Solutions to Attendance Quiz # 2 for Dr. Z.'s Number Theory Course for Sept. 9, 2013

1. (i) Guess a nice formula by inspection (ii) Give a rigorous proof of your guessed formula, using the Fundamental Theorem of Discrete Calculus (iii) Give a fully rigorous Zeilberger-style proof.

$$\sum_{i=1}^{n} (i-1)i \quad .$$

Sol.: (i) The (correct) guess is $S(n) = (n-1)n(n+1)/3$.

APOLOGY: When we did it, on the grass, I gave you a 'hint' that it is $S(n) = (n-1)n(n+1)$. I told you the wrong thing! People who left early, either did not notice, or disproved my wrong 'guess', by showing that $S(n)-S(n-1)$ (with the wrong $S(n)$ that I gave you that was three times the right thing) is not $a(n)$. I thank the people who tried to "prove" what I gave them and realized that I gave them the wrong thing. Anyway, I will now continue with the correct $S(n)$.

(ii) Here $S(n) = (n-1)n(n+1)/3$ and $a(n) = (n-1)n$. First we checn that $S(0) = 0$ (true). Next:

$$S(n)-S(n-1) = (n-1)n(n+1)/3-(n-2)(n-1)n/3 = (n-1)n/3\cdot((n+1)-(n-2)) = ((n-1)n/3)\cdot(3) = (n-1)n \quad = a(n).$$

QED. (by the Fundamental Theorem of Discrete Calculus).

(iii) The **degree** of the right side is 3, so it is enough to check the identity

$$\sum_{i=1}^{n} = \frac{(n-1)n(n+1)}{3}$$

for $3+1=4$ different values. Of course, the simplest are $n = 0, 1, 2, 3$.

$n = 0$: $0 = (0-1)0/3$ is true

$n = 1$: $(1-1)(1) = (1-1)(1)(1+1)/3$ is true

$n = 2$: $(1-1)(1) + (2-1)(2) = (2-1)(2)(2+1)/3$ is true

$n = 3$: $(1-1)(1) + (2-1)(2) + (3-1)(3) = (3-1)(3)(4)/3$ is true.

QED!.

2. Prove the following identity (i) using the Fundamental Theorem of Discrete Calculus (ii) using a Zeilberger-style proof via checking special cases.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad .$$

Sol. of 2. Here $a(i) = i^2$ and $S(n) = n(n+1)(2n+1)/6$. First let's make sure that $S(0) = 0$ (true!).

Next:

$$S(n) - S(n-1) = \frac{n(n+1)(2n+1)}{6} - \frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6} - \frac{(n-1)(n)(2n-1))}{6}$$

$$= \frac{n}{6}((n+1)(2n+1)-(n-1)(2n-1)) = \frac{n}{6}(2n^2+3n+1-(2n^2-3n+1)) = \frac{n}{6}(6n) = n^2 = a(n) \quad .QED.$$

The formula follows from the Fundamental Theorem of Discrete Calculus.

(ii) The degree of the polynomial (in n) that appears up on the right side is 3 so in order to give a fully rigorous proof that the identity is true for **all** positive integers n, we only need to check it for $3 + 1 = 4$ special cases. Of course, the simplest are $n = 0, 1, 2, 3$.

$n = 0$: empty sum is 0 and $S(0) = 0(1)(1)/6 = 0$. ok!

$n = 1$: sum is $1^2 = 1$ and $S(1) = 1(2)(3)/6 = 1$. ok !

$n = 2$: sum is $1^2 + 2^2 = 1 + 4 = 5$ and $S(2) = 2(3)(5)/6 = 5$. ok !

$n = 3$: sum is $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$ and $S(3) = 3(4)(7)/6 = 14$. ok !

QED!