which is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ due to the Iwasawa decomposition. The combination of the Jacquet and the Goodman-Wallach operators allows them to treat all the K-aspects. Similar to [Kuz], the approach of [BM] is also from considering the inner product of *two* certain Poincaré series. It is remarked without proof in [BM, §15] that their Bessel kernel should be interpreted as the Bessel function of an irreducible unitary representation of $\mathrm{PSL}_2(\mathbb{C})$.

Our observation is that, since [CPS, Theorem 4.1] remains valid for an irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{C})$ in view of (3.2.4), one may follow the same lines in [CPS] to obtain the Kuznetsov trace formula for $\Gamma\backslash \mathrm{PGL}_2(\mathbb{C})$, with Γ an arbitrary discrete subgroup in $\mathrm{PGL}_2(\mathbb{C})$ that is cofinite but not cocompact. In this way, we can avoid the very difficult and complicated analysis in [BM]. This will be presented in a subsequent article.

3.2.4. The Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$

Classical Bessel functions that occur in the Kontorovich-Lebedev and the Kuznetsov inversion formulae are exactly the Bessel functions attached to irreducible unitary *tempered* representations of $\mathrm{SL}_2(\mathbb{R})$. Combining these inversion formulae, we obtain an analogue of Harish-Chandra's Plancherel formula, which will be named as *the Bessel-Plancherel formula*.

We have the following inversion formulae due to Kontorovich, Lebedev and Kuznetsov.

Lemma 3.2.8. *Let $f \in C_c^\infty(\mathbb{R}_+)$ be a smooth compactly supported function on $\mathbb{R}_+$.*

(1. The Kontorovich-Lebedev inversion formula) *The Kontorovich-Lebedev transform of f and its inversion are given as below,*

$$\begin{aligned}\theta_{it}(f) &= \int_{\mathbb{R}_+} f(x) K_{2it}(x) d^\times x,\\ f(x) &= \frac{4}{\pi^2}\int_{\mathbb{R}} \theta_{it}(f) K_{2it}(x)\sinh(2\pi t) t dt.\end{aligned}$$

(2. The Kuznetsov inversion formula) *The Kuznetsov transforms of f are given as below,*

$$\begin{aligned}\theta_{\pm, it}(f) &= \int_{\mathbb{R}_+} f(x)\left(J_{2it}(x) \mp J_{-2it}(x)\right) d^\times x,\\ \theta_m(f) &= \int_{\mathbb{R}_+} f(x) J_m(x) d^\times x.\end{aligned}$$

Then, for $x \in \mathbb{R}_+$

$$\begin{aligned}f(x) = -\int_{\mathbb{R}} \theta_{+, it}(f)\left(J_{2it}(x) - J_{-2it}(x)\right)\frac{tdt}{\sinh(2\pi t)}&\\ + \sum_{d=1}^{\infty} 2(2d-1)\theta_{2d-1}(f) J_{2d-1}(x);&\\ f(x) = \int_{\mathbb{R}} \theta_{-, it}(f)\left(J_{2it}(x) + J_{-2it}(x)\right)\frac{tdt}{\sinh(2\pi t)} + \sum_{d=1}^{\infty} 4d\theta_{2d}(f) J_{2d}(x).&\end{aligned}$$

Remark 3.2.9. *Recall the following connection formula* ([Wat, 3.7 (6)])

$$K_\nu(z) = \frac{\pi\left(I_{-\nu}(z) - I_\nu(z)\right)}{2\sin(\pi\nu)}.$$

It follows that the Kontorovich-Lebedev inversion formula may be renormalized, in a similar fashion as the Kuznetsov inversion formula, into

$$\widetilde{\theta}_{it}(f) = \int_{\mathbb{R}_+} f(x)\left(I_{2it}(x) - I_{-2it}(x)\right) d^\times x,$$
$$f(x) = -\int_{\mathbb{R}} \widetilde{\theta}_{it}(f)\left(I_{2it}(x) - I_{-2it}(x)\right)\frac{tdt}{\sinh(2\pi t)}.$$

The irreducible unitary tempered representations of $\mathrm{SL}_2(\mathbb{R})$ are (the restrictions from $\mathrm{GL}_2(\mathbb{R})$ of) the unitary principal series $\pi^+(it)$, $\pi^-(it)$, with $t \in \mathbb{R}$, and the unitary discrete series $\pi(m)$ of weight m, with $m \in \mathbb{N}_+$. It is convenient to view them as representations on $\mathrm{GL}_2(\mathbb{R})$ or rather $\mathrm{GL}_2(\mathbb{R})/\mathrm{Z}_2(\mathbb{R}_+)$ (see §3.2.1), and their associated Bessel functions are given in Proposition 3.2.3. A reformulation of Lemma 3.2.8 is the Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$ given as below.

Corollary 3.2.10 (The Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$)**.** *For a function $f \in C_c^\infty(\mathbb{R}_+)$ we define*

$$\Theta^\delta_{\pm,\, it}(f) = \int_{\mathbb{R}_+} f(x)\mathcal{J}_{\pi^\pm(it)}\left((-)^\delta x\right) d^\times x, \quad \delta \in \mathbb{Z}/2\mathbb{Z},$$
$$\Theta_m(f) = \int_{\mathbb{R}_+} f(x)\mathcal{J}_{\pi(m)}(-x) d^\times x.$$

(1). *We have*

$$4\pi^2 x f(x) = \int_{\mathbb{R}} \Theta^0_{+,\, it}(f)\mathcal{J}_{\pi^+(it)}(x)\tanh(\pi t) tdt,$$
$$4\pi^2 x f(x) = \int_{\mathbb{R}} \Theta^1_{+,\, it}(f)\mathcal{J}_{\pi^+(it)}(-x)\tanh(\pi t) tdt + \sum_{d=1}^\infty (2d-1)\Theta_{2d-1}(f)\mathcal{J}_{\pi(2d-1)}(-x).$$

(2). *We have*

$$4\pi^2 x f(x) = \int_{\mathbb{R}} \Theta^0_{-,\, it}(f)\mathcal{J}_{\pi^-(it)}(x)\coth(\pi t) tdt,$$
$$4\pi^2 x f(x) = -\int_{\mathbb{R}} \Theta^1_{-,\, it}(f)\mathcal{J}_{\pi^-(it)}(-x)\coth(\pi t) tdt - \sum_{d=1}^\infty 2d\Theta_{2d}(f)\mathcal{J}_{\pi(2d)}(-x).$$

3.2.5. The Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{C})$

When studying the Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$, the Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{C})$ is first discovered in [BM, §11] for the principal series representations $\pi_d^+(it)$ and later in [LG, §12] for $\pi_d^-(it)$. Again, for our purpose, we shall see $\pi_d^\pm(it)$ as representations on $\mathrm{GL}_2(\mathbb{C})$ (see §3.2.2).

