Restricted Partitions

Let \( p_N(n) \) is the number of partitions of \( n \), then an analogous argument (only a bit simpler) to the proof of Euler’s theorem gives the generating function formula

\[
\sum_{n=0}^{\infty} p_N(n)q^n = \frac{1}{1-q} \frac{1}{1-q^2} \cdots \frac{1}{1-q^N}.
\]

Also analogously, if \( S \) is any subset of the set of positive integers (either finite or infinite), and \( p_S(n) \) is the number of partitions of \( n \) whose parts always lie in \( S \), then

\[
\sum_{n=0}^{\infty} p_S(n)q^n = \prod_{s \in S} \frac{1}{1-q^s}
\]

Also analogously, if \( S \) is any subset of the set of positive integers (either finite or infinite), and \( p_S(n) \) is the number of distinct partitions of \( n \) whose parts always lie in \( S \), then

\[
\sum_{n=0}^{\infty} p_S(n)q^n = \prod_{s \in S} (1 + q^s)
\]

**Problem 22.1:** Write down the generating function for the sequence, let’s call it \( a(n) \), for the number of distinct partitions of \( n \) whose largest part is \( \leq 3 \). Use it to find \( a(i) \) for all \( i \).

**Sol. to 22.1**

\[
\sum_{n=0}^{\infty} a(n)q^n = (1 + q)(1 + q^2)(1 + q^3)
\]

**Note:** In this case we get a polynomial, i.e. the ‘infinite series’, terminates (after \( q^6 \)).

Opening up parentheses

\[
\sum_{n=0}^{\infty} a(n)q^n = (1 + q)(1 + q^2)(1 + q^3) = (1 + q + q^2 + q^3)(1 + q^3) = 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6
\]

Comparing coefficients, we get

\( a(1) = 1, a(2) = 1, a(3) = 2, a(4) = 1, a(5) = 1, a(6) = 1, \)

and for \( i > 6 \), we have \( a(i) = 0 \). (There is no way that we can get a partition of 7 (or above) into distinct parts with largest part \( \leq 3 \)).
Problem 22.2: Write down the generating function for the sequence, let’s call it \(a(n)\), for the number of partitions of \(n\) that are congruent to 1 or 4 modulo 5. Use it to find \(a(i)\) for \(1 \leq i \leq 6\).

Sol. to 22.2 The generating function is:

\[
\sum_{n=0}^{\infty} a(n)q^n = \prod_{i=0}^{\infty} \frac{1}{1-q^{5i+1}} \frac{1}{1-q^{5i+4}}.
\]

Spelling it out

\[
\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{1-q} \frac{1}{1-q^4} \frac{1}{1-q^6} \cdots
\]

Note: Since we are only interested in \(a(i)\) for \(i \leq 6\) we can replace by \(\cdots\) anything beyond \(q^6\), since the next term would be \((1 + q^9 + q^{18} + q^{27} + \ldots)\) followed by \((1 + q^{11} + q^{22} + q^{33} + \ldots, \text{etc.})\), and none of and these will affect the first six coefficients (i.e. of \(q, q^2, q^3, q^4, q^5, q^6\)). Replacing \(\frac{1}{1-q}\), \(\frac{1}{1-q^4}\) and \(\frac{1}{1-q^6}\) by the geometric series they stand for, using

\[
\frac{1}{1-z} = 1 + z + z^2 + \ldots,
\]

but replacing by \(\ldots\) any power after \(q^6\), we get

\[
(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + \ldots)(1 + q^4 + \ldots)(1 + q^6 + \ldots)
\]

\[
1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + \ldots
\]

\[
+ q^4 + q^5 + q^6 + \ldots
\]

\[
+ q^6 + \ldots
\]

= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots
\]

Reading off the coefficients, we get:

\textbf{Ans. to 22.2:} The generating function is \(\prod_{i=0}^{\infty} \frac{1}{1-q^{5i+1}} \frac{1}{1-q^{5i+4}}\), the first six terms are \(a(1) = 1, a(2) = 1, a(3) = 1, a(4) = 2, a(5) = 2, a(6) = 3\).

