Solutions to Quiz # 8 for Dr. Z.'s Number Theory Course for Nov. 14, 2013

1. Find $\sigma(200)$

Sol. to 1: First we factorize 200

$$200 = 2^3 \cdot 5^2$$

Now

$$\sigma(200) = (1 + 2 + 2^2 + 2^3) \cdot (1 + 5 + 5^2) = 15 \cdot 31 = 465 \quad .$$

Answer to 1: $\sigma(200) = 465$

2 (a) Define a perfect number

Sol to 2: A positive integer is perfect if it equals to the sum of its proper divisors, or equivalenly the sum of the divisors is twice that number. In symbols: $\sigma(n) = 2n$.

(b) Prove that if both p and $2^p - 1$ are primes, then

$$2^{p-1} \cdot (2^p - 1)$$

is a perfect number.

Sol. of 2(b): Since **both** 2 and $2^p - 1$ are prime, it follows from the formula for $\sigma(n)$, that

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (1 + 2 + \ldots + 2^{p-1}) \cdot (1 + (2^p - 1)) \quad .$$

By the famous geometric series formula

$$1 + q + \ldots + q^N = \frac{q^{N+1} - 1}{q - 1} \quad ,$$

with $q = 2$ and $N = p - 1$ we have

$$1 + 2 + \ldots + 2^{p-1} = \frac{2^{p-1+1} - 1}{2 - 1} = 2^p - 1 \quad .$$

By the famous theorem

$$1 + (N - 1) = N \quad ,$$

applied to $N = 2^p - 1$, we have

$$1 + (2^p - 1) = 2^p \quad .$$

Going back above we have

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (2^p - 1) \cdot (2^p) = 2 \cdot (2^{p-1}(2^p - 1)) \quad .$$

Hence $n = 2^{p-1}(2^p - 1)$ satisfies the condition

$$\sigma(n) = 2n \quad ,$$

that this means that n is perfect. QED.