We recall the definition (2.7.14),

$$J_{\mu,m}(z) = J_{-2\mu-\frac{1}{2}m}(z) J_{-2\mu+\frac{1}{2}m}(\overline{z}).$$

Following [BM, LG], we define

$$\begin{aligned} K^+_{\mu,d}(z) &= \frac{1}{\sin(2\pi\mu)} \left(J_{\mu,2d}(z) - J_{-\mu,-2d}(z) \right), \\ K^-_{\mu,d}(z) &= \frac{1}{\cos(2\pi\mu)} \left(J_{\mu,2d+1}(z) + J_{-\mu,-2d-1}(z) \right). \end{aligned}$$

Lemma 3.2.11. *Let $f \in C^\infty_{c,\,even}(\mathbb{C}^\times)$ be an even smooth compactly supported function on $\mathbb{C}^\times$. The Bruggeman-Motohashi transforms of f are given as below,*

$$\theta^\pm_{it,d}(f) = \int_{\mathbb{C}^\times} f(z) K^\pm_{it,d}(z) d^\times z.$$

Then, for $z \in \mathbb{C}^\times$

$$\begin{aligned} f(z) &= \frac{1}{4} \sum_{d\in\mathbb{Z}} \int_{\mathbb{R}} K^+_{it,d}(z)\theta^+_{it,d}(f) \left(d^2 + 4t^2\right) dt, \\ f(z) &= \frac{1}{4} \sum_{d\in\mathbb{Z}} \int_{\mathbb{R}} K^-_{it,d}(z)\theta^-_{it,d}(f) \left(\left(d + \tfrac{1}{2}\right)^2 + 4t^2\right) dt. \end{aligned}$$

According to Proposition 3.2.5, we have

$$\mathcal{J}_{\pi_d^+(it)}(z) = 2\pi^2 |z| K^+_{it,d}(4\pi i\sqrt{z}), \quad \mathcal{J}_{\pi_d^-(it)}(z) = 2\pi^2 \sqrt{|z|\,\overline{z}} K^-_{it,d}(4\pi i\sqrt{z}).$$

The Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{C})$ is as follows.

Corollary 3.2.12 (The Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{C})$)**.** *Let $f \in C_c^\infty(\mathbb{C}^\times)$ be a smooth compactly supported function on $\mathbb{C}^\times$. We define*

$$\Theta_{it,d}^{\pm}(f) = \int_{\mathbb{C}^\times} f(z) \mathfrak{J}_{\pi_d^{\pm}(it)}(z)\, d^\times z.$$

Then, we have

$$\begin{aligned} 32\pi^4 |z|^2 f(z) &= \sum_{d \in \mathbb{Z}} \int_{\mathbb{R}} \mathfrak{J}_{\pi_d^{+}(it)}(z) \Theta_{it,d}^{+}(f) \left(d^2 + 4t^2\right) dt, \\ 32\pi^4 |z| \bar{z} f(z) &= \sum_{d \in \mathbb{Z}} \int_{\mathbb{R}} \mathfrak{J}_{\pi_d^{-}(it)}(z) \Theta_{it,d}^{-}(f) \left(\left(d + \tfrac{1}{2}\right)^2 + 4t^2\right) dt. \end{aligned}$$

3.3. Bessel functions for $\mathrm{GL}_3(\mathbb{F})$

Let us denote $G_2 = \mathrm{GL}_2(\mathbb{F})$ and $G_3 = \mathrm{GL}_3(\mathbb{F})$. Let N_2, respectively $\overline{N}_2$, be the unipotent subgroup of G_2 consisting of upper, respectively lower, triangular matrices with unity on diagonals. Let A_2 be the subgroup of diagonal matrices. Let $X_2 = N_2 w_2 A_2 N_2$ be the open Bruhat cell in G_2. If we use the Bruhat coordinates

$$g = \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} w_2 z \begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$$

on X_2, the measure on X_2 is $dg = \|a\|^{-1} d^\times a d^\times z du dv$.

Let π be an infinite dimensional irreducible unitary admissible generic representation of G_3. The Bessel function $\mathfrak{J}_{\pi,\psi}(g)$ associated with π and ψ is a real analytic function on X_2 which satisfies the bi-ψ^{-1}-variant condition

$$\mathfrak{J}_{\pi,\psi}(vgu) = \psi^{-1}(v)\psi^{-1}(u)\mathfrak{J}_{\pi,\psi}(g), \quad u, v \in N_2, \tag{3.3.1}$$

and the kernel formula

$$W\left(\begin{pmatrix} & 1 \\ g & \end{pmatrix}\right) = \int_{N_2 \backslash G_2} \mathfrak{J}_{\pi,\psi}\left(hg^{-1}\right) W\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right) dh, \quad g \in X_2, \tag{3.3.2}$$

for $W \in \mathcal{W}(\pi, \psi)$, where the definition of $\mathcal{J}_{\pi,\psi}$ is extended trivially onto G_2. By (3.3.1), it suffices to know the values of $\mathcal{J}_{\pi,\psi}$ on $w_2 A_2$. Moreover, for $r \in \mathbb{F}^\times$, we have

$$\mathcal{J}_{\pi,\psi_r}(g) = \omega_\pi(r)\|r\|^{-1}\mathcal{J}_{\pi,\psi_1}\left(r^3 \begin{pmatrix} r & \\ & 1 \end{pmatrix} g \begin{pmatrix} r & \\ & 1 \end{pmatrix}^{-1}\right). \tag{3.3.3}$$

Consequently, we only need to consider the Bessel function $\mathcal{J}_\pi = \mathcal{J}_{\pi,\psi_1}$ associated with ψ_1.

Remark 3.3.1. *In the non-archimedean case, our Bessel functions coincide with those defined in* [Bar1]. *Indeed, if we let $j_{\pi,\psi}$ denote the bi-ψ-variant Bessel function associated with π given in* [Bar1, Theorem 2.3], *then* $\mathcal{J}_{\pi,\psi}(g) = j_{\pi,\psi}\left(\begin{pmatrix} & 1 \\ g^{-1} & \end{pmatrix}\right)$, *and the kernel formula* (3.3.2) *is proved in* [Bar2, Theorem 1.5].

3.3.1. The main identity

Rankin-Selberg zeta integrals for $\mathrm{GL}_3 \times \mathrm{GL}_2$

Let π and π' be a pair of infinite dimensional irreducible unitary admissible generic representations of G_3 and G_2 respectively. Without loss of generality, we assume that π and π' are trivial on the positive centers $\mathrm{Z}_3(\mathbb{R}_+)$ and $\mathrm{Z}_2(\mathbb{R}_+)$ respectively.

Let $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$. The Rankin-Selberg zeta integrals are given by

$$\Psi(s, W, W') = \int_{N_2\backslash G_2} W\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right) W'(h)\|\det h\|^{s-\frac{1}{2}} dh,$$

and

$$\begin{aligned}\widetilde{\Psi}(s, \widetilde{W}, \widetilde{W}') &= \int_{N_2\backslash G_2} \widetilde{W}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) \widetilde{W}'(g)\|\det g\|^{s-\frac{1}{2}} dg \\ &= \int_{N_2\backslash G_2} W\left(w_3 \begin{pmatrix} {}^t g^{-1} & \\ & 1 \end{pmatrix}\right) W'\left(w_2 {}^t g^{-1}\right)\|\det g\|^{s-\frac{1}{2}} dg.\end{aligned}$$

These zeta integrals converge when $\mathfrak{Re}\, s$ is sufficiently large and have meromorphic continuations on the whole complex plane. The local Rankin-Selberg functional equation reads

$$\widetilde{\Psi}(1-s, \widetilde{W}, \widetilde{W}') = \gamma(s, \pi \otimes \pi', \psi)\Psi(s, W, W'). \tag{3.3.4}$$

We assume that $W\left(\begin{pmatrix} h & \\ & 1\end{pmatrix}\right)$ is a compactly supported smooth function on $N_2\backslash G_2$ so that the zeta integral $\Psi(s, W, W')$ on the right in (3.3.4) converges for all s. Therefore, we have the following functional equation of zeta integrals,

$$\begin{aligned}(3.3.5)\qquad & \int_{N_2\backslash G_2} W\left(w_3\begin{pmatrix} {}^t g^{-1} & \\ & 1\end{pmatrix}\right) W'\left(w_2{}^t g^{-1}\right)\|\det g\|^{\frac{1}{2}-s} dg \\ &= \gamma(s, \pi\otimes\pi', \psi)\int_{N_2\backslash G_2} W\left(\begin{pmatrix} h & \\ & 1\end{pmatrix}\right) W'(h)\|\det h\|^{s-\frac{1}{2}} dh,\end{aligned}$$

given that $\mathfrak{Re}\, s$ is sufficiently small.

