Solutions to Math 477 REAL QUIZ #5

Version of Dec. 16, 2017 (Thanks to Mitchell Bregman)

1. (5 points) A coin whose probability of Heads is 0.6 is continually flipped until heads appears for the 5-th time. Let X denote the number of tosses. Compute the probability mass function of X.

Sol. of 1: This calls for the **negative binomial distribution** with parameters $p = 0.6$ and $r = 5$. We have, in general

$$P\{X = n\} \; = \; \binom{n-1}{r-1} (1-p)^{n-r}\, p^r \quad , \quad n = r, r+1, \ldots \quad .$$

Plugging-in $p = 0.6$ and $r = 5$, we get

$$P\{X = n\} \; = \; \binom{n-1}{4} (0.4)^{n-5}\, 0.6^5 \; = \; \binom{n-1}{5} (0.4)^n \, (0.6/0.4)^5$$

$$= \; \binom{n-1}{4} (0.4)^n \, (3/2)^5 \; = \; \frac{243}{32} \binom{n-1}{4} (0.4)^n \quad n \geq 5, \quad .$$

Ans. to 1: The probability mass function of X is $P\{X = n\} \; = \; \frac{243}{32} \binom{n-1}{4} (0.4)^n$, for $n \geq 5$. .

2. (5 points) A company agrees to accept the highest of two sealed bids on property. The two bids are regarded as two independent random variables with common density function

$$f(x) \; = \; 5\,x^4 \quad , \quad 0 < x < 1 \quad .$$

What is the expected value of the accepted bid?

Sol. to 2: We are given the **pdf** for an individual bid. We must first find the **cdf**, let's call it $F(x)$.

$$F(x) = \int f(x)\,dx = \int 5x^4\,dx \; = \; x^5 + C \quad ,$$

where C is an **arbitrary constant**. But $F(0) = 0$ so we get $0 = 0 + C$ and hence $C = 0$, and we know that the cdf of an individual bid if $F(x) = x^5$.

There are **two** sealed bids, let's call them X_1, X_2. We are interested in the random variable

$$X := max(X_1, X_2) \quad .$$

By logic, the event $X \leq x$ is the same as the event $X_1 \leq x$ AND $X_2 \leq x$, for any x. Hence

$$P\{X \leq x\} = P(\{X_1 \leq x\} \; AND \; (\{X_2 \leq x\}) \quad .$$

By **independence**

$$P\{X \leq x\} = P(\{X_1 \leq x\} \cdot P(\{X_2 \leq x\} \quad .$$

But, we know that $P(\{X_1 \leq x\} = P(\{X_2 \leq x\} = F(x)$, so

$$P\{X \leq x\} = F(x)^2 = (x^5)^2 = x^{10} \quad .$$

Hence the **cdf** of X is x^{10}, hence its **pdf** is $(x^{10})' = 10x^9$. Finally

$$E[X] = \int_0^1 x\,(10x^9) \;=\; \int_0^1 10x^{10} \;=\; 10\frac{x^{11}}{11}\Big|_0^1 = \frac{10}{11} \cdot (1^{11} - 0^{11}) = \frac{10}{11} \quad .$$

Ans. to 2: The expected value of the accepted bid is $\frac{10}{11}$.