Solutions to Math 477 "QUIZ" for Lecture 18

1. You go to a strange casino where you have a chance of 0.01 to win 100 dollar, a chance of 0.02 to win 50 dollars, and 0.97 chance to lose a dollar. You do it for n days, and each time is independent of the other times. If X is the random variable denoting your total gain, what is the probability generating function? What is $E[X]$? What is $Var(X)$?

Sol. to 1: The **probability generating function** for **one day** is

$$0.01\, x^{100} \,+\, 0.02\, x^{50} + 0.97\, x^{-1} \quad .$$

By **independence** the probability generating function for the r.v. "total gain" is

$$P(x) = \left(0.01\, x^{100} \,+\, 0.02\, x^{50} + 0.97\, x^{-1}\right)^n \quad .$$

Now for $E[X]$

First way: Using the fact that $E[X] = P'(1)$.

$$P'(x) = n\left(0.01\, x^{100} \,+\, 0.02\, x^{50} + 0.97\, x^{-1}\right)^{n-1} \cdot (0.01\,(100)\, x^{99} \,+\, 0.02\,(50)\, x^{49} + 0.97\,(-1) x^{-2}) \quad .$$

Hence

$$P'(1) = n \cdot (1) \cdot (0.01\,(100)\, 1^{99} \,+\, 0.02\,(50)\, 1^{49} + 0.97\,(-1) 1^{-2})$$
$$= \, n\,(1 + 1 - 0.97) = 1.03 \cdot n \quad .$$

Second Way: Use Linearity of expectation.

The expected gain of one day's winning is $0.01 \cdot 100 + 0.02 \cdot 50 + (0.97)(-1) = 1.03$. Each day has the same expectation, and hence the total expected gain is 1.03 added n times yielding $1.03\, n$.

Note: For $E[X]$ we did not need the fact that each day is independent of other days.

Now for $Var(X)$.

First way: Find $(x\frac{d}{dx})^2 \left((0.01\, x^{100} \,+\, 0.02\, x^{50} + 0.97\, x^{-1})/x^{1.03}\right)^n$ and then plug-in $x = 1$. (You do it!)

Second Way Using the linearity of the variance for a sum of **independent** random variables.

Let X_i be the r.v. of the i-th day's gain.

$$E[X_i^2] = 0.01 \cdot 100^2 + 0.02 \cdot 50^2 + (0.97)(-1)^2 = 150.97 \quad .$$

Hence

$$Var(X_i) = E[X_i^2] - E[X_i]^2 = 150.97 - 1.03^2 = 149.9091 \quad .$$

Hence $Var(X) = Var(X_1) + \ldots + Var(X_n) = n \cdot 149.9091$.

Ans. to 1: The prob. generating function for the total gain is $\left(0.01\,x^{100} + 0.02\,x^{50} + 0.97\,x^{-1}\right)^n$, the expectation of X is $1.03\,n$ and the variance of X is $149.9091\,n$.

2. If you enter a casino with 100 dollars, and wish to make 200 dollars, and the probability, at each round, of winning a dollar is 0.5 and losing a dollar is 0.5, what is the probability of exiting a loser? How long would you expect to stay in the casino?

Sol. to 2: Since this is a **fair** casino (that unfortunately, does not exist in the real world), the probabibilty of winning is $\frac{100}{200} = \frac{1}{2}$, and hence of losing is $1 - \frac{1}{2} = \frac{1}{2}$, and the expected duration of staying in the casino (until you either make 200 dolalrs or get broke) is $100 \cdot (200 - 100) = 100 \cdot 100 = 10000$ rounds.

Ans. to 2: The probability of exiting a loser is $\frac{1}{2}$. amd the expected duration until you get out is 10000 rounds.