Formal derivation of the main identity

Observe that

$$w_3\begin{pmatrix} {}^t g^{-1} & \\ & 1\end{pmatrix} = \begin{pmatrix} & 1 \\ w_2{}^t g^{-1} & \end{pmatrix}$$

and $w_2{}^t g^{-1} \in X_2 = N_2 w_2 A_2 N_2$ if and only if $g \in N_2 A_2 \overline{N}_2$. Since $N_2 A_2 \overline{N}_2$ is an open dense cell in G_2, the left hand side of (3.3.5) is equal to

$$\int_{A_2\overline{N}_2} W\left(\begin{pmatrix} & 1 \\ w_2{}^t g^{-1} & \end{pmatrix}\right) W'\left(w_2{}^t g^{-1}\right)\|\det g\|^{\frac{1}{2}-s} dg,$$

where $N_2\backslash N_2 A_2 \overline{N}_2$ is identified with $A_2\overline{N}_2$. In view of the kernel formula (3.3.2), the left hand side of (3.3.5) further turns into

$$\begin{aligned}&\int_{A_2\overline{N}_2}\int_{N_2\backslash G_2} \mathcal{J}_{\pi,\psi}\left(h{}^t g w_2\right) W\left(\begin{pmatrix} h & \\ & 1\end{pmatrix}\right) dh W'\left(w_2{}^t g^{-1}\right)\|\det g\|^{\frac{1}{2}-s} dg \\ = &\int_{N_2\backslash G_2} W\left(\begin{pmatrix} h & \\ & 1\end{pmatrix}\right)\int_{A_2 N_2} \mathcal{J}_{\pi,\psi}\left(h g^{-1} w_2\right) W'\left(w_2 g\right)\|\det g\|^{s-\frac{1}{2}} dg\, dh,\end{aligned}$$

where we have *interchanged the order of integrations* and made the change of variables from g to ${}^t g^{-1}$. Since $W\left(\begin{pmatrix} h & \\ & 1\end{pmatrix}\right)$ assumes all compactly supported smooth functions on $N_2\backslash G_2$, it follows from comparing this double integral with the right hand side of (3.3.5) that

$$\int_{A_2 N_2} \mathcal{J}_{\pi,\psi}\left(h g^{-1} w_2\right) W'\left(w_2 g\right)\|\det g\|^{s-\frac{1}{2}} dg = \gamma(s, \pi\otimes\pi', \psi) W'(h)\|\det h\|^{s-\frac{1}{2}},$$

which is valid for all $h \in N_2\backslash G_2$. In particular, choosing $h = w_2$, we arrive at

$$\int_{A_2N_2} \mathcal{J}_{\pi,\psi}\left(w_2 g^{-1} w_2\right) W'\left(w_2 g\right) \|\det g\|^{s-\frac{1}{2}} dg = \gamma(s, \pi\otimes\pi', \psi) W'(w_2). \tag{3.3.6}$$

Remark 3.3.2. *If we presume that the Bessel function $\mathcal{J}_{\pi,\psi}$ is of moderate growth, then it should be legitimate to change the order of integrations as above, provided that $\mathfrak{Re}\, s$ is sufficiently small.*

We now recall the kernel formula (3.2.3),

$$W'\left(\begin{pmatrix} & 1 \\ a & \end{pmatrix}\right) = \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(a^{-1}b) W'\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right) d^\times b.$$

From this, the right hand side of (3.3.6) is equal to

$$\gamma(s, \pi\otimes\pi', \psi) \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(b)\, W'\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right) d^\times b. \tag{3.3.7}$$

Let $V_{\pi'}$ denote the underlying space of π'. Suppose that $W' = W'_{v'}$ is the image of $v' \in V_{\pi'}$ in the Whittaker model $\mathcal{W}(\pi', \psi^{-1})$. For convenience, we shall write $g = z^{-1}\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & u \\ & 1\end{pmatrix}$, with $z, a \in \mathbb{F}^\times$ and $u \in \mathbb{F}$, then

$$\begin{aligned}
& W'_{v'}\left(w_2 g\right) \\
&= W'_{v'}\left(w_2 z^{-1}\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & u \\ & 1\end{pmatrix}\right) \\
&= \omega_{\pi'}(z)^{-1} W'_{\pi'\left(\left(\begin{smallmatrix} 1 & u \\ & 1\end{smallmatrix}\right)\right)v'}\left(\begin{pmatrix} & 1 \\ a & \end{pmatrix}\right) \\
&= \omega_{\pi'}(z)^{-1} \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(a^{-1}b) W'_{\pi'\left(\left(\begin{smallmatrix} 1 & u \\ & 1\end{smallmatrix}\right)\right)v'}\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right) d^\times b \\
&= \omega_{\pi'}(z)^{-1} \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(a^{-1}b) W'_{v'}\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & u \\ & 1\end{pmatrix}\right) d^\times b \\
&= \omega_{\pi'}(z)^{-1} \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(a^{-1}b) W'_{v'}\left(\begin{pmatrix} 1 & bu \\ & 1\end{pmatrix}\begin{pmatrix} b & \\ & 1\end{pmatrix}\right) d^\times b \\
&= \omega_{\pi'}(z)^{-1} \int_{\mathbb{F}^\times} \mathcal{J}_{\pi',\psi^{-1}}(a^{-1}b) \psi^{-1}(bu) W'_{v'}\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right) d^\times b.
\end{aligned}$$

Inserting this into the integral in (3.3.6) and *interchanging the order of integrations*, it follows that the left hand side of (3.3.6) is equal to

(3.3.8)
$$\int_{\mathbb{F}^\times}\int_{A_2N_2}\psi^{-1}(bu)\mathfrak{J}_{\pi,\psi}\left(w_2g^{-1}w_2\right)\mathfrak{J}_{\pi',\psi^{-1}}(a^{-1}b)\omega_{\pi'}(z)^{-1}\|\det g\|^{s-\frac{1}{2}}dgW'\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right)d^\times b.$$

Since $W'\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right)$ assumes all compactly supported smooth functions on $\mathbb{F}^\times$, it follows from comparing (3.3.7) and (3.3.8) that

$$\begin{aligned}(3.3.9)\qquad &\gamma(s,\pi\otimes\pi',\psi)\mathfrak{J}_{\pi',\psi^{-1}}(b)\\ &\quad=\int_{A_2N_2}\psi^{-1}(bu)\mathfrak{J}_{\pi,\psi}\left(w_2g^{-1}w_2\right)\mathfrak{J}_{\pi',\psi^{-1}}(a^{-1}b)\omega_{\pi'}(z)^{-1}\|\det g\|^{s-\frac{1}{2}}dg,\end{aligned}$$

for any $b\in\mathbb{F}^\times$.

