Dr. Z.'s Number Theory Lecture 21 Handout: Integer Partitions I

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version of Nov. 17, 2013 (please discard older versions)

An integer partition of the positive integer n is any way of writing it as a sum of positive integers (possibly just n), where order **does not matter**.

So $3+1$ and $1+3$ are the **same** partition.

Since order does not matter,we write a partition as

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_k \quad ,$$

with the convention that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$.

Important Note: 0 is not allowed!

Since it is a nuisance to write plus-signs, we abbreviate and write partitions as vectors of non-decreasing positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \quad .$$

Important Notation: The set of integer partitions of n is denoted by $\mathcal{P}(n)$.

Here are $\mathcal{P}(n)$ for $0 \leq n \leq 7$.

$$\mathcal{P}(0) = \{EmptyPartition\} \quad ,$$

$$\mathcal{P}(1) = \{(1)\} \quad ,$$

$$\mathcal{P}(2) = \{(2), (1,1)\} \quad ,$$

$$\mathcal{P}(3) = \{(3), (2,1), (1,1,1)\} \quad ,$$

$$\mathcal{P}(4) = \{(4), (3,11), (2,2), (2,1,1), (1,1,1,1)\} \quad ,$$

$$\mathcal{P}(5) = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\} \quad ,$$

$$\mathcal{P}(6) = \{(6), (5,1), (4,1,1), (4,2), (3,3), (3,2,1), (3,1,1,1), (2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1)\} \quad .$$

Important Notation: The number of integer partitions of n is denoted by $p(n)$. In other words $p(n) := |\mathcal{P}(n)|$.

More important notation: For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, k is called the **number of parts**, and λ_1 is called the **largest part**.

Example: The number of parts of $(4,3,2,1,1,1)$ is 6, the largest part is 4.

Also Important Notation: The set of integer partitions of n with largest part k is called $\mathcal{P}(n,k)$. $p(n,k) := |\mathcal{P}(n,k)|$ (in other words, $p(n,k)$ is the number of integer partitions of n with largest part k.)

Also Important Notation: The set of integer partitions of n whose number of parts is k is called $\mathcal{Q}(n,k)$. Also $q(n,k) := |\mathcal{Q}(n,k)|$ (in other words, $q(n,k)$ is the number of integer partitions of n whose number of parts is k.)

For example

$$\mathcal{P}(6,6) = \{(6)\} \quad , \quad \mathcal{P}(6,5) = \{(5,1)\} \quad , \quad \mathcal{P}(6,4) = \{(4,1,1),(4,2)\} \quad , \quad \mathcal{P}(6,3) = \{(3,3),(3,2,1),(3,1,1,1)\} \quad ,$$

$$\mathcal{P}(6,2) = \{(2,2,2),(2,2,1,1),(2,1,1,1,1)\} \quad , \quad \mathcal{P}(6,1) = \{(1,1,1,1,1,1)\} \quad , \quad .$$

Counting the number of elements:

$$p(6,6) = 1 \quad , \quad p(6,5) = 1 \quad , \quad p(6,4) = 2 \quad , \quad p(6,3) = 3 \quad , \quad p(6,2) = 3 \quad , \quad p(6,1) = 1 \quad , \quad .$$

Since every partition of n has a unique largest part k, between 1 and n, we have the obvious, but **important** facts

$$\mathcal{P}(n) = \bigcup_{k=1}^{n} \mathcal{P}(n,k) \quad ,$$

$$p(n) = \sum_{k=1}^{n} p(n,k) \quad , \quad .$$

How to systematically generate $\mathcal{P}(n,k)$ (and hence $\mathcal{P}(n)$)

You use recurrence!

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{P}(n,k)$ then $\lambda_1 = k$ and $\lambda_2 \leq k$, so $\lambda' := (\lambda_2, \ldots, \lambda_k)$ is a partition of $n-k$, hence a member of $\mathcal{P}(n-k, \lambda_2)$ for some λ_2 in $[1,k]$. Hence we have the **set recursion**

$$\mathcal{P}(n,k) = \bigcup_{r=1}^{k} k\,\mathcal{P}(n-k,r) \quad , \quad (k < n) \quad ,$$

and $\mathcal{P}(n,n) = \{(n)\}$. Of course $\mathcal{P}(n,k) = \emptyset$ if $k > n$.

Here, for any set of vectors S, kS means the set of vectors obtained by sticking a k in front of each member of S.

