Solutions to Quiz # 8 for Dr. Z.'s Number Theory Course for Nov. 14, 2013

1. (i) (2 pts.) Define $\sigma_2(n)$ (ii) (2 pts.) State the formula for $\sigma_2(n)$ in terms of the expression of n as a product of prime powers (iii) (2 pts.) Verify it for $n = 15$ by using the definition and the formula

Sol. to 1:

(i)

$$\sigma_2(n) := \sum_{d|n} d^2 \quad .$$

(ii) If

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

is the factorization of n into products of prime powers guaranteed by the **Fundamental Theorem of Arithmetic**, then

$$\sigma_2(n) = \prod_{i=1}^{k} (1 + p_i^2 + p_i^4 + \ldots + p_i^{2\alpha_i})$$

Another way (summing the geometric series) is

$$\sigma_2(n) = \prod_{i=1}^{k} \frac{p_i^{2\alpha_i+2} - 1}{p_i^2 - 1} \quad .$$

(iii) Since

$$Div(15) = \{1, 3, 5, 15\} \quad ,$$

From the **definition**

$$\sigma_2(15) = 1^2 + 3^2 + 5^2 + 15^2 = 1 + 9 + 25 + 225 = 260 \quad .$$

From the formula, since $15 = 3^1 \cdot 5^1$,

$$\sigma_2(15) = (1 + 3^2) \cdot (1 + 5^2) = 10 \cdot 26 = 260 \quad .$$

Yea!

2. (4 pts.) Prove that if p is a prime, and $2^p - 1$ is also a prime, then

$$2^{p-1} \cdot (2^p - 1)$$

is a perfect number.

Sol. of 2: Since **both** 2 and $2^p - 1$ are prime, it follows from the formula for $\sigma(n)$, that

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (1 + 2 + \ldots + 2^{p-1}) \cdot (1 + (2^p - 1)) \quad .$$

By the famous geometric series formula

$$1 + q + \ldots + q^N = \frac{q^{N+1} - 1}{q - 1} \quad ,$$

with $q = 2$ and $N = p - 1$ we have

$$1 + 2 + \ldots + 2^{p-1} = \frac{2^{p-1+1} - 1}{2 - 1} = 2^p - 1 \quad .$$

By the famous theorem

$$1 + (N - 1) = N \quad ,$$

applied to $N = 2^p - 1$, we have

$$1 + (2^p - 1) = 2^p \quad .$$

Going back above we have

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (2^p - 1) \cdot (2^p) = 2 \cdot (2^{p-1}(2^p - 1)) \quad .$$

Hence $n = 2^{p-1}(2^p - 1)$ satisfies the condition

$$\sigma(n) = 2n \quad ,$$

that this means that n is perfect. QED.