Remark 3.3.3. *Regarding* $w(b)=W'\left(\begin{pmatrix} b & \\ & 1\end{pmatrix}\right)$ *as a test function, then* (3.3.9) *should be interpreted as an identity in the sense of distributions.*

Conclusion

We choose $\psi=\psi_1$ and note that, in view of (3.2.5), $\mathfrak{J}_{\pi',\psi_{-1}}$ and $\mathfrak{J}_{\pi',\psi_1}$ only differ by $\omega_{\pi'}(-1)$. Therefore, with the notations $\mathfrak{J}_{\pi'}=\mathfrak{J}_{\pi',\psi_1}$ and $\mathfrak{J}_{\pi}=\mathfrak{J}_{\pi,\psi_1}$, the identity (3.3.9) reads

(3.3.10)
$$\gamma(s,\pi\otimes\pi',\psi_1)\mathfrak{J}_{\pi'}(b)=\int_{A_2N_2}\psi_1^{-1}(bu)\,\mathfrak{J}_{\pi}\left(w_2g^{-1}w_2\right)\mathfrak{J}_{\pi'}(a^{-1}b)\omega_{\pi'}(z)^{-1}\|\det g\|^{s-\frac{1}{2}}dg,$$

with $g=z^{-1}\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & u\\ & 1\end{pmatrix}$. The measure on A_2N_2 is $\|a\|^{-1}d^\times a d^\times z du$. Changing the variable z to $\sqrt{|a|}z$, we rewrite (3.3.10) as

$$(3.3.11)\qquad \gamma(s,\pi\otimes\pi',\psi_1)\mathfrak{J}_{\pi'}(b)=\int_{\mathbb{F}^\times}F_{\pi,\pi'}(z,b)\omega_{\pi'}(z)^{-1}\|z\|^{1-2s}d^\times z,$$

with definitions

$$(3.3.12)\qquad F_{\pi,\pi'}(z,b)=\int_{\mathbb{F}^\times}G_\pi(z,a,b)\mathfrak{J}_{\pi'}(a^{-1}b)\|a\|^{-1}d^\times a,$$

$$(3.3.13) \qquad G_\pi(z,a,b) = \int_{\mathbb{F}} \psi_1^{-1}(bu)\mathfrak{J}_\pi\left(w_2 g^{-1} w_2\right) du, \quad g = \frac{1}{\sqrt{|a|}z}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}.$$

Firstly, let us consider (3.3.13). Formal application of the Fourier inversion to (3.3.13) yields

$$(3.3.14) \qquad \mathfrak{J}_\pi\left(w_2 g^{-1} w_2\right) = \int_{\mathbb{F}} G_\pi(z,a,b)\psi_1(bu)db.$$

Let $g = \frac{1}{\sqrt{|a|}z}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$. For $u \neq 0$, we have $w_2 g^{-1} w_2 \in X_2$ and

$$w_2 g^{-1} w_2 = \sqrt{|a|}z\begin{pmatrix} 1 & \\ -u & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -u^{-1} \\ & 1 \end{pmatrix} w_2 \sqrt{|a|}z\begin{pmatrix} -u & \\ & (au)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -(au)^{-1} \\ & 1 \end{pmatrix}.$$

Thus, in view of (3.3.1), we have

$$\mathfrak{J}_\pi\left(w_2 g^{-1} w_2\right) = \psi_1\left(\left(1+\frac{1}{a}\right)\frac{1}{u}\right)\mathfrak{J}_\pi\left(w_2\sqrt{|a|}z\begin{pmatrix} -u & \\ & (au)^{-1} \end{pmatrix}\right).$$

Note that if $a = -1$ then

$$\mathfrak{J}_\pi\left(w_2 g^{-1} w_2\right) = \mathfrak{J}_\pi\left(w_2(-z)\begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}\right),$$

and if $a = 1$ then

$$\mathfrak{J}_\pi\left(w_2 g^{-1} w_2\right) = \psi_1\left(\frac{2}{u}\right)\mathfrak{J}_\pi\left(w_2 z\begin{pmatrix} -u & \\ & u^{-1} \end{pmatrix}\right).$$

Therefore, it follows from (3.3.14) that

$$(3.3.15) \qquad \begin{aligned} \mathfrak{J}_\pi\left(w_2 z\begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}\right) &= \int_{\mathbb{F}} G_\pi(-z,-1,b)\psi_1(bu)db, \\ \mathfrak{J}_\pi\left(w_2 z\begin{pmatrix} -u & \\ & u^{-1} \end{pmatrix}\right) &= \psi_1\left(-\frac{2}{u}\right)\int_{\mathbb{F}} G_\pi(z,1,b)\psi_1(bu)db. \end{aligned}$$

Remark 3.3.4. *We should see the Fourier inverse transforms in* (3.3.15) *from the viewpoint of distributions.*

Remark 3.3.5. *Each diagonal matrix in* A_2 *can be uniquely written as either* $z\begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}$ *or* $z\begin{pmatrix} -u & \\ & u^{-1} \end{pmatrix}$, *if we choose* $z, u \in \mathbb{R}^\times$ *such that* $z > 0$ *in the case* $\mathbb{F} = \mathbb{R}$, *or, if we choose* $z, u \in \mathbb{C}^\times$ *such that* $\arg z \in [0, \pi)$ *in the case* $\mathbb{F} = \mathbb{C}$.

3.3.2. Bessel functions for $\mathrm{GL}_3(\mathbb{R})$

Recall that we defined the character $\eta(a) = \operatorname{sgn}(a)$ on $\mathbb{R}^\times$. We consider two pairs of representations π and $\pi' \otimes \eta^\delta$, with $\delta \in \mathbb{Z}/2\mathbb{Z}$. Since $\omega_{\pi'\otimes\eta^\delta} = \omega_{\pi'} = \omega_{\pi'}^{-1}$ and $\mathfrak{J}_{\pi'\otimes\eta^\delta}(a) = \operatorname{sgn}(a)^\delta \mathfrak{J}_{\pi'}(a)$ (see Lemma 2.A.2 and (3.2.2)), we obtain by replacing π' by $\pi'\otimes\eta^\delta$ in (3.3.11, 3.3.12) that

$$\gamma(s, \pi\otimes\pi' \otimes \eta^\delta, \psi_1)\mathfrak{J}_{\pi'}(b) = \int_{\mathbb{R}^\times} F^\delta_{\pi,\pi'}(z, b)\omega_{\pi'}(z)|z|^{1-2s} d^\times z, \tag{3.3.16}$$

with

$$F^\delta_{\pi,\pi'}(z, b) = \int_{\mathbb{R}^\times} G_\pi(z, a, b)\mathfrak{J}_{\pi'}(a^{-1}b) \operatorname{sgn}(a)^\delta |a|^{-1} d^\times a. \tag{3.3.17}$$

We divide the integral in (3.3.17) according to the partition $\mathbb{R}^\times = \mathbb{R}_+ \cup (-\mathbb{R}_+)$. Similar treatments to the variables z and b will be seen later. More precisely, if we define

$$F_{\pi,\pi',\epsilon}(z, b) = \int_{\mathbb{R}_+} G_\pi\left(z, (-)^\epsilon a, b\right) \mathfrak{J}_{\pi'}\left((-)^\epsilon a^{-1} b\right) a^{-1} d^\times a, \quad \epsilon \in \mathbb{Z}/2\mathbb{Z}, \tag{3.3.18}$$

then we have

$$F^\delta_{\pi,\pi'}(z, b) = \sum_{\epsilon\in\mathbb{Z}/2\mathbb{Z}} (-)^{\epsilon\delta} F_{\pi,\pi',\epsilon}(z, b),$$

and hence

$$F_{\pi,\pi',\epsilon}(z, b) = \frac{1}{2} \sum_{\delta\in\mathbb{Z}/2\mathbb{Z}} (-)^{\epsilon\delta} F^\delta_{\pi,\pi'}(z, b). \tag{3.3.19}$$

