Solutions to Math 477 "QUIZ" for Lecture 16

1. In a certain (not that wealthy) town, the probability that a household has i cars and j bed-rooms is

$$p(i,j) = \begin{cases} \frac{i+j}{16} & if \quad 0 \leq i \leq 1 \quad and \quad 0 \leq j \leq 3; \\ 0 \quad , & otherwise. \end{cases} \quad ,$$

If it is known that a household has 2 bed-rooms, what is the probability that it has a car?

Sol. to 1: The probability that it has 2 bed-rooms is

$$P(TwoBedRooms) = p(0,2) + p(1,2) = \frac{0+2}{16} + \frac{1+2}{16} = \frac{5}{16} \quad .$$

The probability that it has two bedroos and one car is $p(1,2) = \frac{1+2}{16} = \frac{3}{16}$.

Hence the conditional probability, $P(OneCar|TwoBedRooms)$,

$$P(OneCar|TwoBedRooms) = \frac{P(OneCar \quad AND \quad TwoBedRooms)}{P(TwoBedRooms)} = \frac{\frac{3}{16}}{\frac{5}{16}} = \frac{3}{5} \quad .$$

Ans. to 1: The probability that a family has a car if it is known that it has two bed-rooms is $\frac{3}{5}$.

2. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2\,(x+2\,y)}{3} \quad , & if \quad 0 < x < 1\,,\, 0 < y < 1\,; \\ 0 \quad , & otherwise, \end{cases}$$

(i) Compute the conditional density of X given that $Y = y$. (ii) If you know that $Y = 0.5$ what is the probability that $0 \leq X \leq 0.5$.

Sol. to 2(i): We first need to find the **marginal density function** $f_Y(y)$.

$$f_Y(y) \;=\; \int_0^1 f(x,y)\,dx \;=\; \int_0^1 f(x,y)\,dx \;=\; \int_0^1 \frac{2\,(x \;+\; 2\,y)}{3}\,dx$$

$$= \frac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_0^1 = \frac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_0^1 = \frac{2}{3}(\frac{1}{2} \;+\; 2y(1)) \;=\; \frac{1+4y}{3} \quad .$$

Hence the conditional density of X given that $Y = y$, denoted by $f_{X|Y}(x|y)$, that *by definition* is $f(x,y)/f_Y(y)$, is given by

$$\frac{\frac{2}{3}(x+2y)}{\frac{1+4y}{3}} = \frac{2(x+2y)}{1+4y} \quad .$$

Ans. to 2(i): The conditional density of X given that $Y = y$ is $\frac{2(x+2y)}{1+4y}$.

Sol. to 2(ii): Plugging-in $y = 0.5$ into $f_{X|Y}(x|y) = \frac{2(x+2y)}{1+4y}$, we get

$$f_{X|Y}(x|0.5) = \frac{2(x + 2 \cdot 0.5)}{1 + 4 \cdot 0.5} = \frac{2(x+1)}{3} = \frac{2}{3}(x+1) \quad .$$

$$P(0 \leq X \leq \frac{1}{2}\,|\,Y = \frac{1}{2}) \;=\; \int_0^{0.5} \frac{2}{3}(x{+}1)\,dx \;=\; \frac{2}{3}\left(\frac{x^2}{2} + x\right)\Big|_0^{\frac{1}{2}} = \frac{2}{3}\left(\frac{(\frac{1}{2})^2 - 0^2}{2} + (\frac{1}{2} - 0)\right) = \frac{2}{3}(\frac{5}{8}) \;=\; \frac{5}{12} \quad .$$

Ans. to 2(ii): The probability that $0 \leq X \leq 0.5$. if you know that $Y = 0.5$, is $\frac{5}{12}$.