Solutions to Attendance Quiz # 22 for Dr. Z.'s Number Theory Course for Nov. 21, 2013

1. Write down the generating function for the sequence, let's call it $a(n)$, for the number of partitions of n where each part shows up at most three times and whose largest part is ≤ 5. Use it to find $a(i)$ for all $1 \leq i \leq 4$.

Sol. to 1: The set of 'atoms' is $\{1, 2, 3, 4, 5\}$ and each of them can show up either no time, one time, two times, or three times. Using **generating functions**, we have

$$\sum_{i=0}^{\infty} a(i)q^i = (1+q+q^2+q^3)(1+q^2+q^4+q^6)(1+q^3+q^6+q^9)(1+q^4+q^8+q^{12})(1+q^5+q^{10}+q^{15})$$

Note that this is really a **polynomial** of degree $3+6+9+12+15 = 45$ so $a(i) = 0$ when $i > 45$.

Luckily we don't have to compute $a(i)$ all the way to $i = 45$, only up to $i = 4$, so we use high-school algebra, **but** every time we encounter the power q^5, or a higher power, we repalce it by $\ldots$.

Let's first multiply the first two terms:

$$(1+q+q^2+q^3)(1+q^2+q^4+q^6) = (1+q+q^2+q^3)(1+q^2+q^4+\ldots) = (1+q+q^2+q^3)+q^2(1+q+q^2+\ldots)+q^4(1+\ldots)$$

$$= 1+q+q^2+q^3+q^2+q^3+q^4+\ldots+q^4+\ldots == 1+q+2q^2+2q^3+2q^4+\ldots \quad .$$

Now let's multiply by $(1+q^3+q^6+q^9)$, getting

$$(1+q+q^2+q^3)(1+q^2+q^4+q^6)(1+q^3+q^6+q^9) = (1+q+2q^2+2q^3+2q^4+\ldots)(1+q^3+\ldots) = 1+q+2q^2+2q^3+2q^4+q^3+q^4+\ldots =$$

Now let's multiply by $(1+q^4+q^8+q^{12})$, getting

$$(1+q+q^2+q^3)(1+q^2+q^4+q^6)(1+q^3+q^6+q^9)(1+q^4+q^8+q^{12})$$

$$= (1+q+2q^2+3q^3+3q^4)(1+q^4+\ldots) = 1+q+2q^2+3q^3+3q^4+q^4+\ldots = 1+q+2q^2+3q^3+4q^4+\ldots \quad .$$

Now let's multiply by $(1+q^5+q^{10}+q^{15})$, **BUT** this is $1+\ldots$, so we are done.

Extracting coefficients, we get **ans. to second part**:

$$a(0) = 1 \quad , \quad a(1) = 1 \quad , \quad a(2) = 2 \quad , \quad a(3) = 3 \quad , \quad a(4) = 4 \quad .$$

2. i. Apply Glashier's bijection (in the odd $\rightarrow$ distinct direction) to the odd partition $(7, 5, 5, 3, 3, 3, 1, 1, 1)$ to get a distinct partition, call it λ

ii. Apply Glashier's bijection (in the distinct $\rightarrow$ odd direction) to the partition λ and show that you get $(7, 5, 5, 3, 3, 3, 1, 1, 1)$ back.

Sol. to 2i:

We first write the partition in **exponent notation**

$$7^1 5^2 3^3 1^3 \quad .$$

We next express every exponent as a sum of powers of 2 (using the sparse binary representation)

$$7^1 5^2 3^{2+1} 1^{2+1} \quad .$$

We next convert any $s^{2^{i_1}+2^{i_2}+\ldots+2^{i_r}}$ (s odd, of course) into $s \cdot 2^{i_1}, \ldots, s \cdot 2^{i_r}$, getting

$$7, 5 \cdot 2, 3 \cdot 2, 3 \cdot 1, 1 \cdot 2, 1 \cdot 1$$

That comes out to

$$7, 10, 6, 3, 2, 1 \quad .$$

Finally, we arrange it in decreasing order, getting

$$\lambda = (10, 7, 6, 3, 2, 1) \quad .$$

Sol. to 2ii:

We first express every part in the form $2^a s$ where $a \geq 0$ and s is odd:

$$\lambda = (2 \cdot 5, 1 \cdot 7, 2 \cdot 3, 1 \cdot 3, 2 \cdot 1, 1 \cdot 1)$$

we next replace each $2^a \cdot s$ by s^{2^a}.

$$5^2, 7^1, 3^2, 3^1, 1^2, 1^1 \quad .$$

Spelled out

$$5, 5, 7, 3, 3, 3, 1, 1, 1 \quad .$$

We finally sort it:

$$(7, 5, 5, 3, 3, 3, 1, 1, 1) \quad .$$

Notice that we got back the original partition!