Corollary:

$$p(n,k) = \sum_{r=1}^{k} p(n-k,r) \quad , \quad (k < n) \quad .$$

Note: This recurrence enables a fairly fast computation of $p(n,k)$ and hence of $p(n) = \sum_{k=1}^{n} p(n,k)$. Of course $p(n,n) = 1$.

Problem 21.1: Using $\mathcal{P}(n,k) = \bigcup_{r=1}^{k} k\mathcal{P}(n-k,r)$, construct, systematically, $\mathcal{P}(n)$ for $1 \le n \le 3$.

Sol. of 21.1: $\mathcal{P}(1,1) = \{(1)\}$, and hence $\mathcal{P}(1) = \{(1)\}$.

$$\mathcal{P}(2,1) = \bigcup_{r=1}^{1} 1\mathcal{P}(1,1) = 1\{(1)\} = \{(1,1)\} \quad .$$

and $\mathcal{P}(2,2) = \{(2)\}$. So

$$\mathcal{P}(2) = \{(1,1),(2)\} \quad .$$

For $n = 3$,

$$\mathcal{P}(3,1) = \bigcup_{r=1}^{1} 1\mathcal{P}(2,r) = 1\mathcal{P}(2,1) = \{(1,1,1)\} \quad .$$

$$\mathcal{P}(3,2) = \bigcup_{r=1}^{2} 2\mathcal{P}(1,r) = 2\mathcal{P}(1,1)\bigcup 2\mathcal{P}(1,2) = 2\{(1)\} = \{(2,1)\} \quad .$$

Of course $\mathcal{P}(3,3) = \{(3)\}$. So

$$\mathcal{P}(3) = \{(3),(2,1),(1,1,1)\} \quad .$$

Problem 21.1a: Using $p(n,k) = \sum_{r=1}^{k} p(n-k,r)$, compute $p(n)$ for $1 \le n \le 6$.

Sol. to 21.1a:

$$p(1,1) = 1 \quad \textit{hence} \quad p(1) = 1$$

$$p(2,1) = p(1,1) = 1 \quad p(2,2) = 1 \quad \textit{hence} \quad p(2) = 1+1 = 2$$

$$p(3,1) = p(2,1) = 1 \quad p(3,2) = p(1,2)+p(1,1) = 0+1 = 1 \quad p(3,3) = 1 \quad \textit{hence} \quad p(3) = 1+1+1 = 3$$

$$p(4,1) = p(3,1) = 1 \quad p(4,2) = p(2,2)+p(2,1) = 1+1 = 2 \quad p(4,3) = p(1,1) = 1 \quad p(4,4) = 1$$

hence p(4)=1+2+1+1=5

$$p(5,1) = p(4,1) = 1 \quad p(5,2) = p(3,2)+p(3,1) = 1+1 = 2 \quad p(5,3) = p(2,2)+p(2,1) = 1+1 = 2$$

$$p(5,4) = p(1,1) \quad p(5,5) = 1 \quad \textit{hence} \quad p(5) = 1+2+2+1+1 = 7$$

$$p(6,1) = p(5,1) = 1 \quad p(6,2) = p(4,2)+p(4,1) = 2+1 = 3 \quad p(6,3) = p(3,3)+p(3,2)+p(3,1) = 1+1+1 = 3$$

$$p(6,4) = p(2,2)+p(2,1) = 1+1 = 2 \quad p(6,5) = p(1,1) = 1 \quad p(6,6) = 1 \quad \textit{hence} \quad p(6) = 1+3+3+2+1+1 = 11 \quad .$$

The Ferrers Graph of a partition

The **graph** of a partition $(\lambda_1,\ldots,\lambda_k)$ is obtained by drawing a left-alligned diagram with

λ_1 dots on the top row

λ_2 dots on the second row

...

λ_k dots on the k-th (last) row .

Of course, the total number of dots is the number that is being partitioned.

Problem 21.2: Draw the Ferrers graph of the partition $(6, 4, 3, 2)$.

Sol. of 21.2:

$$\begin{array}{cccccc} * & * & * & * & * & * \\ * & * & * & * & & \\ * & * & * & & & \\ * & * & & & & \end{array} .$$

Important Definition: The **conjugate** of partition λ, denoted by λ' is the patition whose Ferrers graph is the **transpose** of the Ferrers graph of λ. In other words

λ'_1 is the number of dots in the first column,

λ'_2 is the number of dots in the second column,

...

λ'_r is the number of dots in the last column .

Of course the number of parts of λ', r, is λ_1.

Problem 21.3: Find the conjugate partition of $(6, 4, 3, 2)$.

