Solutions to Attendance Quiz # 2 for Dr. Z.'s Number Theory

1. (i) Guess a nice formula by inspection (ii) Give a rigorous proof of your guessed formula, using the Fundamental Theorem of Discrete Calculus (iii) Give a fully rigorous Zeilberger-style proof.

$$\sum_{i=1}^{n} (i-1)i \quad .$$

Sol.: (i) The (correct) guess is $S(n) = (n-1)n(n+1)/3$. Defining

$$S(n) := \sum_{i=1}^{n} i(i-1) \quad ,$$

The **summand** is of degree 2 so the sum $S(n)$ is of degree 3 in n, so that general **template** is $a_0 + a_1 n + a_2 n^2 + a_3 n^3$, so we need four data points. we first **collect data**

$$S(0) = 0 \quad , \quad S(1) = 1 \cdot 0 = 0 \quad , \quad S(2) = 1 \cdot 0 + 2 \cdot 1 = 2 \quad , \quad S(3) = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 = 8 \quad .$$

But since $S(0) = 0$ $a_0 = 0$, so it is divisible by n, and since $S(1) = 0$, the polynomial is also divisible by $n-1$. Hence a more efficient template is

$$S(n) = n(n-1)(c_0 + c_1 n)$$

Since $S(2) = 2$ we have

$$2 = S(2) = 2(2-1)(c_0 + 2c_1)$$

Since $S(3) = 8$ we have

$$8 = S(3) = 3(3-1)(c_0 + 3c_1)$$

We have two equations for the two unknowns c_0 and c_1

$$c_0 + 2c_1 = 1 \quad , \quad c_0 + 3c_1 = \frac{4}{3} \quad .$$

Solving, we get $c_0 = \frac{1}{3}$ and $c_1 = \frac{1}{3}$.

Hence $S(n) = n(n-1)(\frac{1}{3} + \frac{1}{3}n)$.

The conjecture is

$$\sum_{i=1}^{n} i(i-1) \;=\; \frac{(n-1)n(n+1)}{3} \quad .$$

(ii) Here $S(n) = (n-1)n(n+1)/3$ and $a(n) = (n-1)n$. First we checn that $S(0) = 0$ (true). Next:

$$S(n) - S(n-1) = (n-1)n(n+1)/3 - (n-2)(n-1)n/3 = (n-1)n/3 \cdot ((n+1)-(n-2)) = ((n-1)n/3) \cdot (3) = (n-1)n \quad =$$

QED. (by the Fundamental Theorem of Discrete Calculus).

(iii) The **degree** of the right side is 3, so it is enough to check the identity

$$\sum_{i=1}^{n} = \frac{(n-1)n(n+1)}{3}$$

for $3+1=4$ different values. Of course, the simplest are $n=0,1,2,3$.

$n=0$: $0=(0-1)0/3$ is true

$n=1$: $(1-1)(1)=(1-1)(1)(1+1)/3$ is true

$n=2$: $(1-1)(1)+(2-1)(2)=(2-1)(2)(2+1)/3$ is true

$n=3$: $(1-1)(1)+(2-1)(2)+(3-1)(3)=(3-1)(3)(4)/3$ is true.

QED!.

2. Prove the following identity (i) using the Fundamental Theorem of Discrete Calculus (ii) using a Zeilberger-style proof via checking special cases.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad .$$

Sol. of 2. Here $a(i)=i^2$ and $S(n)=n(n+1)(2n+1)/6$. First let's make sure that $S(0)=0$ (true!).

Next:

$$S(n)-S(n-1)=\frac{n(n+1)(2n+1)}{6}-\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$$

$$=\frac{n(n+1)(2n+1)}{6}-\frac{(n-1)(n)(2n-1))}{6}$$

$$=\frac{n}{6}((n+1)(2n+1)-(n-1)(2n-1))=\frac{n}{6}(2n^2+3n+1-(2n^2-3n+1))=\frac{n}{6}(6n)=n^2=a(n) \quad .QED.$$

The formula follows from the Fundamental Theorem of Discrete Calculus.

(ii) The degree of the polynomial (in n) that appears up on the right side is 3 so in order to give a fully rigorous proof that the identity is true for **all** positive integers n, we only need to check it for $3+1=4$ special cases. Of course, the simplest are $n=0,1,2,3$.

$n=0$: empty sum is 0 and $S(0)=0(1)(1)/6=0$. ok!

$n=1$: sum is $1^2=1$ and $S(1)=1(2)(3)/6=1$. ok !

$n=2$: sum is $1^2+2^2=1+4=5$ and $S(2)=2(3)(5)/6=5$. ok !

$n = 3$: sum is $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$ and $S(3) = 3(4)(7)/6 = 14$. ok !

QED!