Solutions to Math 477 "QUIZ" for Lecture 11

1. The density function of a continuous random variable, X, is given by

$$f(x) \;=\; \begin{cases} \frac{x^2}{72} & if \quad 0 \leq x \leq 6 \;; \\ 0 & otherwise. \end{cases} \qquad .$$

(i) What is the probability that X is between 0 and 4?

(ii) What is the probability that X is between 0 and 2, if it is known that it is between 0 and 4.

Sol. of 1(i):

$$P\{0 < X < 4\} \;=\; \int_0^4 \frac{x^2}{72}\, dx \;==\; \frac{1}{72}\frac{x^3}{3}\Big|_0^4 = \frac{4^3 - 0^3}{3 \cdot 72} \;=\; \frac{64}{3 \cdot 72} \;=\; \frac{8}{3 \cdot 9} \;=\; \frac{8}{27} \quad .$$

Ans. to 1(i): the probability that X is between 0 and 4 is $\frac{8}{27}$.

Sol. of 1(ii): Using $P(E|F) = P(EF)/P(F)$ where E is the event $\{0 < X < 2\}$ and F is the event $\{0 < X < 4\}$, and since $EF = E$ (E is a subset of F)

$$P(0 < X < 2 \,|\, 0 < X < 4) \;=\; \frac{P\{0 < X < 2\}}{P\{0 < X < 4\}} \quad .$$

We already know the denominator (see above). As for the numerator:

$$P\{0 < X < 2\} \;=\; \int_0^2 \frac{x^2}{72}\, dx \;=\; \frac{1}{72}\frac{x^3}{3}\Big|_0^2 = \frac{2^3 - 0^3}{3 \cdot 72} \;=\; \frac{8}{3 \cdot 72} \;=\; \frac{1}{3 \cdot 9} \;=\; \frac{1}{27} \quad .$$

So

$$P(0 < X < 2 \,|\, 0 < X < 4) \;=\; \frac{\frac{1}{27}}{\frac{8}{27}} = \frac{1}{8} \quad .$$

Ans. to 1(ii): The probability that X is between 0 and 2, if it is known that it is between 0 and 4 is $\frac{1}{8}$.

2. The lifetime of a machine part has continuous distribution on the interval $(0, 10)$, with probability density function f, where $f(x)$ is proportional to $\frac{1}{1+x}$. What is the probability that the lifetime of the machine part is less than 4 ?

Sol. of 2: We **first** need to find the exact expression for the **probability density function** $f(x)$. We are told that

$$f(x) \;=\; \begin{cases} \frac{c}{1+x} & if \quad 0 \leq x \leq 10 \;; \\ 0 & otherwise. \end{cases} \qquad ,$$

for some constant c to be determined. Since, **always**,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \quad .$$

Since $f(x)$ is only non-zero in the interval $(0, 10)$, we have

$$\int_0^{10} \frac{c}{1+x}\, dx = 1 \quad ,$$

So

$$c\int_0^{10} \frac{1}{1+x}\, dx = 1 \quad ,$$

hence

$$c\left(\ln(1+x)\Big|_0^{10} \right) = c\,(\ln(11) - \ln(1)) = c\,(\ln 11) \; = \; 1 \quad .$$

We get $c = 1/(\ln 11)$. Hence

$$P\{0 < X < 4\} = \int_0^4 f(x)\, dx \; = \; \frac{1}{\ln 11} \int_0^4 \frac{1}{1+x} \; = \; \frac{1}{\ln 11}\, (\ln(1+x))\Big|_0^4)$$

$$= \frac{1}{\ln 11}\, (\ln 5 - \ln 1) \; = \; \frac{1}{\ln 11}(\ln 5 - 0) \; = \; \frac{\ln 5}{\ln 11} \quad .$$

Ans. to 2: The probability that the lifetime of the machine part is less than 4 is $\frac{\ln 5}{\ln 11} = 0.6711877412\ldots$.