Sol. to 21.3: The Ferrers graph has 6 columns.

The first column has: 4 dots

The second column has: 4 dots

The third column has: 3 dots

The fourth column has: 2 dots

The fifth column has: 1 dots

The sixth column has: 1 dots

So $\lambda' = (4, 4, 3, 2, 1, 1)$.

Simple but important theorem: The number of partitions of n whose largest part is k equals the number of partitions of n with exactly k parts.

Proof: There is a bijection between the two sets $\lambda \to \lambda'$. It is a bijection since conjutation is an **involution**: $\lambda'' = \lambda$.

Another data structure for integer partitions: If k is the largest part, then

$1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}$

where α_1 is the number of times 1 showed up, α_2 is the number of parts 2 showed up, etc. Most of the time we don't put those parts that show 0 times.

Problem 21.4: Write $(7, 7, 5, 5, 5, 5, 3, 3, 3, 1, 1, 1, 1, 1)$ in exponents notation.

Ans. to 12.4: $1^5 3^3 5^4 7^2$ in short version and $1^5 2^0 3^3 4^0 5^4 6^0 7^2$ in the long version.

Note: If $1^{\alpha_1} \ldots k^{\alpha_k}$ is a partition written in exponent notation, then the number being partitioned is $1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \ldots + k\alpha_k$.

Very important thereom (Euler): Let $p(n)$ be the number of partitions of n then we have the **generating functions**

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} \quad .$$

Proof: Recall that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \quad ,$$

So

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = (1 + q + q^2 + q^3 + ..)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots) \quad .$$

When we open up paranthes, a typical term looks like

$$q^{\alpha_1} \cdot (q^2)^{\alpha_2} \cdot (q^3)^{\alpha_3} \cdots ,$$

making a **decision**

How many copies of 1 shall I pick? (ans. α_1, where α_1 could be $0, 1, 2, 3, \ldots$) ,

How many copies of 2 shall I pick? (ans. α_2, where α_2 could be $0, 1, 2, 3, \ldots$) ,

How many copies of 3 shall I pick? (ans. α_3, where α_3 could be $0, 1, 2, 3, \ldots$) , and so on.

This gives a partition in **exponent notation**. $1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \ldots$ and it contributes to the power of q^n where $n = 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + 3 \cdot \alpha_3 \ldots$, so we get credit for that partition. Every partition of n contributes a 1 to the coefficient of q^n in this 'infinite' product, so we get $p(n)$.

Problem 21.5: Find $p(n)$ for n between 1 and 4 using the generationg function.

Solution of 21.5: Even though the product is 'infinite', since we are only interested in the terms up to q^4 we only need the first four terms. Also we can **discard** with $\ldots$ any powers of q larger than 4. So, let's consider

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^4}$$

We first expand $\frac{1}{1-q}$ up to the fourth power

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \ldots \quad .$$

Now we multiply by $\frac{1}{1-q^2} = 1 + q^2 + q^4 + \ldots$

$$\frac{1}{1-q}\frac{1}{1-q^2} = (1 + q + q^2 + q^3 + q^4 + \ldots)(1 + q^2 + q^4 + \ldots)$$

$$= (1 + q + q^2 + q^3 + q^4 + \ldots + q^2(1 + q + q^2 + \ldots) + q^4(1 + \ldots)$$

$$1 + q + q^2 + q^3 + q^4 + q^2 + q^3 + q^4 + q^4 + \ldots = 1 + q + 2q^2 + 2q^3 + 3q^4 + \ldots$$

Now we multiply by $\frac{1}{1-q^3} = 1 + q^3 + \ldots$

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3} = (1+q+2q^2+2q^3+3q^4+\ldots)(1+q^3+\ldots) = 1+q+2q^2+2q^3+3q^4+q^3(1+q)$$

$$= 1 + q + 2q^2 + 2q^3 + 3q^4 + q^3 + q^4 = 1 + q + 2q^2 + 3q^3 + 4q^4 + \ldots$$

Now we multiply by $\frac{1}{1-q^4} = 1 + q^4 + \ldots$, getting

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^4} = (1 + q + 2q^2 + 3q^3 + 4q^4 +)(1 + q^4) =$$

$$= 1+q+2q^2+3q^3+4q^4+q^4(1+\ldots) = 1+q+2q^2+3q^3+4q^4+q^4+\ldots = 1+q+2q^2+3q^3+5q^4+\ldots$$

Reading off the coefficients, we get

Ans. to 21.5: $p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$.