Dr. Z.'s Number Theory Lecture 12 Handout: Divisibility Rules and Perpetual Calendar

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To find out whether n, given in decimal, is divisible by 3, just add the digits, and see whether it is divisible by 3, why?

Recall from Lecture 4, that our notation

$$d_k d_{k-1} \ldots d_0$$

for a $k+1$-digit positive integer, in base 10 is just *shorthand* for

$$n = \sum_{i=0}^{k} d_i 10^i \quad ,$$

being divisible by 3 means that $n \pmod 3 = 0$. Since $10 \pmod 3 = 1$, $10^i = 1 \pmod 3$, so

$$n \pmod 3 = \sum_{i=0}^{k} d_i \pmod 3 \quad .$$

Same with 9, since $10 \pmod 9 = 1$.

For 11 it is a bit harder, $10 \pmod{11} = -1$ so

$$n \pmod{11} = \sum_{i=0}^{k} (-1)^i d_i \pmod{11} \quad ,$$

so you take the alternating sum of the digits, and see whether it is divisible by 11.

Since $1000 \equiv -1 \pmod 7$, we have a more complicated rule for division by 7. write n in base 1000, meaning that you divide it, from left-to-right, into blocks of 3, i.e.

$$n = \sum_{i=0}^{k} d_i 1000^i \quad , \quad 0 \leq d_i < 1000$$

Then

$$n \pmod 7 = \sum_{i=0}^{k} d_i (-1)^i \pmod 7 \quad .$$

Problem 12.1: Decide whether 123456789 is divisible by 7.

Solution to 12.1:123|345|789. So the alternating sum is

$$123 - 345 + 789 = 567 \quad .$$

Since 567 is divisible by 7 so is 123456789.

Doing things analogously in base b easily produces divisibility tests for division by $b-1$ and $b+1$.

Problem 12.2: Decide whether 587624 (base 9) is divisible by (i)8 (ii) (our) 10 (alias 11 (base 9)).

Solution to 12.2:

(i) The sum of the digits is $5+8+7+6+2+4=32$ (base 10) (that equals 35(base 9), but we don't need that). Since 32 is divisible by 8, 587624 (base 9) is divisible by 8.

(ii) The alternating sum of the digits is $5-8+7-6+2-4=-4$ (base 10) (which is also -4 (base 9)), since this is not divisible by (our) 10, neither is 587624(base 9)).

Perpetual Calendar Formula

Read and explain to yourself why the algorithm in the heading "Perpetual calendar formula" is correct. `http://en.wikipedia.org/wiki/Perpetual_calendar`

Problem 12.3: What day of the week will (or was)

(i) Fourth of July 2015

(ii) Fourth of July 1915

(iii) Fourth of July 1815

Solution of 12.3(i):

1. $15+\lfloor 15/4\rfloor = 15+3=18 \pmod 7 = 4$

2. $4+0=4$ (the July shift is 0)

3. $4+4 \pmod 7 = 1$

4. $1+6 \pmod 7 = 0$

Ans. to 12.3(i): July 4, 2015 will be on Saturday.

Solution of 12.3(ii):

1. $15+\lfloor 15/4\rfloor = 15+3=18 \pmod 7 = 4$

2. $4+0=4$ (the July shift is 0)

3. $4+4 \pmod 7 = 1$

4. $1+0 \pmod 7 = 1$

Ans. to 12.3(ii): July 4, 1915 was on Sunday.

Solution of 12.3(iii):

1. $15 + \lfloor 15/4 \rfloor = 15 + 3 = 18 \pmod 7 = 4$

2. $4 + 0 = 4$ (the July shift is 0)

3. $4 + 4 \pmod 7 = 1$

4. $1 + 2 \pmod 7 = 3$

Ans. to 12.3(iii): July 4, 1815 was on Tuesday.