Solutions to the Attendance Quiz # 3 for Dr. Z.'s Number Theory Course for Sept. 12, 2013

1. List all the 1-2 walks from 0 to 5. Count them and make sure that you get $f_5 = F_6$.

Sol. to 1: Let $W(n)$ be the set of walks from 0 to n. It is easieset to use the rule

$$W(n) = W(n-2)n \cup W(n-1)n \quad .$$

Since for every walk that ended in n, the penultimate stop was either at $n-2$ or $n-1$.

$$W(1) = \{1\} \quad , \quad W(2) = \{12, 2\} \quad .$$

So

$$W(3) = W(1)3 \cup W(2)3 = \{13\} \cup \{123, 23\} = \{13, 123, 23\} \quad .$$

So

$$W(4) = W(2)4 \cup W(3)4 = \{124, 24\} \cup \{134, 1234, 234\} = \{124, 24, 134, 1234, 234\} \quad .$$

$$W(5) = W(3)5 \cup W(4)5 = \{135, 1235, 235\} \cup \{1245, 245, 1345, 12345, 2345\} = \{135, 1235, 235, 1245, 245, 1345, 12345, 2345\}$$

And indeed it has 8 elements.

2. Use the combinatorial model (in terms of paths) to prove that for every positive integer n,

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \quad .$$

Sol. to 2. We first convert to the f_n language using $F_{m+1} = f_m$ for each F_m that shows up.

So we have to prove:

$$f_{2n} = f_n^2 + f_{n-1}^2 \quad .$$

Consider a typical path (with 'atomic' steps of lengths 1 and 2) from 0 to $2n$. By **definition** there are f_{2n} of them.

There are two kinds of paths:

Case I: Those that actually stop at n

Case II: Those that go over n, stopping at $n-1$ and $n+1$.

Let us count them each.

For Case I, you have two segments, the path from 0 to n and the path from n to $2n$. The decision of which path to choose is **independent**, so since there are f_n ways of going from 0 to n and f_n ways of going from n to $2n$ (since the distance from n to $2n$ is also n), there are f_n^2 paths that fall under Case I.

For Case II, you have two segments, the path from 0 to $n-1$ and the path from $n+1$ to $2n$. The decision of which path to choose is **independent**, so since there are f_{n-1} ways of going from 0 to $n-1$ and f_{n-1} ways of going from $n+1$ to $2n$ (since the distance from $n+1$ to $2n$ is also $n-1$), there are f_{n-1}^2 paths that fall under Case II.

Adding these two mutually exclusive sets of paths gives that $f_{2n} = f_n^2 + f_{n-1}^2$. **QED**.