Solutions to Math 477 REAL QUIZ #7

1. (6 points) The return on two investments, X and Y, follows the joint density function

$$f(x,y) \;=\; \begin{cases} \frac{1}{4} \;, & if \quad 0 < x + |y| < 2 \quad and \quad x > 0; \\ 0\,, & otherwise. \end{cases} \quad .$$

Find the marginal density functions $f_X(x)$ and $f_Y(y)$ and use them to find $E[X]$ and $E[Y]$.

Sol. to 1: The region where $f(x,y)$ is not zero is a triangle with vertices $(0,-2)$,$(0,2)$, and $(2,0)$.

$$f_X(x) \;=\; \int_{x-2}^{2-x} f(x,y)\,dy \;=\; \int_{x-2}^{2-x} \frac{1}{4}\,dy = \frac{(2-x)-(x-2)}{4} = 1 - \frac{x}{2} \quad , \quad (0 \le x \le 2) \quad .$$

Of course $f_X(x) = 0$ outside $0 \le x \le 2$.

$$f_Y(y) \;=\; \int_0^{2-|y|} f(x,y)\,dx \;=\; \int_0^{2-|y|} \frac{1}{4}\,dx \;=\; \frac{2-|y|}{4} \quad , \quad (-2 \le y \le 2) \quad .$$

Of course $f_Y(y) = 0$ outside $-2 \le y \le 2$.

$$E[X] \;=\; \int_0^2 x\,f_X(x)\,dx \;=\; \int_0^2 x(1-\frac{x}{2})\,dx \;=\; \int_0^2 (x - \frac{x^2}{2})\,dx = \left(\frac{x^2}{2} - \frac{x^3}{6}\right)\Big|_0^2 =$$

$$\frac{(2^2-0^2)}{2} - \frac{2^3-0^3}{6} = 2 - \frac{4}{3} = \frac{2}{3} \quad .$$

$$E[Y] \;=\; \int_{-2}^2 y\,f_Y(x)\,dx \;=\; \int_{-2}^2 y\frac{2-|y|}{4}\,dy \;=\; \int_{-2}^0 y\frac{2+y}{4}\,dy \;+\; \int_0^2 y\frac{2-y}{4}\,dy$$

$$=\; \int_{-2}^0 \frac{2y+y^2}{4}\,dy \;+\; \int_0^2 \frac{2y-y^2}{4}\,dy \;=\; \left(\frac{y^2}{4} + \frac{y^3}{12}\right)\Big|_{-2}^0 + \left(\frac{y^2}{4} - \frac{y^3}{12}\right)\Big|_0^2 = \;0 \quad .$$

Ans. to 1:

$$f_X(x) \;=\; \begin{cases} 1 - \frac{x}{2} & if \quad 0 \le x \le 2; \\ 0 \quad , & otherwise \end{cases} \quad ,$$

$$f_Y(y) \;=\; \begin{cases} \frac{2-|y|}{2} & if \quad -2 \le y \le 2; \\ 0 \quad , & otherwise \end{cases} \quad ,$$

$E[X] = \frac{2}{3}$, $E[Y] = 0$.

Comments: **(i)** People who are not comfortable with 'absolute value', are welcome to treat separately the cases $y < 0$ and $y > 0$. Then $f_Y(y)$ would look like

$$f_Y(y) \;=\; \begin{cases} \frac{2+y}{2} & if \quad -2 \le y \le 0; \\ \frac{2-y}{2} & if \quad 0 \le y \le 2; \\ 0 \quad , & otherwise \end{cases} \quad .$$

(ii) One can deduce that $E[Y] = 0$ by symmetry, since the integrand at the bottom half is exactly the negative of the integrand at the top half of the region.

2. (4 points) Two friends decide to meet at a certain restaurant. If each of them independently arrives at a time uniformly distributed between 12noon and $12:30$pm. Find the probability that the first to arrive has to wait longer than 10 minutes.

Sol to 2: Since the friends arrive **independently**, and each is **uniformly distributed** (i.e. follows $U(0,30)$), the joint density function is

$$f(x,y) = \begin{cases} \frac{1}{900} & if \quad 0 \le x, y \le 30; \\ 0 & , otherwise \end{cases} \quad .$$

Let X be the arrival time of friend I and Y the arrival time of friend II. The probability that friend I would have to wait more than 10 minutes to friend II is

$$P\{Y - X > 10\} = \int\int_R \frac{1}{900}\, dA \quad ,$$

where R is the region in the xy-plane

$$\{(x,y)\,|\,0 \le x \le 30\,,\, 0 \le y \le 30\,,\, y - x \ge 10\} \quad .$$

In other words the part of the square $[0,30] \times [0,30]$ that is above and to the left of the line $y = x + 10$. Drawing a little diagram shows that this region is a right-angled triangle with vertices

$$(0,10) \quad , \quad (0,30) \quad , \quad (20,30) \quad ,$$

that has base-length 20 and height 20. We know from fourth grade that the area of this triangle is $20 \cdot 20/2 = 200$. Multiplying by the **constant** integrand, $\frac{1}{900}$, we get that the probability that friend I would have to wait more than 10 minutes to friend II is $200 \cdot \frac{1}{900} = \frac{2}{9}$.

By symmetry, the probability that friend II would have to wait more than 10 minutes to friend I is also $\frac{2}{9}$, hence the probability that the first to arrive has to wait longer than 10 minutes is $2 \cdot \frac{2}{9} = \frac{4}{9}$.

Ans. to 2: The probability that the first to arrive has to wait longer than 10 minutes is $\frac{4}{9}$.