Solutions to Math 477 REAL QUIZ #8

1. (4 points) In a certain community, the probability that a family has i boys and j girls is given by

$$p(i,j) = \begin{cases} \frac{c}{(2\,i+3\,j+1)} \quad , \quad if \quad 0 \leq i \leq 1 \quad and \quad 0 \leq j \leq 1; \\ 0 \quad , \quad otherwise. \end{cases} \quad ,$$

for some positive constant c (that would make it a discrete probability mass function). Calculate the conditional probability mass function for the number of boys in families that have exactly one girl.

Sol. to 1:

$$P\{G=1\} = p(0,1) + p(1,1) = \frac{c}{2\cdot 0 + 3 \cdot 1 + 1} + \frac{c}{2 \cdot 1 + 3 \cdot 1 + 1} = \frac{c}{4} + \frac{c}{6}$$

$$= c(\frac{1}{4} + \frac{1}{6}) = \frac{c}{2}(\frac{1}{2} + \frac{1}{3}) = \frac{c}{2} \cdot \frac{5}{6} = 5c/12 \quad .$$

Hence

$$p_{B|G}(i|1) = \frac{p(i,j)}{5c/12} \quad .$$

Hence

$$p_{B|G}(i|1) = \begin{cases} \frac{c}{(2\,i+3\cdot 1+1)(5c/12)} \quad , \quad if \quad 0 \leq i \leq 1; \\ 0 \quad , \quad otherwise. \end{cases} \quad ,$$

$$= \begin{cases} \frac{12}{(2\,i+4)\cdot 5} \quad , \quad if \quad 0 \leq i \leq 1; \\ 0 \quad , \quad otherwise. \end{cases} \quad ,$$

$$= \begin{cases} \frac{6}{(\,i+2)\cdot 5} \quad , \quad if \quad 0 \leq i \leq 1; \\ 0 \quad , \quad otherwise. \end{cases} \quad ,$$

Spelling it out: $p_{B|G}(0|1) = \frac{3}{5}$, $p_{B|G}(1|1) = \frac{2}{5}$, (and $p_{B|G}(i|1) = 0$ if $i \notin \{0,1\}$.)

2. (6 points) Using the linearity of expectation, find the average number of places $1 \leq i \leq n$ satisfying $\pi(i) \in \{i, i+1\}$ taken over all permutations of length n.

Sol. to 2: Let X_i (for integers i satisfying $1 \leq i \leq n$) be the **indicator** r.v. (defined on the set of permutations on $\{1, \ldots, n\}$)

$$X_i(\pi) = \begin{cases} 1, \quad if \quad \pi(i) \in \{i, i+1\}; \\ 0, \quad otherwise \end{cases} \quad .$$

The probability mass function of X_i is very simple . For $1 \leq i \leq n-1$, we have

$$P\{X_i = 1\} = \frac{2}{n} \quad , \quad P\{X_i = 0\} = 0 \quad .$$

This follows from the fact that the probability that $\pi(i) = i$ is $\frac{1}{n}$ and simililarly, that $\pi(i) = i + 1$, and these events are mutually exclusive. Since $\pi(n)$ can never be $n + 1$, we have to single this case out

$$P\{X_n = 1\} = \frac{1}{n} \quad , \quad P\{X_n = 0\} = 0 \quad .$$

Hence, for $1 \leq i \leq n - 1$, we have $E[X_i] = \frac{2}{n} \cdot 1 + (1 - \frac{2}{n}) \cdot 0 = \frac{2}{n}$. Similarly $E[X_n] = \frac{1}{n}$.

We have, of course

$$X = \sum_{i=1}^{n} X_i \quad .$$

By **linearity of expectation**:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n-1} E[X_i] + E[X_n] = (n-1)\frac{2}{n} + \frac{1}{n} = \frac{2n-1}{n} \quad .$$

Ans. to 2: The average number of places $1 \leq i \leq n$ satisfying $\pi(i) \in \{i, i+1\}$ taken over all permutations of length n, is $\frac{2n-1}{n}$.