Dr. Z.'s Number Theory Lecture 22 Handout: Integer Partitions II

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Restricted Partitions

Let $p_N(n)$ is the number of partitions of n, then an analogous argument (only a bit simpler) to the proof of Euler's theorem gives the generating function formula

$$\sum_{n=0}^{\infty} p_N(n) q^n = \frac{1}{1-q} \frac{1}{1-q^2} \cdots \frac{1}{1-q^N} \quad .$$

Also analogously, if S is **any** subset of the set of positive integers (either finite or infinite), and $p_S(n)$ is the number of partitions of n whose parts always lie in S, then

$$\sum_{n=0}^{\infty} p_S(n) q^n = \prod_{s \in S} \frac{1}{1-q^s} \quad .$$

Also analogously, if S is **any** subset of the set of positive integers (either finite or infinite), and $p_S(n)$ is the number of **distinct** partitions of n whose parts always lie in S, then

$$\sum_{n=0}^{\infty} p_S(n) q^n = \prod_{s \in S} (1+q^s) \quad .$$

Problem 22.1: Write down the generating function for the sequence, let's call it $a(n)$, for the number of **distinct** partitions of n whose largest part is ≤ 3. Use it to find $a(i)$ for all i.

Sol. to 22.1

$$\sum_{n=0}^{\infty} a(n) q^n = (1+q)(1+q^2)(1+q^3) \quad .$$

Note: In this case we get a **polynomial**, i.e. the 'infinite series', terminates (after q^6).

Opening up parentheses

$$\sum_{n=0}^{\infty} a(n) q^n = (1+q)(1+q^2)(1+q^3) = (1+q+q^2+q^3)(1+q^3) = 1+q+q^2+2q^3+q^4+q^5+q^6$$

Comparing coefficients, we get

$$a(1) = 1, a(2) = 1, a(3) = 2, a(4) = 1, a(5) = 1, a(6) = 1,$$

and for $i > 6$, we have $a(i) = 0$. (There is no way that we can get a partition of 7 (or above) into distinct parts with largest part ≤ 3).

Problem 22.2: Write down the generating function for the sequence, let's call it $a(n)$, for the number of partitions of n that are congruent to 1 or 4 modulo 5. Use it to find $a(i)$ for $1 \leq i \leq 6$.

Sol. to 22.2 The generating function is:

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{i=0}^{\infty} \frac{1}{1-q^{5i+1}} \frac{1}{1-q^{5i+4}} \quad .$$

Spelling it out

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{1-q} \frac{1}{1-q^4} \frac{1}{1-q^6} \cdots$$

Note: Since we are only interested in $a(i)$ for $i \leq 6$ we can replace by $\cdots$ anything beyond q^6, since the next term would be $(1+q^9+q^{18}+q^{27}+\ldots)$ followed by $(1+q^{11}+q^{22}+q^{33}+\ldots$, etc., and none of and these will affect the first six coefficients (i.e. of $q, q^2, q^3, q^4, q^5, q^6$). Replacing $\frac{1}{1-q}$, $\frac{1}{1-q^4}$ and $\frac{1}{1-q^6}$ by the geometric series they stand for, using

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \quad ,$$

but replacing by $\ldots$ any power after q^6, we get

$$(1+q+q^2+q^3+q^4+q^5+q^6+\ldots)(1+q^4+\ldots)(1+q^6+\ldots)$$

$$\begin{aligned} &1+q+q^2+q^3+q^4+q^5+q^6+\ldots \\ &+q^4+q^5+q^6+\ldots \\ &+q^6+\ldots \\ &= 1+q+q^2+q^3+2q^4+2q^5+3q^6+\ldots \end{aligned}$$

Reading off the coefficients, we get:

Ans. to 22.2: The generating function is $\prod_{i=0}^{\infty} \frac{1}{1-q^{5i+1}} \frac{1}{1-q^{5i+4}}$, the first six terms are $a(1) = 1, a(2) = 1, a(3) = 1, a(4) = 2, a(5) = 2, a(6) = 3$.

The Number of Partitions of n into DISTINCT parts equals The Number of Partitions of n into ODD parts

Let $Odd(n)$ be the set of partitions of n whose parts **only** have **odd** parts (but each odd integer can show up as many times at it wishes, including not showing up at all).

