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MATH 481 (1), Dr. Z. , Exam 1, Fri., Oct. 16, 2026, 10:20-11:40 am, LSH B-117

PUT The FINAL ANSWER TO EACH PROBLEM IN THE AVAILABLE BOX

Do not write below this line

1. (out of 10)
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tot: (out of 100)

EXPLAIN EVERYTHING!

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1. (10 pts., 2 points each)

Three professors share an office:

- Professor 1 has one wife and two children
- Professor 2 had two wives and has no children
- Professor 3 had three wives and has four children

If W is the random variable: ‘Number of wives’ ; and C is the random variable: ‘number of children’ defined on the sample space of the set of occupants of that office, find

$$E[W], E[C], Var(W), Var(C), Cor(W, C) \quad .$$

ans.: $E[W] = 2; \quad E[C] = 2; \quad Var(W) = \frac{2}{3};$

$Var(C) = \frac{8}{3}; \quad Cor(W, C) = \frac{1}{2} \quad .$

$$E[W] = \frac{1 + 2 + 3}{3} = \frac{6}{3} = 2 \quad .$$

$$E[C] = \frac{2 + 0 + 4}{3} = \frac{6}{3} = 2 \quad .$$

$$Var(W) = \frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3} = \frac{1 + 0 + 1}{3} = \frac{2}{3} \quad .$$

$$Var(C) = \frac{(2-2)^2 + (0-2)^2 + (4-2)^2}{3} = \frac{0 + 4 + 4}{3} = \frac{8}{3} \quad .$$

$$Cov(W, C) = \frac{(1-2) \cdot (2-2) + (2-2) \cdot (0-2) + (3-2) \cdot (4-2)}{3} = \frac{0 + 0 + 2}{3} = \frac{2}{3} \quad .$$

$$Cor(W, C) = \frac{Cov(W, C)}{\sqrt{Var(W)}\sqrt{Var(C)}} = \frac{\frac{2}{3}}{\sqrt{\frac{2}{3} \cdot \frac{8}{3}}} = \frac{1}{2} \quad .$$

2. (10 points) With the same data as Problem 1:

- Professor 1 has two children ,
- Professor 2 no children ,
- Professor 3 has four children .

What are the skewness and kurtosis of the random variable: ‘number of children of a professor in that office’.

ans. skewness=0 ; kurtosis= $\frac{3}{2}$.

$$\mu = \frac{2 + 0 + 4}{3} = 2$$

$$m_2 = \frac{(2-2)^2 + (0-2)^2 + (4-2)^2}{3} = \frac{0 + 4 + 4}{3} = \frac{8}{3} .$$

$$m_3 = \frac{(2-2)^3 + (0-2)^3 + (4-2)^3}{3} = \frac{0 - 8 + 8}{3} = 0 .$$

$$m_4 = \frac{(2-2)^4 + (0-2)^4 + (4-2)^4}{3} = \frac{0 + 16 + 16}{3} = \frac{32}{3} .$$

$$skewness = \frac{m_3}{m_2^{3/2}} = 0 .$$

$$kurtosis = \frac{m_4}{m_2^2} = \frac{\frac{32}{3}}{(\frac{8}{3})^2} = \frac{3}{2} . .$$

3. (10 points) Let X be the random variable whose continuous pdf is

$$f(x) = \begin{cases} 4x^3 & , 0 < x < 1 \\ 0 & , \text{otherwise.} \end{cases} ;$$

(i) What is the expectation, $E[X]$?

(ii) What is the variance, $Var(X)$?

ans. (i) $\frac{4}{5}$; (ii) $\frac{2}{75}$.

$$E[X] = \int_0^1 4x^3 \cdot x \, dx = \int_0^1 4x^4 \, dx = 4 \frac{x^5}{5} \Big|_0^1 = 4 \cdot \left(\frac{1}{5} - 0\right) = \frac{4}{5} \quad .$$

$$E[X^2] = \int_0^1 4x^3 \cdot x^2 \, dx = \int_0^1 4x^5 \, dx = 4 \frac{x^6}{6} \Big|_0^1 = 4 \cdot \left(\frac{1}{6} - 0\right) = \frac{2}{3} \quad .$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{3} - \frac{16}{25} = \frac{50 - 48}{75} = \frac{2}{75} \quad .$$

4. (10 points altogether) Let X be the continuous random variable whose probability density function (pdf) is:

$$f(x) = \begin{cases} \frac{2}{x^2} & , \quad 1 < x < 2 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

(i) (4 points) For an integer $k \geq 2$, what is the k -th moment about the **origin**?

