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Solutions to MATH 477 (3), Dr. Z. , Exam 1, Thurs., Oct. 19, 2017, 8:40-10:00am, HLL 116

Version of Oct. 19, 2017, 9:03pm (Thanks to Zhiyun Xu)
PUT The FINAL ANSWER TO EACH PROBLEM IN THE AVAILABLE BOX
EXPLAIN EVERYTHING! Only simple calculaculators are allowed.

1. (10 pts.) The number of injury claims per month is modeled by a random variable N with

$$P\{N = n\} = \frac{6}{\pi^2 n^2}, \quad \textit{where} \quad n \geq 1 \quad ,$$

(and 0 otherwise). Determine the probability of exactly one claim during a particular month, given that there have been at most three claims during that month.

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ans. $\frac{36}{49}$

$$P(N = 1 \,|\, N \leq 3) \;=\; \frac{P\{N = 1\}}{P\{1 \leq N \leq 3\}} = \frac{P\{N = 1\}}{P\{N = 1\} + P\{N = 2\} + P\{N = 3\}}$$

$$\frac{(6/\pi^2) \cdot \frac{1}{1^2}}{(6/\pi^2) \cdot (\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2})}$$

$$\frac{1}{1 + \frac{1}{4} + \frac{1}{9}} = \frac{1}{\frac{36}{36} + \frac{9}{36} + \frac{4}{36}}$$

$$= \frac{1}{\frac{36+9+4}{36}} = \frac{1}{\frac{49}{36}} = \frac{36}{49} \quad .$$

2. (10 points) The alphabet set of certain language is $\{A, B, C\}$. The frequency of A is 0.5, the frequency of B is 0.3, the frequency of C is 0.2. If you make a random word of 9 letters (and draw each letter independently of the other ones), what is the probability that it has 3 A's, 3 B's, and 3 C's ?

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ans. $\frac{9!}{3!^3} \cdot (0.5)^3 \cdot (0.3)^3 \cdot (0.2)^3 = 0.045360$.

Recall that the **multinomial distribution** is

$$\frac{(n_1 + \ldots + n_k)!}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k} \quad .$$

Here $k = 3$ and $p_1 = 0.5, p_2 = 0.3, p_3 = 0.2$, and $n_1 = 3, n_2 = 3, n_3 = 3$.

3. (10 points) A company agrees to accept the highest of five sealed bids on property. The five bids are regarded as five independent random variables with common **probability density function**

$$f(x) \;=\; \begin{cases} 6x^5 & ,0 < x < 1 \quad ; \\ 0 \quad , & otherwise. \end{cases}$$

What is the expected value of the accepted bid?

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ans. $\frac{30}{31}$.

The **first** step is to find the **cumulative** prob. function, let's call it $F(x)$, for a single bid. It is

$$F(x) = \int 6x^5 \, dx \;=\; x^6 + C \quad .$$

But $F(0) = 0$ (since things start at 0, no company will offer to pay for the privilege of doing something), so $C = 0$, and hence

$$F(x) = x^6 + 0 = x^6 \quad .$$

Let X_1, X_2, X_3, X_4, X_5 be the individual bids, we know that

$$P\{X_1 \le x\} = F(x) \quad , \quad P\{X_2 \le x\} = F(x) \quad , \quad P\{X_3 \le x\} = F(x) \quad ,$$

$$P\{X_4 \le x\} = F(x) \quad , \quad P\{X_5 \le x\} = F(x) \quad .$$

Let Y be the maximum bid. Then, by **simple logic**

$$P\{Y \le x\} = P\{X_1 \le x \; AND \; X_2 \le x \; AND \; X_3 \le x \; AND \; X_4 \le x \; AND \; X_5 \le x\} \quad .$$

Since the bids are **independent**, we have

$$P\{Y \le x\} = P\{X_1 \le x\} \cdot P\{X_2 \le x\} \cdot P\{X_3 \le x\} \cdot P\{X_4 \le x\} \cdot P\{X_5 \le x\} = F(x)^5 = (x^6)^5 = x^{30} \quad .$$

Hence the **cumulative prob. function** for Y (the max. bid), let's call it $G(x)$ is

$$G(x) = x^{30} \quad .$$

Hence the **density function** for Y, let's call it $g(x)$ is

$$g(x) = G'(x) = 30x^{29} \quad .$$

Finally

$$E[Y] \;=\; \int_0^1 x g(x) \, dx \;=\; \int_0^1 30x^{30} \, dx = 30 \frac{x^{31}}{31} \Big|_0^1 = 30\left(\frac{1-0}{31}\right) = \frac{30}{31} \quad .$$

4. (10 points) In a certain class, there are Math Majors, CS majors, double-Math-CS majors, and students who major neither in math nor in CS.
• the probability that someone is only a math major (and not a CS major) is 0.4
• the probability that someone is only a CS major (and not a Math major) is 0.2
• the probability that someone is a double-math-CS major is 0.1
Find the conditional probability that someone is math major if he or she is not a CS major.

