Solutions to Quiz # 10 for Dr. Z.'s Number Theory Course for Dec. 5, 2013

1. An *extremely distinct* partition of n is a sequence of integers

$$(\lambda_1, \lambda_2, \ldots, \lambda_t) \quad ,$$

such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_t = n \quad ,$$

and

$$\lambda_1 - \lambda_2 \geq 2 \quad \lambda_2 - \lambda_3 \geq 2 \quad , \ldots , \quad , \lambda_{t-1} - \lambda_t \geq 2$$

and

$$\lambda_t > 0 \quad ,$$

Let $q(n)$ be the number of partitions of n, and $q(n,k)$ be the number of exteremely distinct partitions of n whose largest part is k.

(i) (5 points) Explain why

$$q(n,k) = \sum_{r=1}^{k-2} q(n-k, r) \quad ,$$

and, of course

$$q(n,n) = 1 \quad .$$

(ii) Use the above recurrence, and

$$q(n) = \sum_{k=1}^{n} q(n,k)$$

to compute $q(n)$ for $1 \leq n \leq 5$.

Sol. to 1(i): If you take an extremely disctinct partion of n with largest part k, and chop its head (that equals k), what remains is another such partion, but of the smaller number $n-k$, and whose largest part r, can be either 1, or 2, all the way to $r-2$ (it can't be $r-1$ or r).

Adding up all these options explains the recurrence.

Important Fact: If you have $\sum_a^b$ and $a > b$ then the answer is always 0.

$$q(1,1) = 1$$

$$q(2,1) = \sum_{r=1}^{-1} q(1,r) = 0 \quad , \quad q(2,2) = 1 \quad ;$$

$$q(3,1) = \sum_{r=1}^{-1} q(2,r) = 0 \quad , \quad q(3,2) = \sum_{r=1}^{0} q(1,r) = 0 \quad , \quad q(3,3) = 1 \quad ;$$

$$q(4,1)=\sum_{r=1}^{-1} q(3,r)=0 \quad , \quad q(4,2)=\sum_{r=1}^{0} q(2,r)=0 \quad , \quad q(4,3)=\sum_{r=1}^{1} q(1,r)=q(1,1)=1 \quad ,$$

$$q(4,4)=1 \quad ;$$

$$q(5,1)=\sum_{r=1}^{-1} q(4,r)=0 \quad , \quad q(5,2)=\sum_{r=1}^{0} q(3,r)=0 \quad , \quad q(5,3)=\sum_{r=1}^{1} q(2,r)=q(2,1)=0 \quad ,$$

$$q(5,4)=\sum_{r=1}^{2} q(1,r)=q(1,1)+q(1,2)=1+0=1 \quad ,$$

$$q(5,5)=1 \quad .$$

Finally

$$q(1)=q(1,1)=1 \quad ,$$

$$q(2)=q(2,1)+q(2,2)=0+1=1$$

$$q(3)=q(3,1)+q(3,2)+q(3,3)=0+0+1=1$$

$$q(4)=q(4,1)+q(4,2)+q(4,3)+q(4,4)=0+0+1=1$$

$$q(5)=q(5,1)+q(5,2)+q(5,3)+q(5,4)+q(5,5)=0+0+0+1+1=2 \quad .$$

Ans. to 1(ii): $q(1)=1, q(2)=1, q(3)=1, q(4)=1, q(5)=2$.