Dr. Z.'s Intro to Probability Homework assignment 17

1. Using the linearity of expectation prove that the average number of fixed points of a permutation of length n is 1.

2. (i) For the sample space of all n coin-tosses of a fair coin, let X be the number of occurrences of three heads-in-a-row that come after a tail and is followed by a tail, (i.e. the number of occurrences of $thhht$). Find $E[X]$

(ii) For the sample space of all n coin-tosses of a fair coin, let X be the number of occurrences of two heads-in-a-row (regardless of whether it follows a head or tail or is followed by a head or tail), (for example $X(hhhhh) = 4$, $X(hhthhthhh) = 4$). Find $E[X]$.

3. Let X be the number of Heads upon tossing a coin n times, where the probability of a Head is p. Find $Var(X)$.

4. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} x + y \quad , \quad if \quad 0 < x < 1\,,\, 0 < y < 1; \\ 0 \quad otherwise \end{cases}$$

Find

(i) $E[X]$

(ii) $E[Y]$

(iii) $E[X + Y]$

(iv) $Var(X + Y)$

5. The return on two investments, X and Y, follows the joint probability density function

$$f(x, y) \;=\; \begin{cases} 1/2 \quad , \quad 0 < |x| + |y| < 1 \\ 0 \quad , \quad otherwise. \end{cases}$$

Calculate (i) $E[X + Y]$ and (ii) $Var(X + Y)$.

6. The joint density function of X and Y is

$$f(x, y) \;=\; \begin{cases} \frac{x+y}{8} \quad , \quad for \quad 0 < x < 2\,,\, 0 < y < 2 \\ 0 \quad otherwise. \end{cases}$$

Calculate the variance of $(X + Y)/2$.

7. Let X denote the proportion of employees at a large firm who will choose to be covered under the firm's medical plan, and let Y denote the proportion who will choose to be covered under both the firm's medical and dental plan.

Suppose that for $0 \leq y \leq x \leq 1$, X and Y have the joint cumulative distribution function

$$F(x, y) = y(x^2 + xy - y^2) \quad .$$

Calculate the expected proportion of employees who will choose to be covered under both plans.

8. State and prove the formula for the expected number of coupons one has to buy in order to get a collection of n different coupons.

What is that expected number if you have ten coupons?