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ans. $\frac{4}{7}$.

Let's first gather data (C=CompSci Major, M=Math Major)

$$P(CM) = 0.1 \quad , \quad P(CM^c) = 0.2$$

$$P(C^cM) = 0.4$$

We still need $P(C^cM^c)$, but since all the four probabilities must add-up to 1 we get $P(C^cM^c) = 1 - (0.1 + 0.2 + 0,4) = 1 - 0.7 = 0.3$. Completing the above table we have

$$P(CM) = 0.1 \quad , \quad P(CM^c) = 0.2$$

$$P(C^cM) = 0.4 \quad , \quad P(C^cM^c) = 0.3 \quad .$$

In particular

$$P(C^c) = P(C^cM) + P(C^cM^c) = 0.4 + 0.3 = 0.7 \quad .$$

We need $P(M|C^c)$. Using the famous $P(E|F) = P(EF)/P(F)$, we have

$$P(M|C^c) = \frac{P(MC^c)}{P(C^c)} = \frac{0.4}{0.7} = \frac{4}{7} \quad .$$

5. (10 points) In a certain country $\frac{1}{5}$ of the population are college graduates. $\frac{9}{10}$ of the college graduates wear glasses, but only $\frac{1}{10}$ of the non-college graduates wear glasses. You see a random person in the street wearing glasses, what is the probability that she or he is a college graduate?

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ans. $\frac{9}{13}$.

This calls for **Bayes's Formula**:

$$P(College|Glasses) = \frac{P(Glasses|College) \cdot P(College)}{P(Glasses)} \quad .$$

Now

$$P(Glasses) = P(Glasses|College) \cdot P(College) + P(Glasses|NoCollege) \cdot P(NoCollege)$$

$$\frac{9}{10} \cdot \frac{1}{5} + \frac{1}{10} \cdot \frac{4}{5} = \frac{13}{50} \quad .$$

On the other hand the portion due to College is the first tern above

$$P(Glasses|College) \cdot P(College) = \frac{9}{10} \cdot \frac{1}{5} = \frac{9}{50} \quad .$$

Hence

$$P(College|Glasses) = \frac{\frac{9}{50}}{\frac{13}{50}} = \frac{9}{13} \quad .$$

6. (10 points) State the three axioms of probability (for finite sample spaces).
Sol.:
Given a finite set Ω called the *sample space*, P is a function on the set of subsets (denoted by 2^{Ω}), satisfying

- $P(\Omega) = 1$
- $0 \leq P(A) \leq 1$, for every subset A of Ω.
- If A and B are **disjoint** (i.e. $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B)$.

7. (10 points) The number of letters that I find in my mailbox, in any given day, has a Poisson distribution with mean 4. Today my wife told me that I got strictly more than two letters. What is the probability that I got exactly three letters?

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ans. $0.2564216597\ldots$ (or $\frac{32}{3(e^4-13)}$).

Recall that for a Poisson r.v. with parameter λ,

$$P\{X=k\} = e^{-\lambda}\frac{\lambda^k}{k!} \quad .$$

In this problem $\lambda = 4$.
We have

$$P(X=3|X\geq 3) = \frac{P\{X=3\}}{P\{X\geq 3\}} = \frac{P\{X=3\}}{1-P\{X\leq 2\}} = \frac{P\{X=3\}}{1-(P\{X=0\}+P\{X=1\}+P\{X=2\})}$$

$$\frac{e^{-4}\frac{4^3}{3!}}{1-e^{-4}\cdot(1+4/1!+4^2/2!)}$$

$$= \frac{\frac{64}{6}}{e^4-(1+4+8)} = \frac{32}{3(e^4-13)} = 0.2564216597\ldots \quad .$$

8. (10 points) A study is being conducted in which the health of four independent groups, each consisting of five policyholders, is monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independent of the other participants). What is the probability that there are at least two groups where at least four participants completed the study?

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ans. 0.9417583985...