Mellin inversion

Suppose that the representations π and π' are parametrized by $(\boldsymbol{\mu}, \boldsymbol{\delta}) = (\mu_i, \delta_i)_{i=1,2,3}$ and $(\boldsymbol{\mu}', \boldsymbol{\delta}') = (\mu'_j, \delta'_j)_{j=1,2}$ respectively. We set $(\boldsymbol{\mu}'', \boldsymbol{\delta}'') = (\mu_i + \mu'_j, \delta_i + \delta'_j)_{\substack{i=1,2,3\\ j=1,2}}$, and let $J_{\pi\otimes\pi'} = J_{(\boldsymbol{\mu}'',\boldsymbol{\delta}'')}$ be the fundamental Bessel function defined in §2.4 and $\mathcal{C}_\delta = \mathcal{C}_{(\boldsymbol{\mu}'',\boldsymbol{\delta}''+\delta\boldsymbol{e}^6)}$

be the contour given in Definition 2.4.2. We recall from (2.4.13, 2.4.36, 2.A.9) that, for $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and $y \in \mathbb{R}_+$,

$$J_{\pi\otimes\pi'}((-)^{\epsilon}y) = \frac{1}{4\pi i}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}}(-)^{\epsilon\delta}\int_{\mathcal{C}_\delta}\gamma(1-s,\pi\otimes\pi'\otimes\eta^{\delta},\psi_1)y^{-s}ds. \tag{3.3.20}$$

Combining (3.3.16, 3.3.20), we have

$$\begin{aligned}
&J_{\pi\otimes\pi'}((-)^{\epsilon}y)\mathfrak{J}_{\pi'}(b)\\
&= \frac{1}{4\pi i}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}}(-)^{\epsilon\delta}\int_{\mathcal{C}_\delta}\int_{\mathbb{R}^\times}F^{\delta}_{\pi,\pi'}(z,b)\omega_{\pi'}(z)|z|^{2s-1}d^\times z\, y^{-s}ds\\
&= \frac{1}{4\pi i}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}(-)^{\epsilon\delta}\omega_{\pi'}(-1)^{\gamma}\int_{\mathcal{C}_\delta}\int_{\mathbb{R}_+}F^{\delta}_{\pi,\pi'}\left((-)^{\gamma}z,b\right)z^{2s-1}d^\times z\, y^{-s}ds\\
&= \frac{1}{8\pi i}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}(-)^{\epsilon\delta}\omega_{\pi'}(-1)^{\gamma}\int_{\mathcal{C}_\delta}\int_{\mathbb{R}_+}\frac{1}{\sqrt{z}}F^{\delta}_{\pi,\pi'}\left((-)^{\gamma}\sqrt{z},b\right)z^{s}d^\times z\, y^{-s}ds,
\end{aligned}$$

where for the last equality we have changed the variable z to $\sqrt{z}$. Recall the Mellin inversion

$$\varphi(y) = \frac{1}{2\pi i}\int_{\mathcal{C}}\int_{\mathbb{R}_+}\varphi(z)z^{s}d^\times z\, y^{-s}ds, \quad y \in \mathbb{R}_+.$$

Formally, it follows that

$$J_{\pi\otimes\pi'}((-)^{\epsilon}y)\mathfrak{J}_{\pi'}(b) = \frac{1}{4}\sum_{\delta\in\mathbb{Z}/2\mathbb{Z}}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}\frac{(-)^{\epsilon\delta}\omega_{\pi'}(-1)^{\gamma}}{\sqrt{y}}F^{\delta}_{\pi,\pi'}\left((-)^{\gamma}\sqrt{y},b\right).$$

Now, in view of (3.3.19), we have

$$J_{\pi\otimes\pi'}((-)^{\epsilon}y)\mathfrak{J}_{\pi'}(b) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}\frac{\omega_{\pi'}(-1)^{\gamma}}{\sqrt{y}}F_{\pi,\pi',\epsilon}\left((-)^{\gamma}\sqrt{y},b\right),$$

and, if we let $z = \sqrt{y}$ and insert the definition (3.3.18) of $F_{\pi,\pi',\epsilon}(z,b)$, this identity may be rewritten as

$$zJ_{\pi\otimes\pi'}\left((-)^{\epsilon}z^2\right)\mathfrak{J}_{\pi'}(b) = \int_{\mathbb{R}_+}G_{\pi,\pi'}\left(z,(-)^{\epsilon}a,b\right)\mathfrak{J}_{\pi'}\left((-)^{\epsilon}a^{-1}b\right)a^{-1}d^\times a, \tag{3.3.21}$$

with

$$G_{\pi,\pi'}(z,a,b) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}\omega_{\pi'}(-1)^{\gamma}G_{\pi}\left((-)^{\gamma}z,a,b\right), \quad a,b\in\mathbb{R}^\times, z\in\mathbb{R}_+.$$

Inversion using the Bessel-Plancherel formula

We are now ready to apply the Bessel-Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$. Henceforth, the representation π' will therefore be restricted to irreducible unitary tempered representations of $\mathrm{SL}_2(\mathbb{R})$.

Firstly, let us consider those π' with trivial central character. Then π' is either a unitary principal series $\pi^+(it)$ or a unitary discrete series $\pi(2d-1)$.

If we define

$$G_\pi^+(z, a, b) = \frac{1}{2} \sum_{\gamma \in \mathbb{Z}/2\mathbb{Z}} G_\pi\left((-)^\gamma z, a, b\right), \quad a, b \in \mathbb{R}^\times, z \in \mathbb{R}_+,$$

then $G_{\pi,\pi'}(z, a, b) = G_\pi^+(z, a, b)$ and (3.3.21) reads

$$z J_{\pi\otimes\pi'}\left((-)^\epsilon z^2\right) \mathcal{J}_{\pi'}(b) = \int_{\mathbb{R}_+} G_\pi^+(z, (-)^\epsilon a, b) \mathcal{J}_{\pi'}\left((-)^\epsilon a^{-1} b\right) a^{-1} d^\times a.$$

Upon replacing b by $(-)^\delta b$, with $b \in \mathbb{R}_+$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$, changing the variable a to $a^{-1}b$ and interchanging two sides of the identity, we arrive at

$$\text{(3.3.22)} \quad \int_{\mathbb{R}_+} G_\pi^+\left(z, (-)^\epsilon a^{-1} b, (-)^\delta b\right) \mathcal{J}_{\pi'}\left((-)^{\epsilon+\delta} a\right) da = bz J_{\pi\otimes\pi'}\left((-)^\epsilon z^2\right) \mathcal{J}_{\pi'}\left((-)^\delta b\right).$$

Observe that $\mathcal{J}_{\pi(2d-1)}$ is identically zero on $\mathbb{R}_+$ and so is $J_{\pi\otimes\pi(2d-1)}$ on $-\mathbb{R}_+$. Therefore, except for $(\epsilon, \delta) = (0, 1)$, the right hand side of (3.3.22) vanishes if $\pi' = \pi(2d-1)$, and hence there is no contribution from discrete series.