(ii) (6 points) What is the skewness?

Hint: Recall that if μ_k are the usual **moments-about-the-mean** (aka **moments**) and μ'_k are the easier-to-compute **moments-about-the origin**, and μ is the expectation (aka mean), then,

$$\mu_2 = \mu'_2 - \mu^2 \quad ; \quad \mu_3 = \mu'_3 - 3(\mu'_2)^2 \mu + 2\mu^3 \quad .$$

ans. k -th moment about the **origin** = $\frac{2}{k-1}(2^{k-1} - 1)$

$$\text{skewness} = \frac{3 - 12 \ln 4 + 2(\ln 4)^2}{(2 - (\ln 4)^2)^{3/2}} \quad .$$

$$\mu'_k = \int_1^2 \frac{2}{x^2} \cdot x^k dx = \int_1^2 2x^{k-2} = \frac{2x^{k-1}}{k-1} \Big|_1^2 = \frac{2}{k-1}(2^{k-1} - 1)$$

So

$$\mu'_2 = \frac{2}{2-1}(2^{2-1} - 1) = 2 \quad .$$

$$\mu'_3 = \frac{2}{3-1}(2^{3-1} - 1) = 3 \quad .$$

$$\mu = \int_1^2 x \frac{2}{x^2} = \int_1^2 \frac{2}{x} = 2 \ln x \Big|_1^2 = 2 \ln 2 = \ln 4 \quad .$$

$$\mu_2 = \mu'_2 - \mu^2 = 2 - (\ln 4)^2 \quad .$$

$$\mu_3 = \mu'_3 - 3(\mu'_2)^2 \mu + 2\mu^3 = 3 - 3(2)^2 \cdot \ln 4 + 2(\ln 4)^2 = 3 - 12 \ln 4 + 2(\ln 4)^2$$

$$\text{skewness} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{3 - 12 \ln 4 + 2(\ln 4)^2}{(2 - (\ln 4)^2)^{3/2}} \quad .$$

5. (10 points altogether) If you toss a coin whose probability of Heads is $\frac{2}{3}$ six times,

(j) (4 points) what is the probability that the total number of Heads equals exactly to the expected number of Heads?

(ii) (6 points) what is the probability that the total number of Heads is strictly above the expected number of Heads?

ans. (i) $\frac{80}{243}$ (ii) $\frac{256}{729}$.

$$\mu = 6 \cdot \frac{2}{3} = 4 \quad .$$

$$Pr(H = 4) = \binom{6}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 = \frac{80}{243} \quad .$$

$$Pr(H > 4) = Pr(H = 5) + Pr(H = 6) = \binom{6}{5} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^1 + \binom{6}{6} \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0 = \frac{256}{729} \quad .$$

6. (10 points) The expected number of letters I get each day is one letter.

Assuming that the number of letters I get each day is a Poisson distribution, how likely am I to get:

(i) (3 pts) no letters?

(ii) (3 pts) exactly two letters?

(iii) (4 pts) more than three letters?

ans. (i) $\frac{1}{e}$; (ii) $\frac{1}{2e}$; (iii) $1 - \frac{8}{3e}$.

In general

$$P(X = k) = e^{-\mu} \frac{\mu^k}{k!} .$$

In this problem $\mu = 1$. So

(i)

$$P(X = 0) = e^{-1} \cdot \frac{1^0}{0!} = \frac{1}{e}$$

(ii)

$$P(X = 2) = e^{-1} \cdot \frac{1^2}{2!} = \frac{1}{2e}$$

(iii)

$$\begin{aligned} P(X > 3) &= 1 - (P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)) = 1 - e^{-1} \cdot (1 + 1 + \frac{1}{2} + \frac{1}{3!}) \\ &= 1 - e^{-1} \cdot (1 + 1 + \frac{1}{2} + \frac{1}{6}) = 1 - \frac{8}{3e} . \end{aligned}$$

7. (10 points)

1. In a certain population the height is normally distributed with mean 70 inches and standard-deviation 5 inches.

(i) What is the probability that an individual will be taller than six feet?