This is a **two-step** problem calling for the use of the **Binomial Distribution** twice. Let's define a group to be succesful if at least four participants completed the study. This is $B(5, 0.8)$ since the probability of a person being succesful (i.e. **not** dropping out) is 0.8. Hence the probability of a group being successful, let's call it p is

$$P\{X \geq 4\} = \binom{5}{4}(0.8)^4 \cdot (0.2)^1 + \binom{5}{5}(0.8)^5 \cdot (0.2)^0$$

$$= 5 \cdot (0.8)^4 \cdot (0.2) + (0.8)^5 = (0.8)^4 \cdot (1 + 0.8) = (0.8)^4 \cdot (1.8) = 0.7372800000 \quad .$$

For the second step, we have yet again the binomial r.v. with parameters $(4, p)$. Hence (let's call the new r.v. Y)

$$P\{Y \geq 2\} = 1 - P\{Y \leq 1\} = 1 - (P\{Y = 0\} + P\{Y = 2\}) = 1 - \binom{4}{0} p^0 (1-p)^4 - \binom{4}{1} p^1 (1-p)^3$$

$$1 - (1 - 0.73728)^4 - 4 \cdot 0.73728 \cdot (1 - 0.73728)^3 = 0.9417583985\ldots \quad .$$

9. (10 points) The probability that you win i dollars ($1 \leq i \leq 3$) is proportional to $\frac{1}{i^2}$ (and otherwise it is 0). Let X be the amount won, find the standard deviation of X.

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ans. $\frac{6\sqrt{26}}{49}$ or 0.6243697363583

First we must find c

$$c \cdot (1/1^2 + 1/2^2 + 1/3^2) = 1 \quad .$$

Hence

$$1 = c \cdot (1 + 1/4 + 1/9) = c \cdot (36 + 9 + 4)/36 = c \cdot 49/36 \quad .$$

Hence

$$c = \frac{36}{49} \quad ,$$

and

$$P\{X = i\} = \frac{36}{49} \cdot \frac{1}{i^2} \quad 1 \leq i \leq 3 \quad .$$

Next

$$E[X] = \sum_{i=1}^{3} iP\{X = i\} = \frac{36}{49} \cdot (1 + 1/2 + 1/3) = \frac{66}{49} \quad .$$

Next, we need the **second moment**

$$E[X^2] = \sum_{i=1}^{3} i^2 P\{X = i\} = \frac{36}{49} \cdot (1 + 1 + 1) = \frac{108}{49} \quad .$$

Next

$$Var(X) = E[X^2] - E[X]^2 = \frac{108}{49} - (\frac{66}{49})^2 = \frac{936}{2401} = 0.3898375676801 \quad .$$

Finally

$$SD(X) = \sqrt{\frac{936}{2401}} = 0.6243697363583\ldots \quad .$$

Note: This problem was meant to be a discrete r.v., since dollars are discrete. Some people took it to be a *continuous* r.v.. In that case (replacting i by x to conform to tradition)

$$1 = c \int_1^3 \frac{1}{x^2}\, dx \;=\; c \frac{-1}{x}\Big|_1^3 = c(-1/3 - -1/1) = 2c/3 \quad .$$

So $c = \frac{3}{2}$. Next $E[X] = \frac{3}{2} \int_1^3 \frac{1}{x}\, dx = \frac{3}{2}(\ln 3)$, $E[X^2] = \frac{3}{2} \int_1^3 dx = 3$, and $Var(X) = 3 - (\frac{3}{2}(\ln 3))^2$, and $SD(X) = \sqrt{3 - (\frac{3}{2}(\ln 3))^2}$.

10. (10 points) You toss a coin, whose probability of Tails is 0.3, 1000 times. Let X be the random variable "Number of Heads in the first 700 tosses Plus three times the Number of Tails in the last 300 tosses". Find $E[X]$.

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ans. 760

Let

• X_1 be the r.v. "Number of Heads in the first 700 tosses"

• X_2 be the r.v. "Number of Tails in the first 700 tosses"

We need $E[X_1 + 3X_2]$.

X_1 is a Binomial random variable with parameters $(700, 0.7)$ (since the prob. of a Head is $1 - 0.3 = 0.7$), so

$$E[X_1] = 700 \cdot 0.7 = 490 \quad .$$

X_2 is a Binomial random variable with parameters $(300, 0.3)$, so

$$E[X_2] = 300 \cdot 0.3 = 90 \quad .$$

By **Linearity of Expectation**

$$E[X_1 + 3X_2] = E[X_1] + 3E[X_2] = 490 + 3 \cdot 90 = 490 + 270 = 760 \quad .$$