Dr. Z.'s Number Theory Lecture 2 Handout: Incomplete and Complete Mathematical Induction

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Incomplete Induction: Looking at several cases, detecting a pattern, and *generalizing*.

Problem 2.1: Guess a nice formula for

$$S(n) := 1 + 3 + 5 + \ldots + (2n-1) \quad (= \sum_{i=1}^{n} 2i - 1).$$

Solution to 2.1: $S(1) = 1$, $S(2) = 1 + 3 = 4$, $S(3) = 1 + 3 + 5 = 9$, $S(4) = 1 + 3 + 5 + 9 = 16$, but these are all **perfect squares**! $S(1) = 1$, $S(2) = 2^2$, $S(3) = 3^2$, $S(4) = 4^2$. So we *conjecture* that for every integer $n \geq 0$,

$$S(n) = n^2 \quad .$$

Complete Mathematical Induction

Fundamental Theorem of Discrete Calculus

If $a(i)$ and $S(n)$ are *any* expressions, in order to prove for **every** $n \geq 0$ that

$$S(n) = \sum_{i=1}^{n} a(i)$$

(alias: $S(n) = a(1) + \ldots + a(n)$), all you need is to check

(i) $S(0) = 0$ (the empty sum is zero!)

(ii) $S(n) - S(n-1) = a(n)$

The formal proof is by mathematical induction, but it essentially follows from the definition of $\sum$.

Problem 2.2: Prove rigorously that , for every $n \geq 0$,

$$\sum_{i=1}^{n} (2i-1) = n^2 \quad .$$

Solution of 2.2: Here $a(i) = 2i - 1$, $S(n) = n^2$. By the fundamental theorem of discrete calculus, we need to show

$$S(0) = 0 \quad , \quad S(n) - S(n-1) = a(n) \quad .$$

Using arithmetic and algebra, $S(0) = 0^2 = 0$ and $S(n) - S(n-1) = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = n^2 - n^2 + 2n - 1 = 2n - 1$. But since $a(i) = 2i - 1$ (and i is but a **place-holder**), we have $a(n) = 2n - 1$. Hence $S(n) - S(n-1) = a(n)$. QED!

First Important Fact: If $a(i)$ is a polynomial of i of degree d, then the *indefinite sum*

$$S(n) = \sum_{i=1}^{n} a(i)$$

(alias $a(1) + a(2) + \ldots + a(n)$) is a polynomial of n of degree $d + 1$.

Second Important Fact: If a polynomial of degree $\leq d$ vanishes at $d + 1$ **different** places, it is **identically zero** (i.e. the 0 polynomial).

Immediate Consequence: If two polynomials of degree $\leq d$ coincide at $d + 1$ **different** places, they are **always** the same.

Zeilberger-Style (FULLY RIGOROUS) Proofs of Indefinite Polynomial Sum Identities

In order to rigorously prove an identity of the form

$$S(n) = \sum_{i=1}^{n} a(i) \quad ,$$

where $a(i)$ is a polynomial of degree d and $S(n)$ is a polynomial of degree $\leq d+1$, it is enough to check it for the $d + 2$ special cases ($n = 0, n = 1, \ldots, n = d + 1$).

Note: Of course, it is also OK to check it for any other $d + 2$ special cases, for example $n = 5761, n = 5762, \ldots, n = 5761 + d + 1$, but it would be less convenient.

Problem 2.3: Give a Zeilberger-style (rigorous!) proof of the identity

$$\sum_{i=1}^{n} (2i - 1) = n^2 \quad .$$

Solution of 2.3: The summand $a(i) = 2i - 1$ is a a polynomial in i of degree 1, and the proposed value of the sum, $S(n) = n^2$ is a polynomial in n of degree 2. So here $d = 1$, and in order to prove the identity for *every* positive integer n, it suffices to prove it for $n = 0, 1, 2$.

Now

$$\sum_{i=1}^{0} (2i - 1) = 0 = 0^2 \quad .$$

(the empty-sum is always zero).

$$\sum_{i=1}^{1} (2i - 1) = 1 = 1^2 \quad .$$

$$\sum_{i=1}^{2} (2i - 1) = 1 + 3 = 4 = 2^2 \quad .$$

QED!