Dr. Z.'s Number Theory Homework assignment 21

Version of Nov. 17, 2013 (please discard previous versions).

1. Using $\mathcal{P}(n,k) = \sum_{r=1}^{k} k\mathcal{P}(n-k,r)$, construct, systematically, $\mathcal{P}(n)$ for $1 \leq n \leq 5$.

2. For each of the following partitions λ, (i) Draw the Ferrers graph (ii) Find the conjugate partition λ'

a. $\lambda = (5,4,3,2,1)$

b. $\lambda = (9,4,3,2,1,1,1,1)$

c. $\lambda = (11,7,6,1)$

3. Write the following partitions in exponent notation. First the short version, then the long version (with zero exponents).

a. $(9,5,3,1)$

b. $(9,9,9,4,4,2,2,2,2,2,1,1,1,1,1,1,1)$

c. $(8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1)$

4. a. A partition $(\lambda_1,\dots,\lambda_k)$ is called **distinct** if it has no repeats (i.e. every part shows up at most part), in other words

$$\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0 \quad .$$

Let $dis(n)$ be the number of distinct partitions of n. Use an argument similar to the proof of Euler's theorem about the generating function of $p(n)$ to show that

$$\sum_{n=0} dis(n)q^n = \prod_{i=1}^{\infty}(1+q^i) \quad .$$

b. Use the above generating function to compute $dis(n)$ or $n = 1,2,3,4,5,6$.

5. a. A partition $(\lambda_1,\dots,\lambda_k)$ is called **odd** if it has no even parts, i.e. all its parts are odd. Let $odd(n)$ be the number of odd partitions of n. Use an argument similar to the proof of Euler's theorem about the generating function of $p(n)$ to show that

$$\sum_{n=0} odd(n)q^n = \prod_{i=1}^{\infty}\frac{1}{1-q^{2i+1}} \quad .$$

b. Use the above generating function to compute $odd(n)$ or $n = 1,2,3,4,5,6$.

6. Let $q(n)$ be the number of distinct partitions of n, and $q(n,k)$ be the number of distinct partitions of n whose largest part is k.

(i) Explain why

$$q(n,k) = \sum_{r=1}^{k-1} q(n-k,r) \quad .$$

(ii) Use the above recurrence, and $q(n) = \sum_{k=1}^{n} q(n,k)$ to compute $q(n)$ for $1 \leq n \leq 6$.