(ii) What is the probability that an individual will be shorter than five feet?

ans. (i) $1 - \Phi(0.4)$; (ii) $\Phi(-2)$

(i)

$$P(X > 72) = P(X - 70 > 2) = P\left(\frac{X - 70}{5} > \frac{2}{5}\right) = P\left(Z > \frac{2}{5}\right) = 1 - \Phi(0.4)$$

(ii)

$$P(X < 60) = P(X - 70 < 60 - 70) = P(X - 70 < -10) = P\left(\frac{X - 70}{5} < \frac{-10}{5}\right) = P(Z < -2) = \Phi(-2)$$

8. (10 points) A certain lottery ticket costs one hundred dollars. The probability of winning one million dollars is %1, otherwise you get nothing. What is the expected net gain? Would you buy it? Explain,

ans. 9900. Not unless you are rich and have time.

The expected win is

$$10^6 \cdot \frac{1}{100} + 0 \cdot \frac{99}{100} = 10000 \quad .$$

Subtracting the price of the ticket, the expected gain is 9900, so it seems like a good deal. But almost surely, with prob. 0.99 you will lose one hundred dollars. So it is only a good deal if (i) you have all the time in the world, and you are allowed to either play many times, or buy many lottery tickets. (ii) You are very rich and can afford to keep buying these expensive tickets until you win.

9. (10 points) The probability that you win i dollars (i an integer satisfying $1 \leq i \leq 4$) is proportional to i (and otherwise it is 0). Let X be the amount won, find the standard deviation of X .

ans. s.d. of amount won=1 .

$$P(X = i) = ci \quad ,$$

where c is a positive constant. Since the total probability is 1 we have

$$\sum_{i=1}^4 ci = 1 \quad .$$

So

$$1 = c \sum_{i=1}^4 i = c \cdot (1 + 2 + 3 + 4) = 10c \quad ,$$

so $c = \frac{i}{10}$, and

$$P(X = i) = \frac{i}{10} (1 \leq i \leq 4) \quad .$$

$$E[X] = \sum_{i=1}^4 iP(X = i) = \sum_{i=1}^4 i \frac{i}{10} = \frac{1}{10} \left(\sum_{i=1}^4 i^2 \right) = \frac{1}{10} (1 + 4 + 9 + 16) = 3 \quad .$$

$$E[X^2] = \sum_{i=1}^4 i^2 P(X = i) = \sum_{i=1}^4 i^2 \frac{i}{10} = \frac{1}{10} \left(\sum_{i=1}^4 i^3 \right) = \frac{1}{10} (1 + 8 + 27 + 64) = 10 \quad .$$

$$Var(X) = E[X^2] - E[X]^2 = 10 - 3^2 = 10 - 9 = 1 \quad .$$

Finally

$$s.d(X) = \sqrt{Var(X)} = \sqrt{1} = 1 \quad .$$

10. (10 points) You toss a coin, whose probability of Tails is 0.3, 1000 times. Let X be the random variable:

“Number of Heads in the first 700 tosses Plus three times the Number of Tails in the last 300 tosses”

Find $E[X]$.

ans. $E[X] = 760$

Let X_1 be the r.v. ‘Number of Heads in the first 700 tosses.

Let X_2 be the r.v. ‘Number of Tails in the last 300 tosses.

We have

$$X = X_1 + 3X_2 \quad .$$

We have

$$E[X_1] = 700 \cdot 0.7 = 490 \quad .$$

$$E[X_2] = 300 \cdot 0.3 = 90 \quad .$$

By the **linearity of expectation**

$$E[X] = E[X_1] + 3E[X_2] = 490 + 3 \cdot 90 = 760 \quad .$$