An integer partition of the positive integer $n$ is any way of writing it as a sum of positive integers (possibly just $n$), where order does not matter.

So $3 + 1$ and $1 + 3$ are the same partition.

Since order does not matter, we write a partition as

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$

with the convention that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$.

Important Note: 0 is not allowed!

Since it is a nuisance to write plus-signs, we abbreviate and write partitions as vectors of non-decreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

Important Notation: The set of integer partitions of $n$ is denoted by $P(n)$.

Here are $P(n)$ for $0 \leq n \leq 7$.

$$P(0) = \{EmptyPartition\},$$

$$P(1) = \{(1)\},$$

$$P(2) = \{(2), (1, 1)\},$$

$$P(3) = \{(3), (2, 1), (1, 1, 1)\},$$

$$P(4) = \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\},$$

$$P(5) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\},$$

$$P(6) = \{(6), (5, 1), (4, 1, 1), (4, 2), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}.$$

Important Notation: The number of integer partitions of $n$ is denoted by $p(n)$. In other words $p(n) := |P(n)|$.

More important notation: For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $k$ is called the number of parts, and $\lambda_1$ is called the largest part.

Example: The number of parts of $4, 3, 2, 1, 1, 1$ is 6, the largest part is 4.
**Also Important Notation:** The set of integer partitions of $n$ with largest part $k$ is called $\mathcal{P}(n,k)$. $p(n,k) := |\mathcal{P}(n,k)|$ (in other words, $p(n,k)$ is the number of integer partitions of $n$ with largest part $k$.)

**Also Important Notation:** The set of integer partitions of $n$ whose number of parts is $k$ is called $\mathcal{Q}(n,k)$. Also $q(n,k) := |\mathcal{Q}(n,k)|$ (in other words, $q(n,k)$ is the number of integer partitions of $n$ whose number of parts is $k$.)

For example

\[
\mathcal{P}(6,6) = \{(6)\} , \quad \mathcal{P}(6,5) = \{(5,1)\} , \quad \mathcal{P}(6,4) = \{(4,1,1),(4,2)\} , \quad \mathcal{P}(6,3) = \{(3,3),(3,2,1),(3,1,1,1)\} ,
\]

\[
\mathcal{P}(6,2) = \{(2,2),(2,2,1,1),(2,1,1,1,1)\} , \quad \mathcal{P}(6,1) = \{(1,1,1,1,1,1)\} , .
\]

Counting the number of elements:

\[
p(6,6) = 1 , \quad p(6,5) = 1 , \quad p(6,4) = 2 , \quad p(6,3) = 3 , \quad p(6,2) = 3 , \quad p(6,1) = 1 , .
\]

Since every partition of $n$ has a unique largest part $k$, between 1 and $n$, we have the obvious, but important facts

\[
\mathcal{P}(n) = \bigcup_{k=1}^{n} \mathcal{P}(n,k) ,
\]

\[
p(n) = \sum_{k=1}^{n} p(n,k) , .
\]

**How to systematically generate** $\mathcal{P}(n,k)$ (and hence $\mathcal{P}(n)$)

You use recurrence!

If $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_r) \in \mathcal{P}(n,k)$ then $\lambda_1 = k$ and $\lambda_2 \leq k$, so $\lambda' := (\lambda_2,\ldots,\lambda_k)$ is a partition of $n-k$, hence a member of $\mathcal{P}(n-k,\lambda_2)$ for some $\lambda_2$ in $[1,k]$. Hence we have the set recursion

\[
\mathcal{P}(n,k) = \bigcup_{r=1}^{k} k \mathcal{P}(n-k,r) , \quad (k < n) ,
\]

and $\mathcal{P}(n,n) = \{(n)\}$. Of course $\mathcal{P}(n,k) = \emptyset$ if $k > n$.

Here, for any set of vectors $S$, $kS$ means the set of vectors obtained by sticking a $k$ in front of each member of $S$.

**Corollary:**

\[
p(n,k) = \sum_{r=1}^{k} p(n-k,r) , \quad (k < n) .
\]

**Note:** This recurrence enables a fairly fast computation of $p(n,k)$ and hence of $p(n) = \sum_{k=1}^{n} p(n,k)$. Of course $p(n,n) = 1$. 

Problem 21.1: Using $\mathcal{P}(n,k) = \bigcup_{r=1}^{k} k \mathcal{P}(n-k,r)$, construct, systematically, $\mathcal{P}(n)$ for $1 \leq n \leq 3$.

