MATH 356, Dr. Z., Final Exam, Mon., Dec. 23, 2013, 8-11am, SEC-218

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1. Using the formula (no credit for other methods!), find the unique $x$ between 0 and 98 such that

$$x \equiv 7 \pmod{9}, \quad x \equiv 6 \pmod{11}.$$ 

**Reminder:** The unique solution of the system $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}$ in $0 \leq x < m_1 m_2$, when $m_1$ and $m_2$ are relatively prime

$$x \equiv a_1 m_2 [m_2^{-1}]_{m_1} + a_2 m_1 [m_1^{-1}]_{m_2} \pmod{m_1 m_2}.$$

(Note: you may find the modular inverse by trial-and-error rather than by the ‘official’ way, using the Extended Euclidean Algorithm.)

**Ans.:** $x =$
2. (10 pts.) Prove that if you take any 3-digit positive integer written in decimal positional notation (as usual!)

\[ i_2 i_1 i_0 \ , \quad (1 \leq i_2 \leq 9 \ , \quad 0 \leq i_1 \leq 9 \ , \quad 0 \leq i_0 \leq 9 \ ) \]

then the 6-digit decimal integer obtained by repeating it (for example, if you take 395 you make 395395), namely

\[ i_2 i_1 i_0 i_2 i_1 i_0 \ , \]

is divisible by 13.
3. (10 pts.) State Wilson’s theorem, and verify it empirically for $p = 13$. 
4. (10 pts.) Use the Fermat primality test to investigate whether 13 is prime or composite by picking \textbf{two} random \(a\)'s between 2 and 12
5. (10 pts.) Compute $\phi(18000)$. Explain!

\textbf{Ans.:} $\phi(18000) =$
6. (10 pts.) Suppose Alice used RSA to send you the encrypted message $c$, using the public key $e$ that you gave her. Check that this is an OK message (coprime to $n = pq$). Also check that the key is a valid key. If they are both OK, find her original message?, $m$.

$p = 11$, $q = 13$, $e = 7$, $c = 3$.

Ans.: $m =$
7. (10 pts.) Prove that for every positive integer $n$, the number of partitions of $n$ into odd parts equals the number of partitions of $n$ with distinct parts.
8. (10 pts.) Prove that if $p$ is prime, and $2^p - 1$ is also prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number.
9. (10 pts.) What is $\mu(2002)$? ($\mu(n)$ is the famous Möbius function).

**Ans.:** $\mu(2002) =$
10. (10 pts.) Using the four rules below (most famously the Quadratic Reciprocity Law) decide whether 17 is a quadratic residue modulo 101. Explain everything.

Ans.: 17 is/ is not a quadratic residue mod 101

**Rule 1:** If $p$ is an odd prime and $a$ and $b$ are not multiples of $p$, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$$

**Rule 2:** If $p$ is an odd prime then

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} .$$

**Rule 3:** If $p$ is an odd prime then

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} .$$

**Rule 4:** (The quadratic Reciprocity Law)
If $p$ and $q$ are distinct odd primes, then

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} .$$
11. (10 pts) Apply the famous Bressoud-Zeilberger map to the pair

\[ \lambda = (7, 5, 3, 2, 2, 1, 1) \quad , \quad j = 2 \quad . \]

Call the output \((\lambda', j')\). Then apply it to the output, and show that you get \((\lambda, j)\) back.

**Ans.:** \( \lambda' = \) \( j' = \)

**Reminder:** Let \( \lambda = (\lambda(1), \ldots, \lambda(t)) \), where \( t \) is the number of parts. If \( t + 3j \geq \lambda(1) \) then \( \lambda' = (t + 3j - 1, \lambda(1) - 1, \ldots, \lambda(t) - 1) \) (erasing all zeros at the end, of course), and \( j' = j - 1 \).
Otherwise \( \lambda' = (\lambda(2) + 1, \ldots, \lambda(t) + 1, 1^{\lambda(1) - 3j - t - 1}) \), and \( j' = j + 1 \).
12. (10 pts.) Apply Franklin’s bijection to the distinct partition \( \lambda = (15, 14, 13, 12, 6, 5, 2) \), if it is applicable. If it is indeed applicable, call the output \( \lambda' \), and apply Franklin’s bijection to \( \lambda' \) and show that you get \( \lambda \) back.

\textbf{Ans.:} \( \lambda' = \)
13. (10 pts.) Let $n$ and $k$ be a positive integers, with $k \leq n$. Prove that the number of partitions of $n$ with exactly $k$ parts equals the number of partitions of $n$ whose largest part is $k$. 
14. (10 pts.) Convert \( \frac{37}{55} \) into a simple continued fraction.

Ans.: \( \frac{37}{55} = \)
15. (10 pts.) Evaluate the infinite continued fraction \( x = [5, 1, 3, 1, 3, 1, 3, 1, 3, \ldots] \) (i.e. \( x = [5, (1, 3)^\infty] \) that starts with 5 followed by 1, 3 repeated an infinite number of times) as a quadratic irrationality.

\[\text{Ans.: } x = \ldots\]
16. (10 pts.) In Planet Z there are 9 days in the week, and the year-length is always the same (no leap years!), consisting of 400 days.
If today is 3-Day, what day of the week would it be (on planet Z) at the same date as today, but exactly 1000 years later? Explain!

\[ \text{Ans.: It would be } \text{-Day}. \]
17. (10 pts.) Using the Extended Euclidean algorithm (no credit for other methods!), find out whether it is possible to express 1 as a linear combination

\[ 1 = m \cdot 23 + n \cdot 97 \]

for some integers \( m \) and \( n \), and if it is possible, find \( m \) and \( n \).

**Ans.:** \( m = \) \quad \( n = \).
18. (10 pts.) Express the integer 487 (written in our usual (base 10) notation) in base 7, in (i) sparse notation (4 pts) (ii) dense notation (3 pts) (iii) base-seven positional notation (3 pts)

**Ans.:** (i)

(ii)

(iii)
19. (10 pts.) Let $T_n$ be the Tribonacci numbers, defined by

$$T_1 = 1, \quad T_2 = 1, \quad T_3 = 1,$$

and for $n \geq 4$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}.$$

Give a Zeilberger-style proof of the following identity,

$$T_{n+2} = 4T_{n-1} + 3T_{n-2} + 2T_{n-3},$$

by checking it empirically for $n = 4, 5, 6, 7$. 

20. (10 pts.) Use the Fundamental Theorem of Discrete Calculus to prove the identity

\[
\sum_{i=1}^{n} i(i + 1)(i + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3).
\]