Solutions to Math 477 "QUIZ" for Lecture 14

1. In a certain community the maximum number of boys and the maximum number of girls are both 2.

It is found that the probability density function

$$p(i,j) = Pr(NumberOfBoys = i, NumberOfGirls = j) \; = \; c\, i\, j \quad , \quad 0 \leq i \leq 2 \quad 0 \leq j \leq 2 \quad ,$$

for some constant c.

(i) Find c. (ii) Find the probability that a family has strictly more girls than boys.

(iii) Find the expected number of boys.

Sol. to (1)(i): Since the probabilities must add-up to 1

$$1 = c(0\cdot 0 + 0\cdot 1 + 0\cdot 2 + 1\cdot 0 + 1\cdot 1 + 1\cdot 2 + 2\cdot 0 + 2\cdot 1 + 2\cdot 2) = c(0+1+2)(0+1+2) = c\cdot 9 \quad ,$$

hence $c = \frac{1}{9}$

Sol. to (1)(ii):

$$P\{i < j\} = p(0,1) + p(0,2) + p(1,2) = \frac{1}{9}(0\cdot 1 + 0\cdot 2 + 1\cdot 2) = \frac{2}{9} \quad .$$

Ans. to (1)(ii): The probability that a family has strictly more girls than boys is $\frac{2}{9}$.

Sol. to (1)(iii): Let's first compute the **marginal distribution** of 'number of boys'.

$$P\{B=0\} \; = \; p(0,0) + p(0,1) + p(0,2) = \frac{1}{9}\cdot(0\cdot 0 + 0\cdot 1 + 0\cdot 2) = 0$$

$$P\{B=1\} = p(1,0) + p(1,1) + p(1,2) = \frac{1}{9}\cdot(1\cdot 0 + 1\cdot 1 + 1\cdot 2) = \frac{1}{3}$$

$$P\{B=2\} = p(2,0) + p(2,1) + p(2,2) = \frac{1}{9}\cdot(2\cdot 0 + 2\cdot 1 + 2\cdot 2) = \frac{2}{3}$$

Hence

$$E[B] = \sum_{i=0}^{2} i\cdot P\{B=i\} = 0\cdot 0 + 1\cdot\frac{1}{3} + 2\cdot\frac{2}{3} = \frac{5}{3} \quad .$$

Ans. to (1)(iii): The expected number of boys is $\frac{5}{3}$.

2. A device runs until either of the two components fails, at which point the device stops running. The lifetimes of the two components has a joint probability density function

$$f(x,y) \; = \; \frac{2\,x + y}{96} \quad , \quad for \quad 0 < x < 4 \quad and \quad 0 < y < 4 \quad .$$

What is the probability that the device fails during the first two hours of operation?

Sol. to 2: The probability that **both** devices survive beyond the first two hours is

$$P\{X \geq 2 \quad AND \quad Y \geq 2\} = \int_2^4 \int_2^4 \frac{2x+y}{96} \, dxdy$$

The **inner integral is**

$$\frac{1}{96} \int_2^4 (2x+y) \, dx = \frac{x^2+yx}{96}\Big|_2^4 = \frac{1}{96} \cdot \big((4^2+4y) - (2^2+2y)\big) = \frac{12+2y}{96} \quad .$$

The **outer integral** is

$$\frac{1}{96} \int_2^4 (12+2y) \, dy = \frac{1}{96} \cdot \left(12y+y^2\right)\Big|_2^4 = \frac{1}{96}(12 \cdot (4-2) + (4^2-2^2)) = \frac{1}{96}(24+12) = \frac{36}{96} = \frac{3}{8} \quad .$$

Hence that probability that the two components do **not both** survive beyond the first two hours is the **complementary probability** $1 - \frac{3}{8} = \frac{5}{8} = 0.625$.

Ans. to 2: The probability that the device fails during the first two hours of operation is $\frac{5}{8} = 0.625$.

Comment: Many people set up the integral

$$\int_0^2 \int_0^2 \frac{2x+y}{96} \, dxdy \quad .$$

(and got, correctly for this integral, $\frac{1}{8}$). This is the right answer to the **wrong question**. This is the probability that **both** devices break-down before two hours. But even if one device breaks down, it stops working.