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**How to compute \( p(n) \) Fast**

The generating function

\[
\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i},
\]

is very inefficient for computing the first 100 (or, on a computer, the first 1000000) values of \( p(n) \). Thanks to Leonhard Euler, there is a much better way

**Euler’s Recurrence for \( p(n) \)** If \( n < 0 \), then \( p(n) = 0 \). If \( n = 0 \), \( p(0) = 1 \), if \( n > 0 \), then

\[
p(n) = -\sum_{j=1}^{\infty} (-1)^j p(n - j(3j - 1)/2) - \sum_{j=1}^{\infty} (-1)^j p(n - j(3j + 1)/2)
\]

**Note:** The upper limit of summation above are \( \infty \), but it is only for convenience. The actual summation is for \( j \) for which \( j(3j + 1)/2 \leq n \), since after that the values are all 0.

**Problem 23.1:** Use Euler’s recurrence to compile a table of \( p(n) \) for \( 0 \leq n \leq 7 \)

**Sol. to 23.1:** It convenient first to put a list of the **pentagonal numbers**:

\[1,5,12,22 \ldots ; 2,7,15,26 \ldots\]

By definition:

\[p(0) = 1\]

Now

\[p(1) = -(1)^1p(1 - 1 \cdot 2/2) = 1\]
\[p(2) = -(1)^1p(2 - 1 \cdot 2/2) - (1)^1p(2 - 1 \cdot 4/2) = p(1) + p(0) = 2\]
\[p(3) = -(1)^1p(3 - 1 \cdot 2/2) - (1)^1p(3 - 1 \cdot 4/2) = p(2) + p(1) = 3\]
\[p(4) = -(1)^1p(4 - 1 \cdot 2/2) - (1)^1p(4 - 1 \cdot 4/2) = p(3) + p(2) = 5\]
\[p(5) = -(1)^1p(5 - 1 \cdot 2/2) - (1)^2p(5 - 2 \cdot 5/2) - (1)^1p(5 - 1 \cdot 4/2) = p(4) - p(0) + p(3) = 5 - 1 + 3 = 7\]
\[p(6) = -(1)^1p(6 - 1 \cdot 2/2) - (1)^2p(6 - 2 \cdot 5/2) - (1)^1p(6 - 1 \cdot 4/2) = p(5) - p(1) + p(4) = 7 - 1 + 5 = 11\]
\[p(7) = -(1)^1p(7 - 1 \cdot 2/2) - (1)^2p(7 - 2 \cdot 5/2) - (1)^1p(7 - 1 \cdot 4/2) - (1)^2p(7 - 2 \cdot 7/2)
= p(6) - p(2) + p(5) - p(0) = 11 - 2 + 7 - 1 = 15\]
The Bressoud-Zeilberger Proof of Euler’s Recurrence

An equivalent formula is as follows. Let \( a(j) = (3j^2 + j)/2 \). Then

\[
\sum_{j \text{ even}} \infty \sum_{j \text{ odd}} p(n - a(j)) = \sum_{j} p(n - a(j)) .
\]


Problem 23.2: Apply the Bressoud-Zeilberger mapping \( \phi \) (when \( n = 61 \)), to:

\( j = 2 \), \( \lambda = (14, 12, 11, 5) \),

and then apply it again and make sure that you get it back.

Solution to 23.2: Here, the number of parts, \( t \), equals 5, and the largest part, \( \lambda(1) \), equals 14. Since \( t + 3j = 5 + 6 = 11 \) is less than \( \lambda(1) = 14 \), we apply the second case and get that

\[
\phi((14, 12, 11, 5)) = (12 + 1, 12 + 1, 11 + 1, 5 + 1, 1, 1) = (13, 13, 12, 6, 1, 1) .
\]

And now \( j = 2 + 1 = 3 \). The number of 1’s at the end being \( \lambda(1) - (t + 3j) - 1 = 14 - 11 - 1 = 2 \).

Now let’s apply \( \phi \) again. For the new partition \( j = 3 \), and \( \lambda = (13, 13, 12, 6, 1, 1) \). Now \( \lambda(1) = 13 \) and \( t = 6 \). \( 6 + 3j = 6 + 3 \cdot 3 = 15 \) is \( \geq \) to \( \lambda(1) = 13 \) so the first case applies and since \( t + 3j - 1 = 14 \), we get

\[
\phi((13, 13, 12, 6, 1, 1)) = (14, 13 - 1, 13 - 1, 12 - 1, 6 - 1, 1 - 1 - 1 - 1) = (14, 12, 12, 11, 5) .
\]

(Of course we discard the 0-s). Yea we got it back.

Euler’s Pentagonal Theorem

\[
\prod_{i=1}^{\infty} (1 - q^i) = \sum_{j=-\infty}^{\infty} (-1)^j q^{(3j-1)/2} .
\]

Fabian Franklin’s beautiful proof


Problem 23.3: Apply Franklin’s mapping, \( F \), if possible, to the distinct partition

\( \lambda = (8, 7, 6, 4, 3, 2) \).

Then apply it again and make sure that you get it back.
Sol. to 23.3:

The Ferrers graph is

```
  * * * * * * *
  * * * * * * *
  * * * * * *
  * * * *
  * * *
  * *
```

The length of the downward-diagonal from the top-right is 3, while the smallest part is 2. So it is legal to take the bottom row (the smallest part) and make it into a new downward-diagonal from the top-right. The Ferrers graph is

```
  * * * * * * *
  * * * * * * *
  * * * * * *
  * * *
  * *
```

So

\[ F((8,7,6,4,3,2)) = (9,8,6,4,3) \]

For \( F(9,8,6,4,3) \), the size of the downward-diagonal from the top-right is 2, while the smallest part is 3. So we remove 1 from the top two parts, and stick 2 dots at the bottom, getting

\[ F(9,8,6,4,3) = (8,7,6,4,3,2) \]