Let $Dis(n)$ be the set of partitions of n where every integer can show up, but **at most once**.

Here are the few first sets of both kinds

$$Odd(1) = \{1\} \quad , \quad Dis(1) = \{1\}$$

$$Odd(2) = \{11\} \quad , \quad Dis(1) = \{2\}$$

$$Odd(3) = \{3, 111\} \quad , \quad Dis(3) = \{3, 21\}$$

$$Odd(4) = \{31, 1111\} \quad , \quad Dis(4) = \{4, 31\}$$

$$Odd(5) = \{5, 311, 11111\} \quad , \quad Dis(5) = \{5, 41, 32\}$$

$$Odd(6) = \{51, 3111, 321, 111111\} \quad , \quad Dis(6) = \{6, 51, 42, 321\}$$

$$Odd(7) = \{7, 511, 31111, 3211, 1111111\} \quad , \quad Dis(7) = \{7, 61, 52, 43, 421\}$$

$$Odd(8) = \{71, 5111, 331, 311111, 32111, 11111111\} \quad , \quad Dis(8) = \{8, 71, 62, 521, 53, 431\}$$

Now notice something **amazing**!, they always have the same number of elements. This is not only true for $n \leq 8$ but for **all n**

Euler's Odd=Distinct Theorem: For every positive integer n: $|Odd(n)| = |Dis(n)|$. in words: For every positive integer n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

First Proof (Algebraic proof, due to L. Euler) Let $odd(n)$ be the number of partitions of n into odd parts, and let $dis(n)$ be the number of partitions of n into distint parts. By generatingfunctionology (explained above)

$$\sum_{n=0}^{\infty} dis(n)q^n = \prod_{i=0}^{\infty}(1 + q^i) \quad ,$$

$$\sum_{n=0}^{\infty} odd(n)q^n = \prod_{i=0}^{\infty} \frac{1}{1 - q^{2i+1}} \quad .$$

These do not **look** the same, but thanks to algebra, they are the same! We use the high-school algebra identity

$$1 + z = \frac{1 - z^2}{1 - z} \quad ,$$

that follows from $(1 + z)(1 - z) = 1 - z^2$. So, we have

$$\sum_{n=0}^{\infty} dis(n)q^n = \prod_{i=0}^{\infty}(1 + q^i) = \prod_{i=0}^{\infty} \frac{1 - q^{2i}}{1 - q^i} == \frac{\prod_{i=0}^{\infty}(1 - q^{2i})}{\prod_{i=0}^{\infty}(1 - q^i)}$$

$$= \frac{\prod_{i=0}^{\infty}(1 - q^{2i})}{\prod_{i=0}^{\infty}(1 - q^{2i}) \prod_{i=0}^{\infty}(1 - q^{2i+1})}$$

$$\frac{1}{\prod_{i=0}^{\infty}(1 - q^{2i+1})} = \prod_{i=0}^{\infty} \frac{1}{1 - q^{2i+1}} \quad . \quad QED$$

If this is too *abstract*, here is a more concrete rendition, using ....

$$\sum_{n=0}^{\infty} dis(n)q^n = (1 + q)(1 + q^2)(1 + q^3)(1 + q^4)(1 + q^5) \cdots$$

$$\frac{1-q^2}{1-q}\frac{1-q^4}{1-q^2}\frac{1-q^6}{1-q^3}\frac{1-q^8}{1-q^4}\frac{1-q^{10}}{1-q^5}\frac{1-q^{12}}{1-q^6}\cdots$$
$$=\frac{1}{1-q}\frac{1}{1-q^3}\frac{1}{1-q^5}\cdots,$$

since each term in the numerator of the form $1-q^{even}$ will sooner-or later cancel-out with the same term at the bottom, resulting in only 1 surviving in the numerator, and only terms of the form $1-q^{odd}$ surviving at the bottom.

But the **BEST** proofs are **bijective** proofs.You are supposed to know two bijective proofs. The first one is due to Glashier. The second one is due to my great hero **James Joseph Sylvester**, to be covered in the next lecture.

