Solutions to Math 477 "QUIZ" for Lecture 20

1. Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{x+y}{15} & , \quad if \quad 0 < x < 2\,, \, 0 < y < 3\,; \\ 0 & , \quad otherwise. \end{cases} \quad .$$

Find

(a) $E[X \,|\, Y = y]$

(b) $E[Y \,|\, X = x]$

Sol. to 1(a): For the sake of convenience, we can ignore the constant $\frac{1}{15}$, since it would cancel out

$$E[X|Y=y] \;=\; \frac{\int_0^2 x\,(x+y)\,dx}{\int_0^2 (x+y)\,dx} \quad .$$

The top is

$$\int_0^2 x\,(x+y)\,dx \;=\; \int_0^2 (x^2+xy)\,dx \;=\; \left(\frac{x^3}{3} + y\,\frac{x^2}{2}\right)\Big|_0^2 = \frac{2^3-0^3}{3} + y\,\frac{(2^2-0^2)}{2} \;=\; \frac{8}{3} + 2y$$

The bottom is

$$\int_0^2 (x+y)\,dx \;=\; \left(\frac{x^2}{2} + y\,x\right)\Big|_0^2 = \left(\frac{2^2-0^2}{2} + y\,(2-0)\right) = 2+2y \quad .$$

Hence

$$E[X|Y=y] \;=\; \frac{\frac{8}{3}+2y}{2+2y} = \frac{3y+4}{3(y+1)} \quad .$$

Ans. to 1(a): $E[X|Y=y] \;=\; \frac{3y+4}{3(y+1)}$.

Sol. to 1(b): For the sake of convenience, we can ignore teh constant $\frac{1}{15}$, since it would cancel out

$$E[Y|X=x] \;=\; \frac{\int_0^3 y\,(x+y)\,dy}{\int_0^3 (x+y)\,dy} \quad .$$

The top is

$$\int_0^3 y\,(y+x)\,dy \;=\; \int_0^3 (y^2+x\,y)\,dy \;=\; \left(\frac{y^3}{3} + x\,\frac{y^2}{2}\right)\Big|_0^3 = \frac{3^3-0^3}{3} + x\,\frac{(3^2-0^2)}{2} \;=\; 9 + \frac{9}{2}\,x$$

The bottom is

$$\int_0^3 (y+x)\,dy \;=\; \left(\frac{y^2}{2} + x\,y\right)\Big|_0^3 = \frac{3^2-0^2}{2} + x\,(3-0) \;=\; \frac{9}{2} + 3\,x \quad .$$

Hence

$$E[Y|X = x] \;=\; \frac{9 \,+\, \frac{9}{2}\,x}{\frac{9}{2} \,+\, 3\,x} = \frac{3\,(x \,+\, 2)}{2\,x \,+\, 3} \quad .$$

Ans. to 1(b): $E[Y|X = x] \;=\; \frac{3(x+2)}{2x+3} \quad .$

2. A miner is trapped in a mine containing 2 doors.

• The first door leads to a tunnel that will take him to safety after 2 hours of travel.

• The second door leads to a tunnel that will take him back to the mine in 4 hours of travel.

If the probabilities of him choosing the first door is $\frac{1}{3}$, and of him choosing the second door is $\frac{2}{3}$, what is the expected length of time until he reaches safety?

Sol. to 2: Let T be the expected time to safety. Then,

$$T = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot (T + 4) \quad .$$

since with prob. $\frac{1}{3}$ he would immediately go to safety (that takes 2 hours) , and with prob. $\frac{2}{3}$ he would be back where he started, alas, having wasted 4 hours, so he still expects to have T hours, totalling $T + 4$ hours for that secenario.

Simplifying

$$T = \frac{2}{3} + \frac{2}{3}T \,+\, \frac{8}{3} \quad .$$

$$\frac{T}{3} \;=\; \frac{10}{3} \quad .$$

Giving $T = 10$.

Ans. to 2: The expected length of time until he reaches safety is ten hours.