We define

$$\Theta_{\pi,+,it}^{\epsilon,\delta}(z, b) = bz J_{\pi\otimes\pi^+(it)}\left((-)^\epsilon z^2\right) \mathcal{J}_{\pi^+(it)}\left((-)^\delta b\right),$$

$$\Theta_{\pi,2d-1}(z, b) = bz J_{\pi\otimes\pi(2d-1)}\left(z^2\right) \mathcal{J}_{\pi(2d-1)}(-b).$$

Formally, the Bessel-Plancherel formula (Corollary 3.2.10 (1)) implies the following expression of $G_\pi^+\left(z, (-)^\epsilon a, (-)^\delta b\right)$

$$G_\pi^+\left(z, (-)^\epsilon a, (-)^\delta b\right) = \mathrm{Prin}_{\pi,\epsilon,\delta}^+(z, a^{-1}b, b) + \mathrm{Disc}_{\pi,\epsilon,\delta}^+(z, a^{-1}b, b),$$

with the principal part

$$\mathrm{Prin}^{+}_{\pi,\,\epsilon,\delta}(z,a,b) = \frac{1}{4\pi^2a^2}\int_{\mathbb{R}} \Theta^{\epsilon,\delta}_{\pi,\,+,it}(z,b)\mathcal{J}_{\pi^+(it)}\left((-)^{\epsilon+\delta}a\right)\tanh(\pi t)tdt,$$

and the discrete part

$$\mathrm{Disc}^{+}_{\pi,\,\epsilon,\delta}(z,a,b) = \begin{cases} \dfrac{1}{4\pi^2a^2}\displaystyle\sum_{d=1}^{\infty}(2d-1)\Theta_{\pi,\,2d-1}(z,b)\mathcal{J}_{\pi(2d-1)}(-a), & \text{if } (\epsilon,\delta)=(0,1),\\ 0, & \text{if } (\epsilon,\delta)\neq(0,1).\end{cases}$$

Secondly, we consider those π' with $\omega_{\pi'} = \eta$. Then π' is either a unitary principal series $\pi^-(it)$ or a unitary discrete series $\pi(2d)$. We shall proceed in the same way as above. We define

$$G^{-}_{\pi}(z,a,b) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}(-)^{\gamma}G_{\pi}\left((-)^{\gamma}z,a,b\right), \quad a,b\in\mathbb{R}^{\times}, z\in\mathbb{R}_+.$$

It follows from (3.3.21) that, for $b\in\mathbb{R}_+$ and $\delta\in\mathbb{Z}/2\mathbb{Z}$,

$$\int_{\mathbb{R}_+} G^{-}_{\pi}\left(z,(-)^{\epsilon}a^{-1}b,(-)^{\delta}b\right)\mathcal{J}_{\pi'}\left((-)^{\epsilon+\delta}a\right)da = bzJ_{\pi\otimes\pi'}\left((-)^{\epsilon}z^2\right)\mathcal{J}_{\pi'}\left((-)^{\delta}b\right). \tag{3.3.23}$$

Thus if we define

$$\Theta^{\epsilon,\delta}_{\pi,\,-,it}(z,b) = bzJ_{\pi\otimes\pi^-(it)}\left((-)^{\epsilon}z^2\right)\mathcal{J}_{\pi^-(it)}\left((-)^{\delta}b\right),$$

$$\Theta_{\pi,\,2d}(z,b) = bzJ_{\pi\otimes\pi(2d)}\left(z^2\right)\mathcal{J}_{\pi(2d)}(-b),$$

then it follows formally from the Bessel-Plancherel formula (Corollary 3.2.10 (2)) that

$$G^{-}_{\pi}\left(z,(-)^{\epsilon}a,(-)^{\delta}b\right) = \mathrm{Prin}^{-}_{\pi,\,\epsilon,\delta}(z,a^{-1}b,b) + \mathrm{Disc}^{-}_{\pi,\,\epsilon,\delta}(z,a^{-1}b,b),$$

with the principal part

$$\mathrm{Prin}^{-}_{\pi,\,\epsilon,\delta}(z,a,b) = \frac{(-)^{\epsilon+\delta}}{4\pi^2a^2}\int_{\mathbb{R}} \Theta^{\epsilon,\delta}_{\pi,\,-,it}(z,b)\mathcal{J}_{\pi^+(it)}\left((-)^{\epsilon+\delta}a\right)\coth(\pi t)tdt,$$

and the discrete part

$$\mathrm{Disc}^{-}_{\pi,\,\epsilon,\delta}(z,a,b) = \begin{cases} -\dfrac{1}{4\pi^2a^2}\displaystyle\sum_{d=1}^{\infty}2d\Theta_{\pi,\,2d-1}(z,b)\mathcal{J}_{\pi(2d)}(-a), & \text{if } (\epsilon,\delta)=(0,1),\\ 0, & \text{if } (\epsilon,\delta)\neq(0,1).\end{cases}$$

Finally, it is easily seen that

$$G_\pi\left((-)^\gamma z, a, b\right) = \sum_{\pm} (\pm)^\gamma G_\pi^\pm(z, a, b).$$

Conclusion

For $z, b \in \mathbb{R}_+$, we define

$$K_{\pi,\pm,it}^{\epsilon,\delta}(z,b) = zJ_{\pi\otimes\pi^\pm(it)}\left((-)^\epsilon z^2\right) J_{\pi^\pm(it)}\left((-)^{\delta+1}b\right) J_{\pi^\pm(it)}\left((-)^{\epsilon+\delta+1}b\right),$$

$$K_{\pi,m}(z,b) = zJ_{\pi\otimes\pi(m)}\left(z^2\right) J_{\pi(m)}(b)^2,$$

$$\begin{aligned}
\mathrm{Prin}^+_{\pi,\epsilon,\delta}(z,b) &= \frac{1}{4\pi^2}\int_{\mathbb{R}} K^{\epsilon,\delta}_{\pi,+,it}(z,b)\tanh(\pi t)tdt,\\
\mathrm{Prin}^-_{\pi,\epsilon,\delta}(z,b) &= \frac{1}{4\pi^2}\int_{\mathbb{R}} K^{\epsilon,\delta}_{\pi,-,it}(z,b)\coth(\pi t)tdt,\\
\mathrm{Disc}^+_{\pi,\epsilon,\delta}(z,b) &= \begin{cases} \dfrac{1}{4\pi^2}\displaystyle\sum_{d=1}^\infty (2d-1)K_{\pi,2d-1}(z,b), & \text{if } (\epsilon,\delta) = (0,1),\\ 0, & \text{if } (\epsilon,\delta) \neq (0,1),\end{cases}\\
\mathrm{Disc}^-_{\pi,\epsilon,\delta}(z,b) &= \begin{cases} \dfrac{1}{4\pi^2}\displaystyle\sum_{d=1}^\infty 2dK_{\pi,2d}(z,b), & \text{if } (\epsilon,\delta) = (0,1),\\ 0, & \text{if } (\epsilon,\delta) \neq (0,1),\end{cases}
\end{aligned}$$

and

$$G_\pi^\epsilon\left(z, (-)^\delta b\right) = \sum_{\pm} (-)^\delta \left(\mathrm{Prin}^\pm_{\pi,\epsilon,\delta}(z,b) + \mathrm{Disc}^\pm_{\pi,\epsilon,\delta}(z,b)\right).$$

By the formulation of $G_\pi(z,a,b)$ in the last section, with $\mathcal{J}_\pi(b)$ replaced by $\sqrt{|b|}J_\pi(-b)$ (see (3.2.2)), as well as the pair of identities in (3.3.15), we find that

$$\begin{aligned}
\mathcal{J}_\pi\left(w_2 z\begin{pmatrix} u & \\ & u^{-1}\end{pmatrix}\right) &= \int_{\mathbb{R}} G^1_\pi(z,b)\psi_1(bu)db,\\
\mathcal{J}_\pi\left(w_2 z\begin{pmatrix} -u & \\ & u^{-1}\end{pmatrix}\right) &= \psi_1\left(-\frac{2}{u}\right)\cdot\int_{\mathbb{R}} G^0_\pi(z,b)\psi_1(bu)db,
\end{aligned}$$

with $z \in \mathbb{R}_+$ and $u \in \mathbb{R}^\times$.

