Solutions to Attendance Quiz # 18S for Dr. Z.'s Number Theory Course for Nov. 7, 2013

1. Which of the following are perfect numbers? Explain!

a. 49 ; **b.** 28

Sol. to 1: **a.**: $\sigma(49) = 1 + 7 + 49 \neq 2 \cdot 49$ so it **not** perfect.

b. $28 = 2^2 \cdot 7$, so $\sigma(28) = (1 + 2 + 2^2) \cdot (1 + 7) = 7 \cdot 8 = 56 = 2 \cdot 28$, so it is **perfect**.

2. (Without peeking at your notes), prove that if p is a prime, and $2^p - 1$ is also a prime, then

$$2^{p-1} \cdot (2^p - 1)$$

is a perfect number.

Sol. of 2: Since both 2 and $2^p - 1$ are prime:

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (1 + 2 + \ldots + 2^{p-1}) \cdot (2^p - 1 + 1) \quad .$$

By the famous *sum of geometrical series formula*

$$1 + q + \ldots + q^n = \frac{q^{n+1} - 1}{q - 1} \quad ,$$

we have

$$1 + 2 + \ldots + 2^{p-1} = \frac{2^p - 1}{2 - 1} = 2^p - 1 \quad .$$

Of course

$$2^p - 1 + 1 = 2^p$$

So

$$\sigma(2^{p-1} \cdot (2^p - 1)) = (2^p - 1) \cdot 2^p = 2(2^{p-1}(2^p - 1)) \quad ,$$

so $n = 2^{p-1}(2^p - 1)$ satisfies the condition $\sigma(n) = 2n$ and it is perfect.