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The Fibonacci numbers are defined by the initial conditions

\[ F_0 = 0 \quad , \quad F_1 = 1 \quad , \]

and for \( n \geq 2 \) by

\[ F_n = F_{n-1} + F_{n-2} \quad . \]

**Problem 3.1:** Write down \( F_n \) for \( 1 \leq n \leq 5 \)

**Solution to 3.1:**

\[
\begin{align*}
F_2 &= F_{2-1} + F_{2-2} = F_1 + F_0 = 1 + 0 = 1 \\
F_3 &= F_{3-1} + F_{3-2} = F_2 + F_1 = 1 + 1 = 2 \\
F_4 &= F_{4-1} + F_{4-2} = F_3 + F_2 = 2 + 1 = 3 \\
F_5 &= F_{5-1} + F_{5-2} = F_4 + F_3 = 3 + 2 = 5 \\
\end{align*}
\]

**Combinatorial Meaning of the Fibonacci Numbers**

\( f_n = F_{n+1} \) is the number of ways of walking from 0 to \( n \) where the distance between consecutive stops is either 1 or 2. For example:

1 : \{01\} ,

2 : \{012,02\} ,

3 : \{0123,013,023\} ,

4 : \{01234,0134,0234,0124,024\} , etc. .

Proof: \( f_0 = 1, f_1 = 1 \). Look at the set of ways of going from 0 to \( n \). The previous step before \( n \) was either \( n - 2 \) (\( f_{n-2} \) ways), or \( n - 1 \) (\( f_{n-1} \) ways), adding these we get

\[ f_n = f_{n-1} + f_{n-2} \quad . \]

**Problem 3.2:** List all the 1-2 walks from 0 to 5. Count them and make sure that you get \( f_5 = F_6 \).

**Solution of 3.2:**

\[
\begin{align*}
5 &: \{012345,01345,02345,01245,0245,01235,0135,0235\} \\
\end{align*}
\]

There are 8 walks, and \( f_5 = F_6 = 8 \).
Binet’s formula If $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$, then
\[ F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \ . \]

Important facts:
\[ \alpha \beta = -1 \ , \ \alpha + \beta = 1 \ , \ \alpha - \beta = \sqrt{5} \ , \ \alpha^2 - \alpha - 1 = 0 \ , \ \beta^2 - \beta - 1 = 0 \ . \]

Since $\alpha^2 = \alpha + 1$, we also have $\alpha^3 = \alpha^2 + \alpha = 2\alpha + 1$ $\alpha^4 = 2\alpha^2 + \alpha = 2(\alpha + 1) + \alpha = 3\alpha + 2$, and in general, for any positive integer $k$:
\[ \alpha^k = F_k \cdot \alpha + F_{k-1} \]
and, similarly
\[ \beta^k = F_k \cdot \beta + F_{k-1} \]

**Problem 3.3**: Prove that, for every non-negative integer $n$ we have
\[ F_{n+3} + F_n = 2F_{n+2} \ . \]

**Solution of 3.3**: $F_n$ is a linear combination of $\alpha^n$ and $\beta^n$ so it is enough to prove that
\[ \alpha^{n+3} + \alpha^n - 2\alpha^{n+2} = 0 \ , \]
and similarly for $\beta^n$. But
\[ \alpha^{n+3} + \alpha^n - 2\alpha^{n+2} = \alpha^n(\alpha^3 + 1 - 2\alpha^2) \]

Now $\alpha^2 = \alpha + 1$, $\alpha^3 = 2\alpha + 1$, so this is
\[ \alpha^n(2\alpha + 1 + 1 - 2(\alpha + 1)) = 0 \ . \]
Ditto for $\beta^n$.

**Problem 3.4**: Use the combinatorial model (in terms of paths) to prove that for every positive integer $n$,
\[ F_{2n+1} = F_{n+1}^2 + F_n^2 \ . \]

**Solution of 3.4**: Since $F_n = f_{n-1}$ we have to prove that
\[ f_{2n} = f_n^2 + f_{n-1}^2 \ . \]
The left side counts all the paths from 0 to 2n. There are two cases. Those that skip over n (giving two paths, one from 0 to n−1, and another from n+1 to 2n, altogether \(f_{n-1}^2\) possibilities ), and those that stop at n (giving two paths, one from 0 to n, and another from n to 2n, altogether \(f_n^2\) possibilities). Since these are only two options, we have \(f_{2n} = f_n^2 + f_{n-1}^2\).

Zeilberger-style proofs. For any Fibonacci identity, there is a (usually rather small) integer \(N_0\), such that checking it for \(1 \leq n \leq N_0\) implies it (rigorously!) for \(all n\). How to find the \(N_0\) is beyond the scope of this class, so it will be given to you.

**Problem 3.5:** Give a Zeilberger-style proof of Cassini’s identity

\[
F_{n+1}F_{n-1} - F_n^2 = (-1)^n.
\]

You may take \(N_0 = 4\).

**Solution of 3.5:**

\[
\begin{align*}
F_{1+1}F_{1-1} - F_1^2 &= F_2F_0 - F_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1, \\
F_{2+1}F_{2-1} - F_2^2 &= F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2, \\
F_{3+1}F_{3-1} - F_3^2 &= F_4F_2 - F_3^2 = 3 \cdot 1 - 2^2 = -1 = (-1)^3, \\
F_{4+1}F_{4-1} - F_4^2 &= F_5F_3 - F_4^2 = 5 \cdot 2 - 3^2 = 1 = (-1)^4.
\end{align*}
\]

QED!