3.3.3. Bessel functions for $\mathrm{GL}_3(\mathbb{C})$

Recall that we defined the character $\eta(a) = [a]$ on $\mathbb{C}^\times$. We consider the sequence $\pi' \otimes \eta^m$ of representations of G_2, with $m \in \mathbb{Z}$. Since $\omega_{\pi' \otimes \eta^m} = \omega_{\pi'} \eta^{2m}$ and $\mathcal{J}_{\pi' \otimes \eta^m}(a) = [a]^{-m} \mathcal{J}_{\pi'}(a)$ (see Lemma 2.A.2 and (3.2.2)), we obtain by replacing π' by $\pi' \otimes \eta^m$ in (3.3.11, 3.3.12) that

$$\gamma(s, \pi \otimes \pi' \otimes \eta^m, \psi_1) \mathcal{J}_{\pi'}(b) = \int_{\mathbb{C}^\times} F^m_{\pi,\pi'}(z, b) \omega_{\pi'}(z)^{-1} \|z\|^{1-2s} d^\times z, \tag{3.3.24}$$

with

$$\begin{aligned} F^m_{\pi,\pi'}(z, b) &= \int_{\mathbb{C}^\times} G_\pi(z, a, b) \mathcal{J}_{\pi'}(a^{-1} b) \left[a z^{-2} \right]^m \|a\|^{-1} d^\times a \\ &= \int_{\mathbb{C}^\times} G_\pi \left(z, a[z]^2, b \right) \mathcal{J}_{\pi'} \left(a^{-1} [z]^{-2} b \right) [a]^m \|a\|^{-1} d^\times a. \end{aligned} \tag{3.3.25}$$

Let $a = y e^{i\theta}$, with $y \in \mathbb{R}_+$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and write the integral in the second line of (3.3.25) in the polar coordinate. For this, we define

$$F_{\pi,\pi'} \left(z, e^{i\theta}, b \right) = 2 \int_{\mathbb{R}_+} G_\pi \left(z, y e^{i\theta} [z]^2, b \right) \mathcal{J}_{\pi'} \left(y^{-1} e^{-i\theta} [z]^{-2} b \right) y^{-2} d^\times y, \tag{3.3.26}$$

then we have

$$F^m_{\pi,\pi'}(z, b) = \int_{\mathbb{R}/2\pi\mathbb{Z}} F_{\pi,\pi'} \left(z, e^{i\theta}, b \right) e^{im\theta} d\theta$$

and hence the Fourier series expansion

$$F_{\pi,\pi'} \left(z, e^{i\theta}, b \right) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-im\theta} F^m_{\pi,\pi'}(z, b). \tag{3.3.27}$$

Mellin inversion

Suppose $(\boldsymbol{\mu}, \boldsymbol{m}) = (\mu_i, m_i)_{i=1,2,3}$ and $(\boldsymbol{\mu}', \boldsymbol{m}') = (\mu'_j, m'_j)_{j=1,2}$ are the representation parameters of π and π' respectively. Set $(\boldsymbol{\mu}'', \boldsymbol{m}'') = (\mu_i + \mu'_j, m_i + m'_j)_{\substack{i=1,2,3 \\ j=1,2}}$. Let $J_{\pi \otimes \pi'} = J_{(\boldsymbol{\mu}'', \boldsymbol{m}'')}$ be the fundamental Bessel function defined in 2.4, and $\mathcal{C}_m = \mathcal{C}_{(\boldsymbol{\mu}'', \boldsymbol{m}'' + m e^6)}$ be the contour given in Definition 2.4.2. We recall from (2.4.22, 2.4.45, 2.A.17) that, for $y \in \mathbb{R}_+$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$J_{\pi\otimes\pi'}\left(ye^{i\theta}\right) = \frac{1}{4\pi^2 i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\mathcal{C}_m}\gamma(1-s,\pi\otimes\pi'\otimes\eta^m,\psi_1)y^{-2s}ds. \tag{3.3.28}$$

Combining (3.3.24, 3.3.28), we have

$$\begin{aligned}
&J_{\pi\otimes\pi'}\left(ye^{i\theta}\right)\mathfrak{J}_{\pi'}(b)\\
&=\frac{1}{4\pi^2 i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\mathcal{C}_m}\int_{\mathbb{C}^\times} F^m_{\pi,\pi'}(z,b)\omega_{\pi'}(z)^{-1}\|z\|^{2s-1}d^\times z\, y^{-2s}ds\\
&=\frac{1}{2\pi^2 i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\mathbb{R}/2\pi\mathbb{Z}}\omega_{\pi'}\left(e^{-i\phi}\right)\int_{\mathcal{C}_m}\int_{\mathbb{R}_+} F^m_{\pi,\pi'}\left(xe^{i\phi},b\right)x^{4s-2}d^\times x\, y^{-2s}ds\, d\phi\\
&=\frac{1}{4\pi^2 i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\mathbb{R}/2\pi\mathbb{Z}}\omega_{\pi'}\left(e^{-i\phi}\right)\int_{\mathcal{C}_m}\int_{\mathbb{R}_+} \frac{1}{x}F^m_{\pi,\pi'}\left(\sqrt{x}e^{i\phi},b\right)x^{2s}d^\times x\, y^{-2s}ds\, d\phi,
\end{aligned}$$

where for the last equality we have changed the variable x to $\sqrt{x}$. Recall the Mellin inversion

$$\varphi(y) = \frac{1}{\pi i}\int_{\mathcal{C}}\int_{\mathbb{R}_+}\varphi(x)x^{2s}d^\times x\, y^{-2s}ds, \quad y\in\mathbb{R}_+.$$

Formally, it follows that

$$J_{\pi\otimes\pi'}\left(ye^{i\theta}\right)\mathfrak{J}_{\pi'}(b) = \frac{1}{4\pi}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\mathbb{R}/2\pi\mathbb{Z}}\frac{\omega_{\pi'}\left(e^{-i\phi}\right)}{y}F^m_{\pi,\pi'}\left(\sqrt{y}e^{i\phi},b\right)d\phi.$$

In view of (3.3.27), we have

$$J_{\pi\otimes\pi'}\left(ye^{i\theta}\right)\mathfrak{J}_{\pi'}(b) = \frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}}\frac{\omega_{\pi'}\left(e^{-i\phi}\right)}{y}F_{\pi,\pi'}\left(\sqrt{y}e^{i\phi},e^{-i\theta},b\right)d\phi.$$

Let $x=\sqrt{y}$, $\omega = 2\phi$, and insert the definition (3.3.26) of $F_{\pi,\pi'}\left(z,e^{i\theta},b\right)$, this identity may be rewritten as

(3.3.29)

$$x^2 J_{\pi\otimes\pi'}\left(x^2e^{i\theta}\right)\mathfrak{J}_{\pi'}(b) = 2\int_{\mathbb{R}/2\pi\mathbb{Z}}\int_{\mathbb{R}_+} G_{\pi,\pi'}\left(xe^{i\theta},ye^{-i\omega},b\right)\mathfrak{J}_{\pi'}\left(y^{-1}e^{i(\theta-\omega)}b\right)y^{-2}d\omega d^\times y,$$

with

$$G_{\pi,\pi'}\left(xe^{i\theta},ye^{i\omega},b\right) = \frac{1}{4}\omega_{\pi'}\left(e^{-\frac{1}{2}i\omega}\right)\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}\omega_{\pi'}(-1)^\gamma G_\pi\left((-)^\gamma xe^{\frac{1}{2}i\omega},ye^{i(\omega-\theta)},b\right).$$

