Solutions to the Attendance Quiz # 17 for Dr. Z.'s Number Theory Course for Nov. 4, 2013

1. State the formula for $\sigma_1(n)$, and verify it for $n = 20$.

Sol. to 1: If

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \quad ,$$

Then $\sigma_1(n)$ (aka as $\sigma(n)$) is computed by

$$\sigma(n) = \prod_{i=1}^{k} (1 + p_i + (p_i)^2 + \ldots + (p_i)^{\alpha_i}) \quad .$$

For $n = 20$ we have

$$n = 2^2 \cdot 5^1 \quad ,$$

so

$$\sigma(20) = (1 + 2 + 4) \cdot (1 + 5) = 7 \cdot 6 = 42 \quad .$$

Now let's do it **from the definition**.

$$Div(20) = \{1, 2, 4\} \cdot \{1, 5\} = \{1, 2, 4, 5, 10, 20\}$$

Adding the members of $Div(20)$ together gives:

$$\sigma(20) = 1 + 2 + 4 + 5 + 10 + 20 = 42 \quad .$$

Yea! The formula agrees with the definition, at least for $n = 20$.

2. Verify the Dirichlet series for $\sigma(n)$ for up to $n = 4$

Sol. to 2.

$$\zeta(s)\zeta(s-1) = (1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots)(1 + \frac{2}{2^s} + \frac{3}{3^s} + \frac{4}{4^s} + \ldots)$$

$$1 + \frac{2}{2^s} + \frac{3}{3^s} + \frac{4}{4^s} + \ldots$$

$$+ \frac{1}{2^s} + \frac{2}{4^s} + \ldots$$

$$+ \frac{1}{3^s} + \ldots$$

$$+ \frac{1}{4^s} + \ldots$$

(Note, since we are only worried about $\sigma(n)$ up to $n = 4$, every time we see $1/5^s, 1/6^s, \ldots$ etc. we replace it by $\ldots$).

So this equals (collecting terms)

$$1 + \frac{3}{2^s} + \frac{4}{3^s} + \frac{7}{4^s} + \ldots$$

and indeed $\sigma(1) = 1$, $\sigma(2) = 1 + 2 = 3$, $\sigma(3) = 1 + 3 = 4$, $\sigma(4) = 1 + 2 + 4$.