Glashier's Bijection:

$Dis(n) \to Odd(n)$:

Input: A partition of n into distinct parts.

Output: A partition of n into odd parts.

Description: For each part p, write it as $p = a \cdot 2^s$ where s is the largest s such that $p/2^s$ is an integer, and a is necessarily odd (or else s would not be the largest!). Replace it by 2^s copies of a.

Problem 22.3a Apply the Glashier bijection to the distinct partition $(26, 10, 8, 6, 5, 4, 2)$.

Solution to problem 22.3a:

$$26 = 13 \cdot 2^1 = 13 \cdot 2 \quad ,$$
$$10 = 5 \cdot 2^1 = 5 \cdot 2 \quad ,$$
$$8 = 1 \cdot 2^3 = 1 \cdot 8 \quad ,$$
$$6 = 3 \cdot 2^1 = 3 \cdot 2 \quad ,$$
$$5 = 5 \cdot 2^0 = 5 \quad ,$$
$$4 = 1 \cdot 2^2 = 1 \cdot 4 \quad ,$$
$$2 = 1 \cdot 2^1 = 1 \cdot 2 \quad .$$

.

Hence (in exponent notation)

$$(26, 10, 8, 6, 5, 4, 2) \to 13^2 5^2 1^8 3^2 5^1 1^4 1^2 \quad ,$$

sorting and collecting this equals

$$1^{8+4+2} 3^2 5^{2+1} 13^2 = 1^{14} 3^2 5^3 13^2$$

Finally, going back to the usual notation we get

$$(13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad .$$

Ans. to 22.3a: The odd partition corresponding to the distinct partition $(26, 10, 8, 6, 5, 4, 2)$, under Glashier's bijection, is $(13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.

$Odd(n) \to Dis(n)$:

Input: A partition of n into odd parts.

Output: A partition of n into distinct parts.

Description: First convert the partition into **exponent** notation

$$1^{a_1} 3^{a_3} 5^{a_5} \quad ...$$

For each part (that must be odd, of course), $(2i+1)^a$, that shows up (i.e. where $a > 0$), write a in (sparse) binary representation

$$a = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_r} \quad , \quad j_1 > j_2 > \ldots > j_r \geq 0,$$

and replace it by the following parts

$$(2i+1) \cdot 2^{j_1} \quad , \quad (2i+1) \cdot 2^{j_2} \quad , \quad (2i+1) \cdot 2^{j_3} \quad , \quad \ldots \quad , \quad (2i+1) \cdot 2^{j_r} \quad .$$

Obviously all the parts are distinct, and at the end just sort them.

Problem 22.3b Apply the Glashier bijection to the odd partition

$$(13, 13, 5, 5, 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad .$$

We first convert it to exponent notation

$$13^2\, 5^3\, 3^2\, 1^{14} \quad .$$

Next, for each (necessarily odd!) part $(2i+1)^m$, we take the multiplicity and express it as a sum of powers of 2:

Regarding 13^2:

$$2 = 2^1 \quad ,$$

so 13^2 only yields **one** part in the output, namely $13 \cdot 2 = 26$.

Regarding 5^3 :

$$3 = 2^1 + 2^0 = 2 + 1 \quad ,$$

so 5^3 yields **two** parts in the output, namely $5 \cdot 2 = 10$, and $5 \cdot 1 = 5$.

Regarding 3^2 :

$$2 = 2^1 = 2 \quad ,$$

so 3^2 yields only **one** part in the output, namely $3 \cdot 2 = 6$.

Regarding 1^{14} :

$$14 = 2^3 + 2^2 + 2^1 = 8 + 4 + 2 \quad ,$$

so 1^{14} yields **three** parts in the output, namely $1 \cdot 8 = 8$, $1 \cdot 4 = 4$, $1 \cdot 2 = 2$.

Combining them, we get:

$$26, 10, 5, 6, 8, 4, 2 \quad .$$

Finally, for the sake of politeness, we sort the parts from largest to smallest

$$(26, 10, 8, 6, 5, 4, 2) \quad .$$

Ans. to 22.3b: The distinct partition corresponding to the odd partition (13,13,5,5,5,3,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1), under Glashier's bijection, is $(26, 10, 8, 6, 5, 4, 2)$.