Sol. of 21.1: $\mathcal{P}(1,1) = \{(1)\}$, and hence $\mathcal{P}(1) = \{(1)\}$.

$$\mathcal{P}(2,1) = \bigcup_{r=1}^{1} 1 \mathcal{P}(1,1) = 1 \{1\} = \{(1)\} \ .$$

and $\mathcal{P}(2,2) = \{(2)\}$. So

$$\mathcal{P}(2) = \{(1,1),(2)\} \ .$$

For $n = 3$,

$$\mathcal{P}(3,1) = \bigcup_{r=1}^{1} 1 \mathcal{P}(2,1) = 1 \mathcal{P}(2,1) = \{(1,1,1)\} \ .$$

$$\mathcal{P}(3,2) = \bigcup_{r=1}^{2} 2 \mathcal{P}(1,1) = 2 \mathcal{P}(1,1) = 2 \{1\} = \{(2,1)\} \ .$$

Of course $\mathcal{P}(3,3) = \{(3)\}$. So

$$\mathcal{P}(3) = \{(3),(2,1),(1,1,1)\} \ .$$

Problem 21.1a: Using $p(n,k) = \sum_{r=1}^{k} p(n-k,r)$, compute $p(n)$ for $1 \leq n \leq 6$.

Sol. to 21.1a:

$$p(1,1) = 1 \quad \text{hence} \quad p(1) = 1$$
$$p(2,1) = p(1,1) = 1 \quad p(2,2) = 1 \quad \text{hence} \quad p(2) = 1 + 1 = 2$$

$p(3,1) = p(2,1) = 1 \quad p(3,2) = p(1,2)+p(1,1) = 0+1 = 1 \quad p(3,3) = 1 \quad \text{hence} \quad p(3) = 1+1+1 = 3$

$p(4,1) = p(3,1) = 1 \quad p(4,2) = p(2,2)+p(2,1) = 1 + 1 = 2 \quad p(4,3) = p(1,1) = 1 \quad p(4,4) = 1$

hence $p(4)=1+2+1+1=5$

$p(5,1) = p(4,1) = 1 \quad p(5,2) = p(3,2)+p(3,1) = 1 + 1 = 2 \quad p(5,3) = p(2,2)+p(2,1) = 1 + 1 = 2$

$p(5,4) = p(1,1) \quad p(5,5) = 1 \quad \text{hence} \quad p(5) = 1 + 2 + 2 + 1 + 1 = 7$

$p(6,1) = p(5,1) = 1 \quad p(6,2) = p(4,2)+p(4,1) = 2+1 = 3 \quad p(6,3) = p(3,3)+p(3,2)+p(3,1) = 1+1+1 = 3$

$p(6,4) = p(2,2)+p(2,1) = 1+1 = 2 \quad p(6,5) = p(1,1) = 1 \quad p(6,6) = 1 \quad \text{hence} \quad p(6) = 1+3+3+2+1+1 = 11$ .

The Ferrers Graph of a partition

The graph of a partition $(\lambda_1, \ldots, \lambda_k)$ is obtained by drawing a left-alligned diagram with
$\lambda_1$ dots on the top row

$\lambda_2$ dots on the second row

\[ \cdots \]

$\lambda_k$ dots on the $k$-th (last) row.

Of course, the total number of dots is the number that is being partitioned.

**Problem 21.2:** Draw the Ferrers graph of the partition $(6, 4, 3, 2)$.

**Sol. of 21.2:**

```
  * * * * * *
  * * * *    
  * * *      
  * * 
```

**Important Definition:** The *conjugate* of partition $\lambda$, denoted by $\lambda'$ is the partition whose Ferrers graph is the transpose of the Ferrers graph of $\lambda$. In other words

$\lambda'_1$ is the number of dots in the first column,

$\lambda'_2$ is the number of dots in the second column,

\[ \cdots \]

$\lambda'_r$ is the number of dots in the last column.

Of course the number of parts of $\lambda'$, $r$, is $\lambda_1$.

**Problem 21.3:** Find the conjugate partition of $(6, 4, 3, 2)$.

**Sol. to 21.3:** The Ferrers graph has 6 columns.

The first column has: 4 dots

The second column has: 4 dots

The third column has: 3 dots

The fourth column has: 2 dots

The fifth column has: 1 dots

The sixth column has: 1 dots

So $\lambda' = (4, 4, 3, 2, 1, 1)$. 

4
Simple but important theorem: The number of partitions of \( n \) whose largest part is \( k \) equals the number of partitions of \( n \) with exactly \( k \) parts.

