Solutions to Math 477 REAL QUIZ #10

1. (5 points) Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{x+y}{15} & , \quad if \quad 0 < x < 2\,, \, 0 < y < 3\,; \\ 0 & , \quad otherwise. \end{cases} \quad .$$

Find

(a) $E[X \,|\, Y = y]$

(b) $E[Y \,|\, X = x]$

Sol. to 1(a) When $0 < y < 3$, we have

The **denominator** is $f_Y(y)$:

$$\int_0^2 \frac{x+y}{15}\, dx = \frac{1}{15}\left(\frac{x^2}{2} + y\, x\right)\Big|_0^2 = \frac{1}{15}\left((2^2 - 0^2)/2 + y(2-0)\right)$$

$$= \frac{1}{15}(2 + 2y) \quad .$$

The **numerator** is

$$\int_0^2 x\frac{x+y}{15}\, dx \; = \; \int_0^2 \frac{x^2 + yx}{15}\, dx = \frac{1}{15}\left(\frac{x^3}{3} + y\,\frac{x^2}{2}\right)\Big|_0^2 = \frac{1}{15}\left((2^3 - 0^3)/3 + y(2^2 - 0^2)/2\right)$$

$$= \frac{1}{15}(\frac{8}{3} + 2y) \quad .$$

Hence, for $0 < y < 3$,

$$E[X \,|\, Y = y] \; = \; \frac{\frac{8}{3} + 2y}{2 + 2y} = \frac{3y+4}{3(y+1)} \quad .$$

Ans. to 1(a):

$$E[X \,|\, Y = y] \; = \begin{cases} \frac{3y+4}{3(y+1)} & , \quad if \quad 0 < y < 3\,; \\ 0 & , \quad otherwise \end{cases} \quad .$$

Sol. to 1(b) When $0 < x < 2$, we have

The **denominator** is $f_X(x)$:

$$\int_0^3 \frac{x+y}{15}\, dy = \frac{1}{15}\left(x\, y + \frac{y^2}{2}\right)\Big|_0^3 = \frac{1}{15}\left(x(3-0) + (3^2 - 0^2)/2\right)$$

$$= \frac{1}{15}(3x + \frac{9}{2}) \quad .$$

The **numerator** is

$$\int_0^3 y\frac{x+y}{15}\,dy = \frac{1}{15}\int_0^3 (xy+y^2)\,dy \;=\; \frac{1}{15}\left(x\,\frac{y^2}{2}+\frac{y^3}{3}\right)\Big|_0^3 = \frac{1}{15}\left(x(3^2-0^2)/2+(3^3-0^3)/3\right)$$

$$= \frac{1}{15}(\frac{9}{2}\,x\,+\,9) \quad .$$

Hence, for $0 < x < 2$,

$$E[Y\,|\,X=x] \;=\; \frac{\frac{9}{2}x+9}{3x+\frac{9}{2}} = \frac{3(x+2)}{2x+3} \quad .$$

Ans. to 1(b):

$$E[Y\,|\,X=x] \;=\; \begin{cases} \frac{3\,(x+2)}{2\,x+3} \quad , & if \quad 0<x<2\,; \\ 0 \quad , & otherwise \end{cases} \quad .$$

2. (5 points) Find the first two moments, and the variance, of a random variable X whose moment generating function is given by

$$M_X(t) = (1-3t)^{-2} \quad .$$

Sol. to 2: By the **Binomial theorem**

$$(1-3t)^{-2} \;=\; 1+(-2)(-3t)+\frac{(-2)(-3)}{2}(-3t)^2\;+\ldots\;=\;1\;+\;6\,t\;+\;27\,t^2+\ldots \quad .$$

Hence

$$m_1 = 6 \quad , \quad \frac{m_2}{2} = 27 \quad .$$

Hence $m_1 = 6$, $m_2 = 54$ and $Var(X) = m_2 - m_1^2 = 54 - 6^2 = 54 - 36 = 18$.

Ans. to 2: The first moment (alias mean, alias expectation) is 6, the second moment is 54 and $Var(X) = 18$.