Dr. Z.'s Number Theory Lecture 3 Handout: Fibonacci numbers

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The **Fibonacci numbers** are defined by the **initial conditions**

$$F_0 = 0 \quad , \quad F_1 = 1 \quad ,$$

and for $n \geq 2$ by

$$F_n = F_{n-1} + F_{n-2} \quad .$$

Problem 3.1: Write down F_n for $1 \leq n \leq 5$

Solution to 3.1:

$$F_2 = F_{2-1} + F_{2-2} = F_1 + F_0 = 1 + 0 = 1 \quad .$$

$$F_3 = F_{3-1} + F_{3-2} = F_2 + F_1 = 1 + 1 = 2 \quad .$$

$$F_4 = F_{4-1} + F_{4-2} = F_3 + F_2 = 2 + 1 = 3 \quad .$$

$$F_5 = F_{5-1} + F_{5-2} = F_4 + F_3 = 3 + 2 = 5 \quad .$$

Combinatorial Meaning of the Fibonacci Numbers

$f_n = F_{n+1}$ is the number of ways of walking from 0 to n where the distance between consecutive stops is either 1 or 2. For example:

$1 : \{01\} \quad ,$

$2 : \{012, 02\} \quad ,$

$3 : \{0123, 013, 023\} \quad ,$

$4 : \{01234, 0134, 0234, 0124, 024\} \quad$, etc. .

Proof: $f_0 = 1, f_1 = 1$. Look at the set of ways of going from 0 to n. The previous step before n was either $n-2$ (f_{n-2} ways), or $n-1$ (f_{n-1} ways), adding these we get

$$f_n = f_{n-1} + f_{n-2} \quad .$$

Problem 3.2: List all the 1-2 walks from 0 to 5. Count them and make sure that you get $f_5 = F_6$.

Solution of 3.2:

$5 : \{012345, 01345, 02345, 01245, 0245, 01235, 0135, 0235\}$

There are 8 walks, and $f_5 = F_6 = 8$.

Binet's formula If $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, then

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \quad .$$

Important facts:

$$\alpha\beta = -1 \quad , \quad \alpha + \beta = 1 \quad , \quad \alpha - \beta = \sqrt{5} \quad , \quad \alpha^2 - \alpha - 1 = 0 \quad , \quad \beta^2 - \beta - 1 = 0 \quad .$$

Since $\alpha^2 = \alpha + 1$, we also have $\alpha^3 = \alpha^2 + \alpha = 2\alpha + 1$ $\alpha^4 = 2\alpha^2 + \alpha = 2(\alpha + 1) + \alpha = 3\alpha + 2$, and in general, for any positive integer k:

$$\alpha^k = F_k \cdot \alpha + F_{k-1}$$

and, similarly

$$\beta^k = F_k \cdot \beta + F_{k-1}$$

Problem 3.3: Prove that, for every non-negative integer n we have

$$F_{n+3} + F_n = 2F_{n+2} \quad .$$

Solution of 3.3: F_n is a linear combination of α^n and β^n so it is enough to prove that

$$\alpha^{n+3} + \alpha^n - 2\alpha^{n+2} = 0 \quad ,$$

and similarly for β^n. But

$$\alpha^{n+3} + \alpha^n - 2\alpha^{n+2} = \alpha^n(\alpha^3 + 1 - 2\alpha^2)$$

Now $\alpha^2 = \alpha + 1$, $\alpha^3 = 2\alpha + 1$, so this is

$$\alpha^n(2\alpha + 1 + 1 - 2(\alpha + 1)) = 0 \quad .$$

Ditto for β^n.

Problem 3.4: Use the combinatorial model (in terms of paths) to prove that for every positive integer n,

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \quad .$$

Solution of 3.4: Since $F_n = f_{n-1}$ we have to prove that

$$f_{2n} = f_n^2 + f_{n-1}^2 \quad .$$

The left side counts all the paths from 0 to $2n$. There are two cases. Those that skip over n (giving two paths, one from 0 to $n-1$, and another from $n+1$ to $2n$, altogether f_{n-1}^2 possibilties), and those that stop at n (giving two paths, one from 0 to n, and another from n to $2n$, altogether f_n^2 possibilities). Since these are only two options, we have $f_{2n} = f_n^2 + f_{n-1}^2$.

Zeilberger-style proofs. For any Fibonacci identity, there is a (usually rather small) integer N_0, such that checking it for $1 \leq n \leq N_0$ implies it (rigorously!) for *all* n. How to find the N_0 is beyond the scope of this class, so it will be given to you.

Problem 3.5: Give a Zeilberger-style proof of Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad .$$

You may take $N_0 = 4$.

Solution of 3.5:

$$F_{1+1}F_{1-1} - F_1^2 = F_2F_0 - F_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1 \quad ,$$

$$F_{2+1}F_{2-1} - F_2^2 = F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2 \quad ,$$

$$F_{3+1}F_{3-1} - F_3^2 = F_4F_2 - F_3^2 = 3 \cdot 1 - 2^2 = -1 = (-1)^3 \quad ,$$

$$F_{4+1}F_{4-1} - F_4^2 = F_5F_3 - F_4^2 = 5 \cdot 2 - 3^2 = 1 = (-1)^4 \quad .$$

QED!