Solutions t tendance Quiz # 24 for Dr. Z.'s Number Theory Course for Dec. 5, 2013

1. Convert the $\frac{27}{23}$ into a simple continued fraction.

Sol.:

$$\frac{27}{23} = 1 + \frac{4}{23} = 1 + \frac{1}{\frac{23}{4}} \quad ,$$

$$\frac{23}{4} = 5 + \frac{3}{4} = 5 + \frac{1}{\frac{4}{3}}$$

$$\frac{4}{3} = 1 + \frac{1}{3} \quad .$$

Now it is time to go to the **back journey**

$$\frac{23}{4} = 5 + \frac{3}{4} = 5 + \frac{1}{1 + \frac{1}{3}} \quad ,$$

$$\frac{27}{23} = 1 + \frac{4}{23} = 1 + \frac{1}{5 + \frac{1}{1+\frac{1}{3}}} \quad .$$

This is the answer. In compact (one-line) notation it is $[1, 5, 1, 3]$.

2. Find a representation in the form $a + b\sqrt{Q}$ for rational numbers a and b and positive integer Q, for the following infinite, ultimately periodic, continued fractions x.

$$x = [3, 1, 2, 1, 2, 1, 2, 1, 2, \dots \quad] \quad ,$$

where $1, 2$ repeat for ever.

Sol. to 2: Let's call The periodic part y:

$$y = [1, 2, 1, 2, 1, 2, 1, 2, \dots \quad]$$

We have

$$y = [1, 2, y] \quad .$$

Spelling out the right side, we have

$$y = 1 + \frac{1}{2 + \frac{1}{y}} = 1 + \frac{1}{\frac{2y+1}{y}} = 1 + \frac{y}{2y+1} = \frac{3y+1}{2y+1} \quad .$$

So we have to solve

$$y = \frac{3y+1}{2y+1} \quad .$$

This means

$$y - \frac{3y+1}{2y+1} = 0$$

Simplifying

$$y(2y+1)-(3y+1)=0 \quad ,$$

$$2y^2-2y-1=0 \quad .$$

Using the quadratic formula

$$y=\frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-1)}}{2\cdot 2}=\frac{2\pm\sqrt{12}}{4}=\frac{2\pm 2\sqrt{3}}{4}=\frac{1\pm\sqrt{3}}{2} \quad .$$

But y is obviously positive, so

$$y=\frac{1+\sqrt{3}}{2} \quad .$$

Now it is time to find x. We have

$$x=[3,y]=3+\frac{1}{y}=3+\frac{2}{1+\sqrt{3}}=3+\frac{2(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}=3+\frac{2(\sqrt{3}-1)}{\sqrt{3}^2-1}$$

$$=3+\frac{2(\sqrt{3}-1)}{2}=3+\sqrt{3}-1=2+\sqrt{3}$$

Ans. to 2: $x=2+\sqrt{3}$.