Dr. Z.'s Number Theory Homework assignment 2

Version of Sept. 12, 2013.

1. For the following (i) Guess a nice formula by inspection (ii) Give a rigorous proof of your guessed formula, using the Fundamental Theorem of Discrete Calculus (iii) Give a fully rigorous Zeilberger-style proof.

Added Sept. 12, 2013: It turns out that to guess the right formula, you need an IQ of 170 or up, so you could cheat and first figure out the answer using algebra and the famous formulas for $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$, and $\sum_{i=1}^{n} i^3$ (that you can look up). Another way is by *undetermined coefficients*, since you know the degree of the right side (it is one more than the degree of the summand) and since $S(0) = 0$ you can write $S(n) = a_1 n + \ldots + a_d n^d$ (if d is the degree), and plug-in $n = 1, 2, \ldots, d$ and use linear algebra to solve for $a_1, \ldots, a_d$.

a.

$$\sum_{i=1}^{n} 3i - 1 \quad ,$$

b.

$$\sum_{i=1}^{n} (2i-1)^2 \quad ,$$

c.

$$\sum_{i=1}^{n} (2i-1)^3 \quad .$$

2. Prove the following identities (i) using the Fundamental Theorem of Discrete Calculus (ii) using a Zeilberger-style proof via checking special cases.

a.

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2 \quad .$$

b.

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4} n(n+1)(n+2)(n+3) \quad .$$

3. Use the Fundamental Theorem of Discrete Calculus to prove

a.

$$\sum_{i=1}^{n} q^i = \frac{q(1-q^n)}{1-q} \quad ,$$

for any number q, and any positive integer n .

b.

$$\sum_{i=1}^{n} iq^i = \frac{nq^{n+2} - nq^{n+1} - q^{n+1} + q}{(q-1)^2}$$

for any number q, and any positive integer n .

4. In a small Italian village there are 100 married couples, except for the priest (so there are 100 married men and 100 married women, and one priest). One Sunday, the priest announces in Church:

It came to my knowledge that **at least** one woman is unfaithful. It is a great sin to shoot your cheating wife unless you are **completely** sure that she is cheating on you. On the other hand, if you are sure, by all means, you should shoot her exactly at **midnight** (after you found out for sure).

Of course, due to gossip, everyone knows about everybody else's wife, **but not** about one's own wife.

For the first 99 midnights, nothing happened, but at the 100th midnight, there were heard 100 shots, meaning that all the husbands knew for sure that their wives were cheating on them. Use mathematical induction to explain this.