Solutions to Math 477 "QUIZ" for Lecture 17

1. For the sample space of all n coin-tosses of a loaded coin with probability p of landing on Head, let X be the number of occurrences of two heads-in-a-row that come after a tail and is followed by a tail, (i.e. the number of occurrences of $thht$). Find $E[X]$

Sol. to 1: Let $\{c_i\}_i^n$ be the sequence of coin-tosses. For $1 \leq i \leq n-3$, let Let X_i be the r.v. that is 1 if $c_i = t, c_{i+1} = h, c_{i+2} = h, c_{i+3} = t$. For each such i, $E[X_i] = 1 \cdot (1-p)pp(1-p) = p^2(1-p)^2$. By **Linearity of Expectation**,

$$E[X] = \sum_{i=1} n - 3 = p^2(1-p)^2 = (n-3)\, p^2(1-p)^2 \quad .$$

Ans. to 1: $E[X] = (n-3)\, p^2(1-p)^2$.

2. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{x+y}{8} \quad , & if \quad 0 < x < 2\,,\, 0 < y < 2; \\ 0 & otherwise \end{cases}$$

Find

(i) $E[X+Y]$

(ii) $Var(X+Y)$

Sol. to 2:

$$E[X+Y] = \int_0^2 \int_0^2 (x+y)\, f(x,y)\, dx\, dy = \int_0^2 \int_0^2 (x+y)\, \frac{x+y}{8}\, dx\, dy \quad .$$

Hence

$$E[X+Y] = \frac{1}{8} \int_0^2 \int_0^2 (x+y)^2\, dx\, dy.$$

Let's do $\int_0^2 \int_0^2 (x+y)^2\, dx\, dy$ (and later multiply by $\frac{1}{8}$).

$$\int_0^2 \int_0^2 (x+y)^2\, dx\, dy = \int_0^2 \int_0^2 (x^2 + 2xy + y^2)\, dx\, dy,$$

$$= \int_0^2 \int_0^2 x^2\, dx\, dy + 2 \int_0^2 \int_0^2 xy\, dx\, dy + \int_0^2 \int_0^2 y^2\, dx\, dy$$

$$(\int_0^2 x^2\, dx)(\int_0^2 dy) + 2(\int_0^2 x\, dx)(\int_0^2 ydy_) + (\int_0^2 dx)(\int_0^2 y^2\, dy)$$

$$= \frac{8}{3} \cdot 2 + 2(\frac{2^2}{2})^2 + 2 \cdot \frac{8}{3} = \frac{16}{3} + 8 + \frac{16}{3} = \frac{32+24}{3} = \frac{56}{3} \quad .$$

Multiplying by $\frac{1}{8}$, we get

$$E[X+Y] = \frac{1}{8} \cdot \frac{56}{3} = \frac{7}{3} \quad .$$

We next need $E[(X+Y)^2]$.

$$E[(X+Y)^2] \;=\; \frac{1}{8} \int_0^2 \int_0^2 (x+y)^3 \, dx \, dy.$$

Let's do $\int_0^2 \int_0^2 (x+y)^3 \, dx \, dy$ (and later multiply by $\frac{1}{8}$).

$$\int_0^2 \int_0^2 (x+y)^3 \, dx \, dy \;=\; \int_0^2 \int_0^2 (x^3 + 3x^2y + 3xy^2 + y^3) \, dx \, dy.$$

By symmetry, this equals

$$= \; 2 \left(\int_0^2 \int_0^2 x^3 \, dx \, dy \; + 3 \int_0^2 \int_0^2 x^2 y \, dx \, dy \right)$$

$$2 \left(\int_0^2 x^3 \, dx \right) \left(\int_0^2 dy \right) + 6 \left(\int_0^2 x^2 \, dx)(\int_0^2 y dy \right)$$

$$= 2 \cdot \frac{16}{4} \cdot 2 + 6 \frac{2^3}{3} \cdot \frac{2^2}{2} \; = \; 16 + 32 = 48 \quad .$$

Multiplying by $\frac{1}{8}$, we get

$$E[(X+Y)^2] = \frac{1}{8} \cdot 48 = 6 \quad .$$

Hence

$$Var(X+Y) \quad = \quad E[(X+Y)^2] - E[X+Y]^2 \quad = \quad 6 - (\frac{7}{3})^2 = \frac{5}{9} \quad .$$

Ans. to 2: $E[X+Y] = \frac{7}{3}$ and $Var(X+Y) = \frac{5}{9}$.