Incomplete Induction: Looking at several cases, detecting a pattern, and generalizing.

Problem 2.1: Guess a nice formula for

\[ S(n) := 1 + 3 + 5 + \ldots + (2n - 1) = \sum_{i=1}^{n} 2i - 1. \]

Solution to 2.1: \( S(1) = 1, S(2) = 1 + 3 = 4, S(3) = 1 + 3 + 5 = 9, S(4) = 1 + 3 + 5 + 9 = 16, \) but these are all perfect squares! \( S(1) = 1, S(2) = 2^2, S(3) = 3^2, S(4) = 4^2. \) So we conjecture that for every integer \( n \geq 0, \)

\[ S(n) = n^2. \]

Complete Mathematical Induction

Fundamental Theorem of Discrete Calculus

If \( a(i) \) and \( S(n) \) are any expressions, in order to prove for every \( n \geq 0 \) that

\[ S(n) = \sum_{i=1}^{n} a(i) \]

(alias: \( S(n) = a(1) + \ldots + a(n) \)), all you need is to check

(i) \( S(0) = 0 \) (the empty sum is zero!)

(ii) \( S(n) - S(n-1) = a(n) \)

The formal proof is by mathematical induction, but it essentially follows from the definition of \( \sum. \)

Problem 2.2: Prove rigorously that, for every \( n \geq 0, \)

\[ \sum_{i=1}^{n} (2i - 1) = n^2. \]

Solution of 2.2: Here \( a(i) = 2i - 1, S(n) = n^2. \) By the fundamental theorem of discrete calculus, we need to show

\[ S(0) = 0, \quad S(n) - S(n-1) = a(n). \]

Using arithmetic and algebra, \( S(0) = 0^2 = 0 \) and \( S(n) - S(n-1) = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1. \) But since \( a(i) = 2i - 1 \) (and \( i \) is but a place-holder), we have \( a(n) = 2n - 1. \) Hence \( S(n) - S(n-1) = a(n). \) QED!
First Important Fact: If \( a(i) \) is a polynomial of \( i \) of degree \( d \), then the indefinite sum
\[
S(n) = \sum_{i=1}^{n} a(i)
\]
(alias \( a(1) + a(2) + \ldots + a(n) \)) is a polynomial of \( n \) of degree \( d + 1 \).

Second Important Fact: If a polynomial of degree \( \leq d \) vanishes at \( d + 1 \) different places, it is identically zero (i.e. the 0 polynomial).

Immediate Consequence: If two polynomials of degree \( \leq d \) coincide at \( d + 1 \) different places, they are always the same.

Zeilberger-Style (FULLY RIGOROUS) Proofs of Indefinite Polynomial Sum Identities

In order to rigorously prove an identity of the form
\[
S(n) = \sum_{i=1}^{n} a(i)
\]
where \( a(i) \) is a polynomial of degree \( d \) and \( S(n) \) is a polynomial of degree \( \leq d + 1 \), it is enough to check it for the \( d + 2 \) special cases \( (n = 0, n = 1, \ldots, n = d + 1) \).

Note: Of course, it is also OK to check it for any other \( d + 2 \) special cases, for example \( n = 5761, n = 5762, \ldots, n = 5761 + d + 1 \), but it would be less convenient.

Problem 2.3: Give a Zeilberger-style (rigorous!) proof of the identity
\[
\sum_{i=1}^{n} (2i - 1) = n^2 .
\]

Solution of 2.3: The summand \( a(i) = 2i - 1 \) is a polynomial of degree 1, and the proposed value of the sum, \( S(n) = n^2 \) is a polynomial of degree 2. So here \( d = 1 \), and in order to prove the identity for every positive integer \( n \), it suffices to prove it for \( n = 0, 1, 2 \).

Now
\[
\sum_{i=1}^{0} (2i - 1) = 0 = 0^2 .
\]
(the empty-sum is always zero).
\[
\sum_{i=1}^{1} (2i - 1) = 1 = 1^2 .
\]
\[
\sum_{i=1}^{2} (2i - 1) = 1 + 3 = 4 = 2^2 .
\]
QED!