Inversion using the Bessel-Plancherel formula

Firstly, let $\pi' = \pi_d^+(it)$. We define

$$G_\pi^+\left(xe^{i\theta}, ye^{i\omega}, b\right) = \frac{1}{4}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}} G_\pi\left((-)^\gamma xe^{\frac{1}{2}i\omega}, ye^{i(\omega-\theta)}, b\right).$$

then (3.3.29) reads

$$x^2 J_{\pi\otimes\pi_d^+(it)}\left(x^2e^{i\theta}\right)\mathcal{J}_{\pi_d^+(it)}(b) = 2\int_{\mathbb{R}/2\pi\mathbb{Z}}\int_{\mathbb{R}_+} G_\pi^+\left(xe^{i\theta}, ye^{i\omega}, b\right)\mathcal{J}_{\pi_d^+(it)}\left(y^{-1}e^{i(\theta-\omega)}b\right) y^{-2}d\omega d^\times y.$$

Let $b = re^{i\phi}$, with $r\in\mathbb{R}_+$ and $\phi\in\mathbb{R}/2\pi\mathbb{Z}$. We change the variables y and ω to ry^{-1} and $\theta+\phi-\omega$, respective. After interchanging two sides of the identity, we arrive at

$$\begin{aligned}2\int_{\mathbb{R}/2\pi\mathbb{Z}}\int_{\mathbb{R}_+} G_\pi^+\left(xe^{i\theta}, ry^{-1}e^{i(\theta+\phi-\omega)}, re^{i\phi}\right)\mathcal{J}_{\pi_d^+(it)}\left(ye^{i\omega}\right) yd\omega dy&\\ = r^2x^2 J_{\pi\otimes\pi_d^+(it)}\left(x^2e^{i\theta}\right)\mathcal{J}_{\pi_d^+(it)}\left(re^{i\phi}\right)&.\end{aligned}\tag{3.3.30}$$

We denote the right hand side of (3.3.30) by $\Theta_{it,d}^+\left(xe^{i\theta}, re^{i\phi}\right)$ and define

$$\mathrm{Prin}_\pi^+\left(xe^{i\theta}, ye^{i\omega}, re^{i\phi}\right) = \frac{1}{32\pi^4y^4}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}}\mathcal{J}_{\pi_d^+(it)}\left(ye^{i\omega}\right)\Theta_{it,d}^+\left(xe^{i\theta}, re^{i\phi}\right)\left(d^2+4t^2\right)dt.$$

Then, in a formal manner, the Bessel-Plancherel formula (Corollary 3.2.12) implies that

$$G_\pi^+\left(xe^{i\theta}, ye^{i\omega}, re^{i\phi}\right) = \mathrm{Prin}_\pi^+\left(xe^{i\theta}, ry^{-1}e^{i(\theta+\phi-\omega)}, re^{i\phi}\right).$$

Secondly, let $\pi' = \pi_d^-(it)$. We define

$$G_\pi^-\left(xe^{i\theta}, ye^{i\omega}, b\right) = \frac{1}{4}e^{-\frac{1}{2}i\omega}\sum_{\gamma\in\mathbb{Z}/2\mathbb{Z}}(-)^\gamma G_\pi\left((-)^\gamma xe^{\frac{1}{2}i\omega}, ye^{i(\omega-\theta)}, b\right).$$

Applying similar arguments as above, it follows from (3.3.29) that

$$\begin{aligned}2\int_{\mathbb{R}/2\pi\mathbb{Z}}\int_{\mathbb{R}_+} G_\pi^-\left(xe^{i\theta}, ry^{-1}e^{i(\theta+\phi-\omega)}, re^{i\phi}\right)\mathcal{J}_{\pi_d^-(it)}\left(ye^{i\omega}\right) yd\omega dy&\\ = r^2x^2 J_{\pi\otimes\pi_d^-(it)}\left(x^2e^{i\theta}\right)\mathcal{J}_{\pi_d^-(it)}\left(re^{i\phi}\right)&.\end{aligned}\tag{3.3.31}$$

Again, we denote the right hand side of (3.3.31) by $\Theta^-_{it,d}\left(xe^{i\theta}, re^{i\phi}\right)$ and define

$$\mathrm{Prin}^-_\pi\left(xe^{i\theta}, ye^{i\omega}, re^{i\phi}\right) = \frac{e^{i\omega}}{32\pi^4 y^4}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}} \mathcal{J}_{\pi^-_d(it)}\left(ye^{i\omega}\right)\Theta^-_{it,d}\left(xe^{i\theta}, re^{i\phi}\right)\left(\left(d+\tfrac{1}{2}\right)^2 + 4t^2\right) dt.$$

It follows formally from the Bessel-Plancherel formula that

$$G^-_\pi\left(xe^{i\theta}, ye^{i\omega}, re^{i\phi}\right) = \mathrm{Prin}^-_\pi\left(xe^{i\theta}, ry^{-1}e^{i(\theta+\phi-\omega)}, re^{i\phi}\right).$$

Finally, if we let $\omega \in [0, 2\pi)$, then

$$G_\pi\left((-)^\gamma xe^{\frac{1}{2}i\omega}, ye^{i\theta}, re^{i\phi}\right) = 2G^+_\pi\left(xe^{i(\omega-\theta)}, ye^{i\omega}, re^{i\phi}\right) + (-)^\gamma 2e^{\frac{1}{2}i\omega}G^-_\pi\left(xe^{i(\omega-\theta)}, ye^{i\omega}, re^{i\phi}\right).$$

Conclusion

For $z, b \in \mathbb{C}^\times$ such that $\arg z \in [0, \pi)$, we define

$$K^\epsilon_{\pi,\pm,it,d}(z,b) = |z|^2 J_{\pi\otimes\pi^\pm_d(it)}\left((-)^\epsilon z^2\right) J_{\pi^\pm_d(it)}(-b) J_{\pi^\pm_d(it)}\left((-)^\epsilon b\right),$$

$$\mathrm{Prin}^+_{\pi,\epsilon}(z,b) = \frac{1}{32\pi^4}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}} K^\epsilon_{\pi,+,it,d}(z,b)\left(d^2+4t^2\right) dt,$$
$$\mathrm{Prin}^-_{\pi,\epsilon}(z,b) = \frac{1}{32\pi^4}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}} K^\epsilon_{\pi,-,it,d}(z,b)\left(\left(d+\tfrac{1}{2}\right)^2+4t^2\right) dt,$$

and

$$G^\epsilon_\pi(z,b) = 2\mathrm{Prin}^+_{\pi,\epsilon}(z,b) + 2\,[bz]\,\mathrm{Prin}^-_{\pi,\epsilon}(z,b).$$

Then we have the formulae

$$\mathcal{J}_\pi\left(w_2 z\begin{pmatrix} u & \\ & u^{-1}\end{pmatrix}\right) = \int_{\mathbb{C}}\psi_1(bu)G^1_\pi(z,b)db,$$
$$\mathcal{J}_\pi\left(w_2 z\begin{pmatrix} -u & \\ & u^{-1}\end{pmatrix}\right) = \psi_1\left(-\frac{2}{u}\right)\cdot\int_{\mathbb{C}}\psi_1(bu)G^0_\pi(z,b)db,$$

with $z, u \in \mathbb{C}^\times$ such that $\arg z \in [0, \pi)$.

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