The Number of Partitions of \(n\) into DISTINCT parts equals The Number of Partitions of \(n\) into ODD parts

Let \(\text{Odd}(n)\) be the set of partitions of \(n\) whose parts only have odd parts (but each odd integer can show up as many times at it wishes, including not showing up at all).

Let \(\text{Dis}(n)\) be the set of partitions of \(n\) where every integer can show up, but at most once.

Here are the few first sets of both kinds

\[\text{Odd}(1) = \{1\}, \quad \text{Dis}(1) = \{1\}\]
Now notice something amazing!, they always have the same number of elements. This is not only true for \( n \leq 8 \) but for all \( n \)!

**Euler’s Odd=Distinct Theorem**: For every positive integer \( n \): \(|\text{Odd}(n)| = |\text{Dis}(n)|\). In words: For every positive integer \( n \), the number of partitions of \( n \) into odd parts equals the number of partitions of \( n \) into distinct parts.

**First Proof** (Algebraic proof, due to L. Euler) Let \( \text{odd}(n) \) be the number of partitions of \( n \) into odd parts, and let \( \text{dis}(n) \) be the number of partitions of \( n \) into distinct parts. By generatingfunctionology (explained above)

\[
\sum_{n=0}^{\infty} \text{dis}(n)q^n = \prod_{i=0}^{\infty} (1 + q^i),
\]

\[
\sum_{n=0}^{\infty} \text{odd}(n)q^n = \prod_{i=0}^{\infty} \frac{1}{1 - q^{2i+1}}.
\]

These do not look the same, but thanks to algebra, they are the same! We use the high-school algebra identity

\[
1 + z = \frac{1 - z^2}{1 - z},
\]

that follows from \((1 + z)(1 - z) = 1 - z^2\). So, we have

\[
\sum_{n=0}^{\infty} \text{dis}(n)q^n = \prod_{i=0}^{\infty} (1 + q^i) = \prod_{i=0}^{\infty} \frac{1 - q^{2i}}{1 - q^i} = \prod_{i=0}^{\infty} \frac{(1 - q^{2i})}{(1 - q^i)} = \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} \frac{(1 - q^{2i-j})}{(1 - q^{i-j})} = \prod_{i=0}^{\infty} \frac{1}{1 - q^{2i+1}} = QED.
\]

If this is too abstract, here is a more concrete rendition, using . . . .

\[
\sum_{n=0}^{\infty} \text{dis}(n)q^n = (1 + q)(1 + q^2)(1 + q^3)(1 + q^4)(1 + q^5) \ldots
\]
\[
\frac{1 - q^2 1 - q^4 1 - q^6 1 - q^8 1 - q^{10} 1 - q^{12} \ldots}{1 - q 1 - q^2 1 - q^3 1 - q^4 1 - q^5 \ldots} = \frac{1}{1 - q} \frac{1}{1 - q^3} \frac{1}{1 - q^6} \ldots,
\]

since each term in the numerator of the form \(1 - q^{2\text{even}}\) will sooner-or later cancel-out with the same term at the bottom, resulting in only 1 surviving in the numerator, and only terms of the form \(1 - q^{\text{odd}}\) surviving at the bottom.

But the **BEST** proofs are bijective proofs. You are supposed to know two bijective proofs. The first one is due to Glashier. The second one is due to my great hero **James Joseph Sylvester**, to be covered in the next lecture.

**Glashier’s Bijection:**

\[\text{Dis}(n) \rightarrow \text{Odd}(n):\]

**Input:** A partition of \(n\) into distinct parts.

**Output:** A partition of \(n\) into odd parts.

**Description:** For each part \(p\), write it as \(p = a \cdot 2^s\) where \(s\) is the largest \(s\) such that \(p/2^s\) is an integer, and \(a\) is necessarily odd (or else \(s\) would not be the largest!). Replace it by \(2^s\) copies of \(a\).