Proof: There is a bijection between the two sets \( \lambda \to \lambda' \). It is a bijection since conjugation is an involution: \( \lambda'' = \lambda \).

Another data structure for integer partitions: If \( k \) is the largest part, then

\[
1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}
\]

where \( \alpha_1 \) is the number of times 1 showed up, \( \alpha_2 \) is the number of parts 2 showed up, etc. Most of the time we don’t put those parts that show 0 times.

Problem 21.4: Write \((7, 7, 5, 5, 5, 3, 3, 3, 1, 1, 1, 1)\) in exponents notation.

Ans. to 12.4: \(1^5 3^3 5^4 7^2\) in short version and \(1^5 2^0 3^4 5^4 6^0 7^2\) in the long version.

Note: If \(1^{\alpha_1} \ldots k^{\alpha_k}\) is a partition written in exponent notation, then the number being partitioned is \(1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \ldots + k \alpha_k\).

Very important theorem (Euler): Let \(p(n)\) be the number of partitions of \( n \) then we have the generating functions

\[
\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.
\]

Proof: Recall that

\[
\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots,
\]

So

\[
\prod_{i=1}^{\infty} \frac{1}{1-q^i} = (1 + q + q^2 + q^3 + \ldots)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots)\.
\]

When we open up parentheses, a typical term looks like

\[
q^{\alpha_1} \cdot (q^2)^{\alpha_2} \cdot (q^3)^{\alpha_3} \ldots,
\]

making a decision

How many copies of 1 shall I pick? (ans. \(\alpha_1\), where \(\alpha_1\) could be 0, 1, 2, 3, \ldots)

How many copies of 2 shall I pick? (ans. \(\alpha_2\), where \(\alpha_2\) could be 0, 1, 2, 3, \ldots)

How many copies of 3 shall I pick? (ans. \(\alpha_3\), where \(\alpha_3\) could be 0, 1, 2, 3, \ldots), and so on.

This gives a partition in exponent notation. \(1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \ldots\) and it contributes to the power of \(q^n\) where \(n = 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + 3 \cdot \alpha_3 \ldots\), so we get credit for that partition. Every partition of \( n \) contributes a 1 to the coefficient of \( q^n \) in this ‘infinite’ product, so we get \(p(n)\).
Problem 21.5: Find \( p(n) \) for \( n \) between 1 and 4 using the generationg function.

Solution of 21.5: Even though the product is ‘infinite’, since we are only interested in the terms up to \( q^4 \) we only need the first four terms. Also we can discard with ... any powers of \( q \) larger than 4. So, let’s consider

\[
\frac{1}{1-q} \frac{1}{1-q^2} \frac{1}{1-q^3} \frac{1}{1-q^4}
\]

We first expand \( \frac{1}{1-q} \) up to the fourth power

\[
\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \ldots .
\]

Now we multiply by \( \frac{1}{1-q^2} = 1 + q^2 + q^4 + \ldots \)

\[
\frac{1}{1-q} \frac{1}{1-q^2} = (1 + q + q^2 + q^3 + q^4 + \ldots)(1 + q^2 + q^4 + \ldots)
\]

\[
= (1 + q + q^2 + q^3 + q^4 + \ldots + q^2(1 + q + q^2 + \ldots) + q^4(1 + \ldots)
\]

\[
1 + q + q^2 + q^3 + q^4 + \ldots = 1 + q + 2q^2 + 2q^3 + 3q^4 + \ldots
\]

Now we multiply by \( \frac{1}{1-q^3} = 1 + q^3 + \ldots \)

\[
\frac{1}{1-q} \frac{1}{1-q^2} \frac{1}{1-q^3} \frac{1}{1-q^4} = (1 + q + 2q^2 + 3q^3 + 4q^4 + \ldots)(1 + q^3 + \ldots) = 1 + q + 2q^2 + 2q^3 + 3q^4 + q^5(1 + q)
\]

\[
= 1 + q + 2q^2 + 2q^3 + 3q^4 + q^5 = 1 + q + 2q^2 + 3q^3 + 4q^4 + q^5
\]

Now we multiply by \( \frac{1}{1-q^4} = 1 + q^4 + \ldots \), getting

\[
\frac{1}{1-q} \frac{1}{1-q^2} \frac{1}{1-q^3} \frac{1}{1-q^4} = (1 + q + 2q^2 + 3q^3 + 4q^4 + \ldots)(1 + q^4 + \ldots) = 1 + q + 2q^2 + 3q^3 + 4q^4 + q^5 + q^6 + \ldots
\]

Reading off the coefficients, we get

Ansv. to 21.5: \( p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5. \)