**Problem 22.3a** Apply the Glashier bijection to the distinct partition \((26, 10, 8, 6, 5, 4, 2)\).

**Solution to problem 22.3a:**

\[
26 = 13 \cdot 2^1 = 13 \cdot 2, \\
10 = 5 \cdot 2^1 = 5 \cdot 2, \\
8 = 1 \cdot 2^3 = 1 \cdot 8, \\
6 = 3 \cdot 2^1 = 3 \cdot 2, \\
5 = 5 \cdot 2^0 = 5, \\
4 = 1 \cdot 2^2 = 1 \cdot 4, \\
2 = 1 \cdot 2^1 = 1 \cdot 2.
\]

Hence (in exponent notation)

\[(26, 10, 8, 6, 5, 4, 2) \rightarrow 13^2 5^2 1^8 3^2 5^1 1^4 1^2,\]

sorting and collecting this equals

\[1^{8+4+2} 3^2 5^2 1^3 13^2 = 1^{14} 3^2 5^3 13^2\]
Finally, going back to the usual notation we get

\[(13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \]

**Ans. to 22.3a:** The odd partition corresponding to the distinct partition \((26, 10, 8, 6, 5, 4, 2)\), under Glashier’s bijection, is \((13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\).

**Odd** \(n \rightarrow \text{Dis}(n):**

**Input:** A partition of \(n\) into odd parts.

**Output:** A partition of \(n\) into distinct parts.

**Description:** First convert the partition into exponent notation

\[1^{a_1}3^{a_3}5^{a_5} \ldots\]

For each part (that must be odd, of course), \((2i + 1)^a\), that shows up (i.e. where \(a > 0\)), write \(a\) in (sparse) binary representation

\[a = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_r}, \quad j_1 > j_2 > \ldots > j_r \geq 0,
\]

and replace it by the following parts

\[(2i + 1) \cdot 2^{j_1}, \quad (2i + 1) \cdot 2^{j_2}, \quad (2i + 1) \cdot 2^{j_3}, \quad \ldots, \quad (2i + 1) \cdot 2^{j_r}.
\]

Obviously all the parts are distinct, and at the end just sort them.

**Problem 22.3b** Apply the Glashier bijection to the odd partition

\[(13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \]

We first convert it to exponent notation

\[13^2 5^3 3^2 1^{14}.
\]

Next, for each (necessarily odd!) part \((2i + 1)^m\), we take the multiplicity and express it as a sum of powers of 2:

Regarding \(13^2\):

\[2 = 2^1,
\]

so \(13^2\) only yields one part in the output, namely \(13 \cdot 2 = 26\).

Regarding \(5^3\):

\[3 = 2^1 + 2^0 = 2 + 1,
\]

5
so $5^3$ yields two parts in the output, namely $5 \cdot 2 = 10$, and $5 \cdot 1 = 5$.

Regarding $3^2$:

$$2 = 2^1 = 2,$$

so $3^2$ yields only one part in the output, namely $3 \cdot 2 = 6$.

Regarding $1^{14}$:

$$14 = 2^3 + 2^2 + 2^1 = 8 + 4 + 2,$$

so $1^{14}$ yields three parts in the output, namely $1 \cdot 8 = 8$, $1 \cdot 4 = 4$, $1 \cdot 2 = 2$.

Combining them, we get:

$$26, 10, 5, 6, 8, 4, 2.$$

Finally, for the sake of politeness, we sort the parts from largest to smallest

$$(26, 10, 8, 6, 5, 4, 2).$$

Ans. to 22.3b: The distinct partition corresponding to the odd partition $(13,13,5,5,3,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, under Glashier’s bijection, is $(26, 10, 8, 6, 5, 4, 2)$. 